GERMS OF CHARACTERS

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Let F be a p-adic field of characteristic zero, $\mathbf{G}$ a a reductive group defined over F, and $G = \mathbf{G}(F)$. If π is an irreducible admissible representation of G, Θ_π denotes the character of π, viewed either as a distribution or as a locally integrable function which is locally constant on the set G_{reg} of regular semisimple elements in G. By the germ of Θ_π (at the identity), we mean the restriction of (the function) Θ_π to $V \cap G_{reg}$, where V is some open neighbourhood of the identity in G. In this talk, we discuss the relation between germs of characters and linear combinations of Fourier transforms of measures on $\mathrm{Ad}\, G$-orbits in the Lie algebra $\mathfrak{g}$ of G.

The symbol $\mathcal{O}$ will be used for an $\mathrm{Ad}\, G$-orbit in $\mathfrak{g}$. Given $X \in \mathfrak{g}$, $\mathcal{O}_G(X)$ or $\mathcal{O}(X)$ will be used for the $\mathrm{Ad}\, G$-orbit of X in $\mathfrak{g}$. Let $C_c^\infty(\mathfrak{g})$ be the set of of locally constant compactly supported complex valued functions on $\mathfrak{g}$. Given an orbit $\mathcal{O}$ and a function $f \in C_c^\infty(\mathfrak{g})$, let $\mu_{\mathcal{O}}(f)$ denote the integral of f with respect to a G-invariant measure on $\mathcal{O}$.

Recall that if $B(\cdot,\cdot)$ is a non-degenerate symmetric G-invariant bilinear form on $\mathfrak{g}$, ψ is a non trivial character of F, and dX is a Haar measure on $\mathfrak{g}$, the Fourier transform $\widehat{f}$ of f is given by:

$$\widehat{f}(X) = \int_{\mathfrak{g}} \psi((B(X,Y))\, f(Y)\, dY, \qquad f \in C_c^\infty(\mathfrak{g}).$$

Given an $\mathrm{Ad}\, G$-orbit $\mathcal{O}$, the Fourier transform $\widehat{\mu}_{\mathcal{O}}$ of the distribution $\mu_{\mathcal{O}}$ is defined by $\widehat{\mu}_{\mathcal{O}}(f) = \mu_{\mathcal{O}}(\widehat{f})$, $f \in C_c^\infty(\mathfrak{g})$. The distribution $\widehat{\mu}_{\mathcal{O}}$ is given by a locally integrable function on $\mathfrak{g}$ which is locally constant on $\mathfrak{g}_{reg}$ (regular semisimple elements in $\mathfrak{g}$). This function will also be denoted by $\widehat{\mu}_{\mathcal{O}}$.

Let π be an irreducible admissible representation of G. Recall ([HC]) that Harish-Chandra's local character expansion (at the identity) expresses the germ of Θ_π as a linear combination of $\{\, \widehat{\mu}_{\mathcal{O}} \mid \mathcal{O} \in (\mathcal{N}_G)\,\}$, where $(\mathcal{N}_G)$ is the set of nilpotent $\mathrm{Ad}\, G$-orbits: there exists an open neighbourhood V_π of zero in $\mathfrak{g}$, and complex numbers $c_{\mathcal{O}}(\pi)$, $\mathcal{O} \in (\mathcal{N}_G)$, such that

$$\Theta_\pi(\exp X) = \sum_{\mathcal{O} \in (\mathcal{N}_G)} c_{\mathcal{O}}(\pi)\, \widehat{\mu}_{\mathcal{O}}(X), \qquad X \in V_\pi \cap \mathfrak{g}_{reg}.$$

Here exp denotes the exponential map. In some examples a different map can be used in place of exp. For example, if $G = \mathbf{GL}_n(F)$, resp. G is a classical group, $X \mapsto 1 + X$, resp. the Cayley transform, is sometimes used.

