LOCAL THETA CORRESPONDENCE AND UNREFINED MINIMAL K-TYPES*

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0. Introduction

0.1. Unrefined minimal K-types of irreducible admissble representations of p-adic reductive groups are defined by A. Moy and G. Prasad [MP1, MP2]. In particular, a non-negative rational number can be associated to an irreducible admissble representation of a p-adic reductive group. This rational number is called the *depth* of the representation.

0.2. Let (G_1, G_2) be a type I (irreducible) reductive dual pair in a symplectic group $\mathrm{Sp}_{2n}(F)$. Let ψ be a fixed non-trivial additive character of F. Let ω_ψ be the Weil representation associated to ψ of the metaplectic cover $\widetilde{\mathrm{Sp}_{2n}}(F)$ of $\mathrm{Sp}_{2n}(F)$. By restricting ω_ψ to $\widetilde{G}_1 \cdot \widetilde{G}_2$, where $\widetilde{G}_i$ is the inverse image of G_i under the map $\widetilde{\mathrm{Sp}_{2n}}(F) \to \mathrm{Sp}_{2n}(F)$, we obtain a one-to-one correspondence between some irreducible admissible representations of $\widetilde{G}_1$ and irreducible admissible representations of $\widetilde{G}_2$. The correspondence is called the *local theta correspondence*.

0.3. Let I be an Iwahori subgroup of a classical group G which is a member of some reductive dual pair. It is known the map $\widetilde{I} \to I$ splits. By fixing an embedding $I \to \widetilde{G}$, the concepts of unrefined minimal K-types and depths can be generalized to irreducible admissible representations of $\widetilde{G}$. We prove that the depths of the representations paired by the local theta correspondence for any (type I) reductive dual pair are equal.

1. Moy-Prasad filtrations

1.1. Basic constructions. Let F be a p-adic local field, $\mathcal{O}$ the ring of integers, ϖ a prime element, $\mathfrak{p}$ the maximal ideal, $\mathbf{f}$ the residue field of F. Let G be the group of F-rational points of a classical group over F. Given a point x in the Bruhat-Tits building of G, A. Moy and G. Prasad [MP1, MP2] define a decreasing filtration subgroups $\{G_{x,r}\}_{r\in\mathbf{R}^+\cup\{0\}}$ of the parahoric subgroup G_x associated to x (i.e., $G_x \supseteq G_{x,r} \supseteq G_{x,s}$ if $0 \le r \le s$). Let $G_{x,r^+} = \bigcup_{s>r} G_{x,s}$. It is known that $G_{x,r}$ and G_{x,r^+} are normal subgroups of G_x, and $G_{x,r}/G_{x,r^+}$ is a finite reductive group when $r = 0$ and it is a finite abelian group if $r > 0$. We can associate to x a sequence of non-negative real numbers $\{r_i\}_{i\ge 0}$ such that $r_0 = 0$ and r_i satisfies the conditions that $G_{x,r_i} \ne G_{x,r_i^+}$ and $G_{x,r_i^+} = G_{x,r_{i+1}}$.

1.2. filtration of a Lie algebra and its dual space. The similar filtration $\mathfrak{g}_{x,r}$ of Lie algebra $\mathfrak{g}$ of G and the filtration $\mathfrak{g}^*_{x,r}$ of the dual space $\mathfrak{g}^*$ are also defined for x in the building and r any real number. We normalize the discrete valuation of F such that $\varpi\mathfrak{g}_{x,r} = \mathfrak{g}_{x,r+1}$, $\varpi\mathfrak{g}^*_{x,r} = \mathfrak{g}^*_{x,r+1}$. There is a natural isomorphism between the Pontrjagin dual $(G_{x,r_i}/G_{x,r_{i+1}})^\wedge$ and the quotient $\mathfrak{g}^*_{x,-r_i}/\mathfrak{g}^*_{x,-r_{i-1}}$.