It can be difficult to compute the coefficients $c_{\mathcal{O}}(\pi)$ and the functions $\widehat{\mu}_{\mathcal{O}}$. For certain representations π, it might be convenient to express the germ of Θ_π in terms of Fourier transforms of non nilpotent $\mathrm{Ad}\,G$-orbits. Suppose that π is a tamely ramified irreducible supercuspidal representation of a classical group ([A], [H], [Mo 1-3], [My1]). As shown in [Mu1-4], for many such π, there exists a regular elliptic $s_\pi \in \mathfrak{g}$ such that

$$\Theta_\pi(\exp X) = d(\pi)\,\widehat{\mu}_{\mathcal{O}(s_\pi)}(X), \qquad X \in V \cap \mathfrak{g}_{reg},$$

for some open neighbourhood V of zero in $\mathfrak{g}$. Here, $d(\pi)$ is the formal degree of π. For convenience, we could renormalize the measure on $\mathcal{O}(s_\pi)$ so that $d(\pi) = 1$. The orbit $\mathcal{O}(s_\pi)$ is defined using the construction of π via induction from a representation of some open compact mod centre subgroup of G. That is, $\mathcal{O}(s_\pi)$ depends on the K-type(s) of π. The relation of the above to the local character expansion is given by:

$$c_{\mathcal{O}}(\pi) = d(\pi)\,\Gamma_{\mathcal{O}}(s_\pi), \qquad \mathcal{O} \in (\mathcal{N}_G).$$

Here, $\Gamma_{\mathcal{O}}$ is the Shalika germ corresponding to the nilpotent orbit $\mathcal{O}$.

Let π be an irreducible admissible representation of G. We are interested in expressing the germ of Θ_π as a linear combination of $\{\, \mu_{\mathcal{O}} \mid \mathcal{O} \in \mathcal{S}_\pi \,\}$, where $\mathcal{S}_\pi$ is some finite collection of $\mathrm{Ad}\,G$-orbits in $\mathfrak{g}$. Of course there may be several choices for $\mathcal{S}_\pi$. It follows from Harish-Chandra's local character expansion that we can always take $\mathcal{S}_\pi = (\mathcal{N}_G)$. For the supercuspidal representations mentioned above, we can also take $\mathcal{S}_\pi = \{\, \mathcal{O}(s_\pi) \,\}$. Given a semisimple element $s \in \mathfrak{g}$, let $\Omega_G(s)$ denote the set of $\mathrm{Ad}\,G$-orbits $\mathcal{O} \subset \mathfrak{g}$ which contain elements whose semisimple part is conjugate to s; that is, orbits $\mathcal{O}$ whose closure contains s. In particular, $\Omega_G(0) = (\mathcal{N}_G)$. We will discuss some cases where $\mathcal{S}_\pi = \Omega_G(s)$ for some non zero s. A different approach would be to look for an $\mathcal{S}_\pi$ consisting of orbits in $\mathfrak{g}_{reg}$.

Results which we use include:

1. Hecke algebra isomorphisms and K-types
2. Uniformity of local character expansions - relation between the depth of π and the size of the neighbourhood V_π.
3. Descent properties of orbital integrals.

The idea is to use 1–3 to relate the germ of Θ_π to the germ of $\Theta_{\pi'}$ for an irreducible admissible representation π' of a reductive subgroup G' of G. In particular, if we know the local character expansion for π', what does that imply about the germ of Θ_π? More generally, if we have some collection $\mathcal{S}_{\pi'}$ as above for π', do we get a corresponding collection $\mathcal{S}_\pi$ for π? If so, what is the relation between the coefficients in the two linear combinations? This is work in progress. We will discuss examples.

Example 1:

- $G = \mathbf{GL}_n(F)$
- d a positive integral divisor of n
- E/F an unramified extension of degree d
- ε_d a root of unity in O_E whose image in the residue class field $\mathbb{E} = O_E/\mathfrak{p}_E$ generates $\mathbb{E}/\mathbb{F}$, where $\mathbb{F} = O_F/\mathfrak{p}_F$
- $\mathbf{M}_d = \mathbf{GL}_d \times \cdots \times \mathbf{GL}_d$ (n/d times), viewed as a standard Levi subgroup of $\mathbf{GL}_n$, and $M_d = \mathbf{M}_d(F)$
- $s_d = (\varepsilon_d, \ldots, \varepsilon_d) \in M_d \subset G$, ε_d identified with an element of $\mathbf{GL}_d(F)$ via a basis for E/F consisting of roots of unity
- $G' \simeq \mathbf{GL}_{n/d}(E)$, the centralizer of s_d in G
- K_d the parahoric subgroup of G which is equal to the inverse image in $\mathbf{GL}_n(O_F)$ of a standard parabolic subgroup of $\mathbf{GL}_n(\mathbb{F})$ having Levi component $\mathbf{M}_d(\mathbb{F})$
- K_d^u the pro unipotent radical of K_d
- σ an irreducible cuspidal representation of $\mathbf{GL}_d(\mathbb{F})$
- ρ the pullback to K_d of the representation $(\otimes\sigma)^{n/d}$ of $\mathbf{M}_d(\mathbb{F}) \simeq K_d/K_d^u$
- π an admissible representation of G such that the restriction of π to K_d contains ρ, written $(K_d, \rho) \hookrightarrow \pi$.
- $\mathcal{H}(G//K_d, \rho)$ the ρ-spherical Hecke algebra of compactly supported functions f from G to the endomorphisms of the space of ρ such that

$$f(k_1 x k_2) = \rho(k_1) f(x) \rho(k_2), \qquad \forall\, k_1, k_2 \in K_d,\ x \in G.$$

- I' an Iwahori subgroup of G'
- $\mathcal{H}(G'//I') = \mathcal{H}(G'/I', 1_{I'})$, $1_{I'}$ being the trivial representation of I'

Howe and Moy ([HM1]) proved that $\mathcal{H}(G//K_d, \rho)$ is isomorphic to $\mathcal{H}(G'//I')$ in such a way that the isomorphism preserves the natural inner products on the two algebras up to multiples and takes Plancherel measure to Plancherel measure. This Hecke algebra isomorphism induces a bijection between the set of (equivalence classes of) irreducible admissible representations π of G such that $(K_d, \rho) \hookrightarrow \pi$ and the set of (equivalence classes of) irreducible admissible representations π' of G' such that $(I', 1_{I'}) \hookrightarrow \pi'$. We will write $\pi \leftrightarrow \pi'$ for this correspondence.

Given $\mathcal{O} \in \Omega_G(s_d)$, choose an element of $\mathcal{O}$ whose semisimple part is s_d. The nilpotent part of this element belongs to $\mathfrak{g}'$, hence defines an $\mathcal{O}' \in \Omega_{G'}(0)$. The map $\mathcal{O} \mapsto \mathcal{O}'$ is a bijection, and we write $\mathcal{O} \leftrightarrow \mathcal{O}'$. Given a choice of G'-invariant measure on $\mathcal{O}'$, we normalize the G-invariant measure on $\mathcal{O}$ so that

$$(*) \qquad \mu_{\mathcal{O}}(ch(\mathfrak{gl}_n(O_F))) = |\mathbf{GL}_n(\mathbb{F})|_{p'} |\mathbf{GL}_{n/d}(\mathbb{E})|_{p'}^{-1}\, \mu_{\mathcal{O}'}(ch(\mathfrak{gl}_{n/d}(O_E))).$$

Here $|\cdot|_{p'}$ denotes p'-cardinality (p the characteristic of $\mathbb{F}$), and $ch(S)$ the characteristic function of the set S. When evaluating Fourier transforms, we take $B(X,Y) = \mathrm{trace}(XY)$ and we assume that the character ψ has conductor $\mathfrak{p}_F$. Let $\mathfrak{g}_{tn}$ be the set of topologically nilpotent elements in $\mathfrak{g}$. Our main result in this setting is:

Theorem. *Suppose that π is irreducible and admissible and $(K_d, \rho) \hookrightarrow \pi$.*
(1) There exist complex numbers $c_{\mathcal{O}}(\pi)$, $\mathcal{O} \in \Omega_G(s_d)$, such that

$$\Theta_\pi(1+X) = \sum_{\mathcal{O} \in \Omega_G(s_d)} c_{\mathcal{O}}(\pi)\, \widehat{\mu}_{\mathcal{O}}(X) \qquad X \in \mathfrak{g}_{reg} \cap \mathfrak{g}_{tn}.$$

(2) Let π' be an irreducible admissible representation of G' such that $\pi \leftrightarrow \pi'$. Given $\mathcal{O}' \in \Omega_{G'}(0)$, let $c_{\mathcal{O}'}(\pi')$ denote the coefficient of $\widehat{\mu}_{\mathcal{O}'}$ in the local character expansion of π' at 1. If $\mathcal{O} \leftrightarrow \mathcal{O}'$ and the measures on $\mathcal{O}$ and $\mathcal{O}'$ are related as above in $(*)$, then

$$c_{\mathcal{O}}(\pi) = c_{\mathcal{O}'}(\pi'), \qquad \mathcal{O} \in \Omega_G(s_d).$$

The orbits in $\Omega_G(s_d)$ can be parametrized by ordered partitions of n/d. Fix $\mathcal{O} \in \Omega_G(s_d)$. Let $\mathcal{O}' \in \Omega_{G'}(0)$ be such that $\mathcal{O} \leftrightarrow \mathcal{O}'$. Choose a parabolic subalgebra $\mathcal{P}_{\mathcal{O}'}$ of $\mathfrak{g}'$ such that the intersection of $\mathcal{O}'$ with the nilradical of $\mathcal{P}_{\mathcal{O}'}$ is open in that nilradical (that is, $\mathcal{O}'$ is the Richardson class attached to $\mathcal{P}_{\mathcal{O}'}$). The Levi component of $\mathcal{P}_{\mathcal{O}'}$ is isomorphic to $\mathfrak{gl}_{\alpha_1}(E) \oplus \cdots \oplus \mathfrak{gl}_{\alpha_r}(E)$ for some partition $\alpha = (\alpha_1, \ldots, \alpha_r)$ of n/d. Up to re-ordering, α is determined by $\mathcal{O}'$, and conversely. Set $\ell(\alpha) = r$ and let $|St(\alpha)|$ be the cardinality of the stabilizer of α in the symmetric group on r letters.

Let $P_{\mathcal{O}'}$ be the parabolic subgroup of G' having Lie algebra $\mathcal{P}_{\mathcal{O}'}$. We normalize $\mu_{\mathcal{O}'}$ in such a way that $\widehat{\mu}_{\mathcal{O}'}$ equals the germ of the character of $\mathrm{Ind}_{P_{\mathcal{O}'}}^{G'} 1_{P_{\mathcal{O}'}}$. This then determines a normalization of $\mu_{\mathcal{O}}$ via $(*)$.

Corollary. *Let π be a generalized special (discrete series) representation of G such that $(K_d, \rho) \hookrightarrow \pi$. Given $\mathcal{O} \in \Omega_G(s_d)$, define α as above.*

$$c_{\mathcal{O}}(\pi) = \frac{(-1)^{(n/d)-\ell(\alpha)}\ell(\alpha)!}{|St(\alpha)|}, \qquad \mathcal{O} \in \Omega_G(s_d)$$

Remarks:

(i) For the corollary, if $\pi \leftrightarrow \pi'$, then $\pi' = St_{G'}$, the Steinberg (or special) representation of G'. As the character $\Theta_{St_{G'}}$ is known, (3) is obtained by substituting the value for $c_{\mathcal{O}'}(St_{G'})$ into (2). In order to do that, it was necessary to specify a normalization for $\mu_{\mathcal{O}'}$.