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1.3. Optimal points. A certain finite set of points in a (closed) chamber, called *a set of optimal points*, satisfying some optimal properties is defined by Moy and Prasad. We know that for $\mathrm{SL}_n(F)$ (or $\mathrm{GL}_n(F)$) the optimal points can be chosen to be the *barycenters* of symplices which are facets of a chamber.

1.3.1. Generalized barycenters. Define δ to be the smallest positive rational number such that $\phi + t\delta$ is an affine root for any affine root ϕ and any integer t. Let $\Delta_{\mathrm{af}} = \{\alpha_0, \ldots, \alpha_n\}$ be a system of simple affine roots, identified with the vertices of the local Dynkin diagram of G. There are unique integers c_i such that $\sum_i c_i\alpha_i = \delta$. A vertex α_i is called *terminal* if there is at most one other vertex jointed to α_i. A vertex α_i is called *exceptional* if it is terminal and $c_i = 2$. A pair of vertices $\{\alpha_i, \alpha_j\}$ is called an *exceptional pair* if $\alpha_i \neq \alpha_j$ and α_i and α_j form an orbit under the action of G on the local Dynkin diagram. Let Ξ be a non-empty subset of Δ_{af}, then Ξ determines a subsimplex C_Ξ of a chamber. Assign weights w_i to $\alpha_i \in \Xi$ in the following way: if $\{\alpha_i, \alpha_j\}$ be an exceptional pair such that both α_i, α_j are in Ξ, $(w_i, w_j) = (1,1)$, $(1,2)$, or $(2,1)$; if α_i is an exceptional vertex, $w_i = 1$ or 2; if α_i is not an exceptional vertex and there is no α_j such that $\{\alpha_i, \alpha_j\}$ is an exceptional pair contained in Ξ, $w_i = 2$. Then the weighted barycenter of the points $\alpha_i \in \Xi$ with weights w_i is called a *generalized barycenter* of C_Ξ.

1.3.2. There are at most 9 generalized barycenters of any C_Ξ. Many groups such as special linear groups, symplectic groups do not have any exceptional vertices or exceptional pairs, so all generalized barycenters are indeed (genuine) barycenters. For these groups there is only one (generalized) barycenter of C_Ξ.

1.4. (Unrefined) minimal K-types. An irreducible representation χ of $G_{x,r_i}/G_{x,r_i^+}$ is called a *non-degenerate representation* if:

(i) χ is a cuspidal representation of $G_{x,r_i}/G_{x,r_i^+}$ when $r_i = 0$;
(ii) χ, realized as a residue class in $\mathfrak{g}^*_{x,-r_i}/\mathfrak{g}^*_{x,-r_i^+}$, contains no nilpotent elements when $r_i > 0$.

An (unrefined) *minimal K-type* of an irreducible admissible representation (π, V) of G is a pair $(G_{x,r}, \chi)$, where χ is a non-degenerate representation of $G_{x,r}/G_{x,r^+}$, and $V^{G_{x,r^+}}$, the space fixed by G_{x,r^+}, is non-trivial and, as a representation of $G_{x,r}/G_{x,r^+}$, contains χ.

1.5. Theorem (Moy-Prasad). *Let (π, V) be an irreducible admissible representation of a p-adic reductive group G. Then there is a unique non-negative rational number ρ such that ρ is the smallest number r such that there exists an optimal point y with $V^{G_{y,r^+}} \neq \{0\}$. Moreover, if $\rho > 0$ and y is a point such that $V^{G_{y,\rho^+}} \neq \{0\}$, then the space $V^{G_{y,\rho^+}}$ decomposes into non-degenerate representations of $G_{y,\rho}/G_{y,\rho^+}$. Conversely, if $r > 0$ and $V^{G_{x,r^+}}$ contains a non-degenerate representation of $G_{x,r}/G_{x,r^+}$ for some x, then r is equal to ρ.*

This unique rational number ρ is called the *depth* of the representation (π, V).