(ii) The condition $(K_d, \rho) \hookrightarrow \pi$ guarantees the existence of non zero K_d^u-invariant vectors in the space of π. Denote the pro unipotent radical of $K_n = \mathbf{GL}_n(O_F)$ by K_n^u. Note that $K_n^u \subset K_d^u$. let V be the space of K_n^u-fixed vectors in the space of π. As K_n^u is normal in

K_n, V is $\pi\,|\,K_n$-stable. Hence we get a representation κ of $\mathbf{GL}_n(\mathbb{F}) \simeq K_n/K_n^u$ on V. We assume that p is sufficiently large that results of Waldspurger ([W]) concerning uniformity of local character expansions can be applied. Then the local character expansion of π holds on $\mathfrak{g}_{tn} \cap \mathfrak{g}_{reg}$, and the expansion is determined by the values of the character χ_κ of κ on the unipotent set $\mathcal{U}_{\mathbf{GL}_n(\mathbb{F})}$.

Similarly, there exists a representation κ' of $\mathbf{GL}_{n/d}(\mathbb{E})$ whose character values on $\mathcal{U}_{\mathbf{GL}_{n/d}(\mathbb{E})}$ determine the local character expansion of π' (which holds on $\mathfrak{g}'_{tn} \cap \mathfrak{g}'_{reg}$).

The cuspidal representation σ of $\mathbf{GL}_d(\mathbb{F})$ corresponds to a character θ of $\mathbb{E}^\times$, via Deligne-Lusztig induction: $\chi_\sigma = (-1)^{d-1}\, R_{\mathbb{E}^\times}^{\mathbf{GL}_d(\mathbb{F})}(\theta)$. The Hecke algebra isomorphism of Howe and Moy restricts to an isomorphism of finite group Hecke algebras which we can use to show that $\chi_\kappa = (-1)^{n-(n/d)}\, R_{\mathbf{GL}_{n/d}(\mathbb{E})}^{\mathbf{GL}_n(\mathbb{F})}((\theta \circ \det')\,\chi_{\kappa'})$. Here, $\det'$ is the determinant on $\mathbf{GL}_{n/d}(\mathbb{E})$ and $R_{\mathbf{GL}_{n/d}(\mathbb{E})}^{\mathbf{GL}_n(\mathbb{F})}$ denotes Deligne-Lusztig induction ([L]). This relation between χ_κ and $\chi_{\kappa'}$ leads to relations between the germs of Θ_π and $\Theta_{\pi'}$. These relations are obtained by evaluating Θ_π and $\Theta_{\pi'}$ on characteristic functions of pro unipotent radicals of parahoric subgroups: up to scalar multiples, this is equivalent to computing the dimension of the space of κ or κ'-fixed vectors for the unipotent radical of a parabolic subgroup of $\mathbf{GL}_n(\mathbb{F})$, resp. $\mathbf{GL}_{n/d}(\mathbb{E})$.

Let K be a parahoric subgroup of G with pro unipotent radical K^u. Let $\mathcal{K}$ be the corresponding parahoric O_F-subalgebra of $\mathfrak{g}$, with pro nilpotent radical $\mathcal{K}^u$. In proving the theorem, we compare the values $\Theta_\pi(ch(K^u))$ and $\Theta_{\pi'}(ch(K'^u))$ for those parahoric subgroups K of G such that $K' = K \cap G'$ is a parahoric subgroup of G'. The descent formulas for orbital integrals which are useful for comparing the germs of Θ_π and $\Theta_{\pi'}$ are those relating $\mu_{\mathcal{O}}(ch(\mathcal{K}))$ and $\mu_{\mathcal{O}'}(ch(\mathcal{K}'))$ for parahoric subalgebras $\mathcal{K}$ and $\mathcal{K}'$ of $\mathfrak{g}$ and $\mathfrak{g}'$ respectively, such that $\mathcal{K}' = \mathcal{K} \cap \mathfrak{g}'$.

(iii) The relations between the values of χ_κ and $\chi_{\kappa'}$ on unipotent elements can be expressed in terms of Hall algebras of finite general linear groups.

(iv) There is a finite collection $\mathcal{S}'$ of regular semisimple unramified $\mathrm{Ad}\,G'$ orbits having the property that, for any π' such that $(I', 1_{I'}) \hookrightarrow \pi'$, the germ of $\Theta_{\pi'}$ can be expressed as a linear combination of $\{\,\widehat{\mu}_{\mathcal{O}'} \mid \mathcal{O}' \in \mathcal{S}'\,\}$. Given $X \in \mathcal{O}' \in \mathcal{S}'$, let $\mathcal{O} = \mathcal{O}_G(s_d + X)$. Let $\mathcal{S}$ be the set of all such $\mathcal{O}$. There is a result analogous to the above theorem which gives the germ of Θ_π, $\pi \leftrightarrow \pi'$, as a linear combination of $\{\,\widehat{\mu}_{\mathcal{O}} \mid \mathcal{O} \in \mathcal{S}\,\}$, including relations between the coefficients corresponding to the orbits $\mathcal{O}' \in \mathcal{S}'$ and $\mathcal{O} \in \mathcal{S}$ such that $\mathcal{O}' \leftrightarrow \mathcal{O}$.

(v) Barbasch and Moy ([BM]) have given a recursive method for computing the coefficients in the local character expansions of depth zero irreducible admissible representations of reductive groups. The method uses generalized Gelfand-Graev representations of finite reductive groups. The method can be applied to calculate the coefficients for representations of groups of low rank (some examples are given in [BM]). In particular, Moy computed the values of the coefficients for a depth zero generalized special representation of $\mathbf{GL}_4(F)$ corresponding to $d = 2$ in our example. Using the corollary, after expressing each $\widehat{\mu}_{\mathcal{O}}$, $\mathcal{O} \in \Omega_{\mathbf{GL}_4(F)}(s_2)$ as a linear combination of $\widehat{\mu}_{\mathcal{O}}$, $\mathcal{O} \in \Omega_{\mathbf{GL}_4(F)}(0)$, we obtain the same result.

Example 2:

- $G = \mathbf{GL}_n(F)$. We assume $p > n$.
- d a positive integral divisor of n
- σ an irreducible supercuspidal representation of $\mathbf{GL}_d(F)$
- π_{gs} the discrete series representation of G corresponding to σ
- E the degree d extension and θ the admissible quasi character of $E^\times$ which parametrizes the supercuspidal representation σ via Howe's construction ([H])
- $G' = \mathbf{GL}_{n/d}(E)$
- (J, ρ) the Howe-Moy refined minimal K-type attached to π_{gs} ([CMS],[HM2])
- $\mathcal{H}(G//J, \rho)$ the ρ-spherical Hecke algebra, as for example 1 but with J replacing K_d

The group J is an open compact subgroup of G having the property that $I' = J \cap G'$ is an Iwahori subgroup of G'. Howe and Moy ([HM2]) proved that $\mathcal{H}(G//J, \rho)$ is isomorphic to $\mathcal{H}(G'//I')$ via an isomorphism which preserves support and is an L^2 isometry. As for example 1, if $(J, \rho) \hookrightarrow \pi$ and π corresponds via the Hecke algebra isomorphism to π' such that $(I', 1_{I'}) \hookrightarrow \pi'$, we write $\pi \leftrightarrow \pi'$.