1.5.1. Locations of optimal points. Let $\mathcal{Q}$ be the set of generalized barycenters in a fixed chamber of the building of a classical group G. The first result in [PY] is the following.

Theorem. *Suppose that the local Dynkin diagram of a classical group G does not have any ramified ends, then the point y in Theorem 1.5 can be chosen from $\mathcal{Q}$.*

1.6. Good lattices. From now on, we shall assume that D is a divison algebra over F with an involution τ. Let $\mathcal{O}_D$ be the local ring in D, ϖ_D be a prime element. Let $\mathcal{V}$ be a right space over D. We assume that $\mathcal{V}$ is equipped with a non-degenerate ϵ-hermitian form $\langle , \rangle$, where ϵ is 1 or -1. Let G be the group of isometries $U(\mathcal{V})$ of $(\mathcal{V}, \langle , \rangle)$. If L is a lattice in $\mathcal{V}$, define

$$L^* = \{ v \in \mathcal{V} \mid \langle v, l \rangle \in \mathcal{O}_D \text{ for all } l \in L \}.$$

L is said to be *self-dual* if $L = L^*$, and is said to be *almost-self-dual* if $L^* = L\varpi_D^t$ for some integer t. A lattice L is called a *good lattice* if $L^*\varpi_D \subseteq L \subseteq L^*$.

1.6.1. Write $\mathcal{V}$ as an orthogonal direct sum $\mathcal{V}^\circ \oplus \mathcal{V}_1$, where $\mathcal{V}^\circ$ is anisotropic, and $\mathcal{V}_1$ is a direct sum of hyperbolic planes. Then there is a unique good lattice A_0 in $\mathcal{V}^\circ$. Choose a complete polarization $\mathcal{V}_1 = X + Y$, a basis $x_1, \dots, x_n$ of X, the dual basis $y_1, \dots, y_n$ of Y. Then

$$N_i = x_1\mathcal{O}_D\varpi_D + \cdots + x_i\mathcal{O}_D\varpi_D + x_{i+1}\mathcal{O}_D + \cdots + x_n\mathcal{O}_D + y_n\mathcal{O}_D + \cdots y_1\mathcal{O}_D$$

is a good lattice in $\mathcal{V}_1$ for each $0 \le i \le n$. Moreover, $B_i = A_0 + N_i$ is a good lattice in $\mathcal{V}$.

1.6.2. Regular self-dual lattice chains and barycenters. Two lattices L_1, L_2 in $\mathcal{V}$ are said to be *similar* if $L_1 = L_2\varpi_D^k$ for some integer k. A non-empty collection of lattices $\mathcal{L} = \{L_i\}_{i \in \mathbf{Z}}$ in $\mathcal{V}$ is called a *lattice chain* if it satisfies the following conditions:

(i) $\mathcal{L}$ is totally ordered by inclusion i.e., $L_i \subseteq L_{i+1}$ for each i;
(ii) each lattice L_i is similar to a good lattice or the dual lattice of a good lattice;
(iii) $L \in \mathcal{L}$ implies that the whole similarity class of L is contained in $\mathcal{L}$ i.e., if $L \in \mathcal{L}$ then $L\varpi^k \in \mathcal{L}$ for any k;
(iv) there exists a number m such that $L_{i+m} = L_i\varpi$ for all i.

Let $\mathcal{L}$ be a lattice chain, then the number m is called the *period* of $\mathcal{L}$. $\mathcal{L}$ is said to be *regular* if $L_i \neq L_j$ whenever $i \neq j$. Let m be a positive integer and $m_0 = 0$ or 1. A lattice chain $\mathcal{L} = \{L_i\}_{i \in \mathbf{Z}}$ is said to be *self-dual with numerical invariant* (m, m_0) if it has period m and $L_i^* = L_{-i-m_0}$ for all i. We have a bijection between the following two sets:

(1) regular self-dual lattice chains in $\mathcal{V}$;
(2) (genuine) barycenters in a chamber.

For each parahoric subgroup we have a *standard filtration* defined by Prasad-Raghunathan. This filtration is equal to the Moy-Prasad filtration at the corresponding barycenter.

1.6.3. Small admissible lattice chains. A lattice in $\mathcal{V}$ is called a *big* (resp. *small*) *critical lattice* if it is similar to B_0^* or B_n (resp. B_0 or B_n^*) defined in subsection 1.6.1. $\mathcal{L}$ is called a *small* (resp. *big*) *admissible lattice chain with numerical invariant* (m, m_0) if it has period m and satisfies one of the following conditions:

(i) when m is even and $m_0 = 1$, $L_i^* = L_{-i-1}$ for all i;
(ii) when m is even and $m_0 = 0$, $L_i^* = L_{-i}$ for all $i \not\equiv 0$ or $m/2 \pmod m$ and L_i is small (res. big) critical if $i \equiv 0$ or $m/2 \pmod m$;

(iii) when m is odd and $n_0 = 1$, $L_i^* = L_{-i-1}$ for all $i \not\equiv 0 \pmod m$ and L_i is small (res. big) critical if $i \equiv 0 \pmod m$;

(iv) when m is odd and $m_0 = 0$, $L_i^* = L_{-i}$ for all $i \not\equiv (m-1)/2 \pmod m$ and L_i is small (res. big) critical if $i \equiv (m-1)/2 \pmod m$.

1.7. Filtration subgroups by lattice chains. Let $\mathcal{L}$ be a regular small admissible lattice chain in $\mathcal{V}$ with numerical invariant (m, m_0). Define $\mathcal{L}^\sharp := \{ L_i^\sharp \mid L_i^\sharp := L^*_{-i-m_0} \}_{i \in \mathbf{Z}}$. Then we define

$$\begin{cases} G_{\mathcal{L},d/m} := \{ g \in G \mid (g-1).L_i \subseteq L_{i+d},\ (g-1).L_i^\sharp \subseteq L_{i+d}^\sharp \text{ for all } i \} & \text{for } d \geq 1; \\ G_{\mathcal{L},(d/m)+} := \{ g \in G \mid (g-1).L_i \subseteq L_{i+d+1}^\sharp,\ (g-1).L_i^\sharp \subseteq L_{i+d} \text{ for all } i \} & \text{for } d \geq 0. \end{cases}$$

If $\mathcal{L}$ is self-dual then $L_i = L_i^\sharp$, and $G_{\mathcal{L},d/m} = \{ g \in G \mid (g-1).L_i \subseteq L_{i+d} \text{ for all } i \}$ and $G_{\mathcal{L},(d/m)+} = G_{\mathcal{L},(d+1)/m}$.

1.7.1. Similarly, we can define the filtration $\mathfrak{g}_{\mathcal{L},d/m}$, $\mathfrak{g}_{\mathcal{L},(d/m)}$, $\mathfrak{g}^*_{\mathcal{L},-d/m}$, and $\mathfrak{g}^*_{\mathcal{L},(-d/m)+}$.

1.7.2. Comparison of two filtrations. Let $G = U(\mathcal{V})$ be a classical group. Then essentially we have a bijective correspondence between the following two sets:

(1) regular small admissible lattice chains in $\mathcal{V}$;

(2) generalized barycenters in a chamber.

Proposition. *Let x be a generalized barycenter, $\mathcal{L}$ be the corresponding regular small admissible lattice chain and m be the period of $\mathcal{L}$, $\{r_i\}_{i \geq 0}$ be the sequence associated x in subsection 1.1. Then*

(i) *If $\mathcal{L}$ is self-dual, then $r_i = i/m$ for any i. If $\mathcal{L}$ is not self-dual, then $r_i = i/2m$.*

(ii) *Every residue class in $\mathfrak{g}^*_{x,-r_i}/\mathfrak{g}^*_{x,-r_{i-1}}$ contains a nilpotnet element if $\mathcal{L}$ is not self-dual, $m \geq 2$, and i is odd.*

(iii) $G_{x,i/m} = G_{\mathcal{L},i/m}$ *and* $G_{x,(i/m)+} = G_{\mathcal{L},(i/m)+}$.