There exists $s_\sigma \in E \hookrightarrow \mathfrak{gl}_d(E)$ such that the germ of Θ_σ is given by $d(\sigma)\,\widehat{\mu}_{\mathcal{O}_{\mathbf{GL}_d(F)}(s_\sigma)}$ ([Mu1]). Identifying Let $s = (s_\sigma, \ldots, s_\sigma) \in \mathbf{M}_d(F) \times \cdots \times \mathbf{M}_d(F) \subset G$. As s_σ generates E/F, it follows that G' is isomorphic to the centralizer of s in G. There is a bijection between $\Omega_G(s)$ and $\Omega_{G'}(0)$ resulting from the Jordan decomposition. We expect that if $\mu_{\mathcal{O}}$, $\mathcal{O} \in \Omega_G(s)$ and $\mu_{\mathcal{O}'}$, $\mathcal{O}' \in \Omega_{G'}(0)$ are suitably normalized, the analogue of the theorem and corollary in example 1 will hold in this more general setting. In particular, the germ of $\Theta_{\pi_{gs}}$ should be a linear combination of the $\widehat{\mu}_{\mathcal{O}}$, $\mathcal{O} \in \Omega_G(s)$, with the coefficient of $\widehat{\mu}_{\mathcal{O}}$ as in the corollary.

Remarks:

(i) The Hecke algebra isomorphisms of Howe and Moy ([HM2]) relate admissible representations of $GL_n(F)$ and unipotent representations of products of general linear groups. It appears that these relations tranlate into relations between the germs of characters of admissible representations and germs of characters of unipotent representations. It is natural to ask whether Hecke algebra isomorphisms for other reductive groups give rise to similar relations between germs of characters.

(ii) In [CMS], Corwin, Moy and Sally used properties of the Hecke algebra isomorphisms discussed in example 2 to compute the formal degrees of all discrete series representations of $\mathbf{GL}_n(F)$.

Example 3:

- $G = \mathbf{GSp}_4(F)$; p odd
- E a quadratic unramified extension of F
- ε a root of unity in O_E whose image in $\mathbb{E}$ generates $\mathbb{E}$ over $\mathbb{F}$; ε will also denote an image of this root of unity in $\mathbf{GL}_2(O_F)$
- $\mathbf{M} = \mathbf{GL}_2 \times \mathbf{GL}_1$, viewed as a standard Levi subgroup of $\mathbf{GSp}_4$, and $M = \mathbf{M}(F)$
- $s = (\varepsilon, 0) \in \mathfrak{g}l_2(F) \oplus F = Lie(M)$; $G' \simeq \{\, g = (x, \alpha) \in \mathbf{GL}_2(F) \times E^\times \mid \det(g) = N_{E/F}(\alpha) \,\}$, the stabilizer of s in G.
- K the parahoric subgroup of G which is equal to the inverse image in $\mathbf{GSp}_4(O_F)$ of a standard parabolic subgroup of $\mathbf{GSp}_4(\mathbb{F})$ having Levi component $\mathbf{M}(\mathbb{F})$, K^u the pro unipotent radical of K
- $I' = K \cap G'$, an Iwahori subgroup of G'
- σ an irreducible cuspidal representation of $\mathbf{GL}_2(\mathbb{F})$, ρ the pullback to K of the representation $\sigma \otimes 1$ of $\mathbf{M}(\mathbb{F}) \simeq K/K^u$

In [My2], Moy classified the depth zero admissible representations of G via nondegenerate represetations and Hecke algebra isomorphisms. This example corresponds to case 3.62(i) of [My2]. Moy obtained an isomorphism of $\mathcal{H}(G//K, \rho)$ with the Iwahori spherical Hecke algebra of a group which he called G', which differs from the one defined above. However, the Iwahori spherical Hecke algebra for Moy's G' coincides with the Iwahori spherical Hecke algebra for the above G'. Hence $\mathcal{H}(G//K, \rho) \simeq \mathcal{H}(G'//I')$. Preliminary calculations indicate that the analogues of the theorem and corollary obtained for our first example will hold in this setting. In particular, if π is the discrete series representation corresponding via the Hecke algebra isomorphism to the Steinberg representation $St_{G'}$ of G', the germ of Θ_π belongs to the span of $\{\, \widehat{\mu}_{\mathcal{O}},\ \mathcal{O} \in \Omega_G(s) \,\}$, with the coefficients equal to the corresponding coefficients in the local character expansion for the germ of $\Theta_{St_{G'}}$.

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