1.8. We have the following refinement of Theorem 1.5.1 and Proposition 1.7.2.

Theorem. *Suppose that the local Dynkin diagram of G does not have any ramified ends, then an irreducible admissible representation (π, V) of positive depth of G contains a minimal K-type of the form $(G_{\mathcal{L},d/m}, \chi)$, where $\mathcal{L}$ is a regular small admissible lattice chain of period m such that $\mathcal{L}, d, m$ satisfy one of the following two conditions:*

(i) $\gcd(d,m) = 1$; *or*

(ii) $\gcd(d,m) = 2$ *and $\mathcal{L}$ is self-dual.*

2. Local theta correspondence

2.1. Lattice model of the Weil representations. From now on, we assume that the characteristic of $\mathbf{f}$ is odd. Let $\mathcal{W}$ be a finite dimensional symplectic space over F, with the skew-symmetric bilinear form $\langle\!\langle , \rangle\!\rangle$. Let ψ be a fixed non-trivial (addivtive) character of F of conductor $\mathcal{O}$. Let A_0 be a self-dual lattice in $\mathcal{W}$. Let $\mathcal{S}(A_0)$ be the space of locally constant, compactly supported functions $f\colon \mathcal{W} \to \mathbf{C}$ satisfying the condition

$$f(a+w) = \psi(\frac{1}{2}\langle\!\langle w, a\rangle\!\rangle)f(w),$$

where $a \in A_0$, and $w \in \mathcal{W}$. The (projective) action ω_ψ of $\mathrm{Sp}(\mathcal{W})$ on the space $\mathcal{S}(A_0)$ is given by

$$(\omega_\psi(g)\cdot f)(w) = \int_{A_0} \psi(\frac{1}{2}\langle\!\langle w, a\rangle\!\rangle)f(g^{-1}.(w+a))\, da,$$

where $g \in \mathrm{Sp}(\mathcal{W})$, $f \in \mathcal{S}(A_0)$, $w \in \mathcal{W}$, and da is some Haar measure on A_0. We can lift ω_ψ to get a representation, also denoted by ω_ψ, of the metaplectic cover $\widetilde{\mathrm{Sp}(\mathcal{W})}$ of $\mathrm{Sp}(\mathcal{W})$. $(\omega_\psi, \mathcal{S}(A_0))$ is called a *lattice model* of the *Weil representation* of $\widetilde{\mathrm{Sp}(\mathcal{W})}$. If B is a union of A_0-cosets, let $\mathcal{S}(A_0)_B$ denote the subspace of functions supported in B. The (one-dimensional) space $\mathcal{S}(A_0)_{A_0+w}$ is also denoted by $\mathcal{S}(A_0)_w$.

2.2. Reductive dual pairs. Let $(\mathcal{V}_1, \langle,\rangle_1)$ be a right ϵ_1-hermitian D-space, $(\mathcal{V}_2, \langle,\rangle_2)$ a right ϵ_2-hermitian D-space where $\epsilon_1, \epsilon_2 = 1, -1$ and $\epsilon_1\epsilon_2 = -1$. Let $\mathcal{W} := \mathcal{V}_1 \otimes_D \mathcal{V}_2$ and $\langle\!\langle , \rangle\!\rangle := \mathrm{Trd}_{D/F}(\langle,\rangle_1 \otimes \tau \circ \langle,\rangle_2)$ where $\mathrm{Trd}_{D/F}$ denotes the reduced trace from D to F. Then $(U(\mathcal{V}_1), U(\mathcal{V}_2))$ forms a *reductive dual pair* in $\mathrm{Sp}(\mathcal{W})$. The reductive dual pair $(U(\mathcal{V}_1), U(\mathcal{V}_2))$ is called *unramified* if D is F or a unramified quadratic extension of F and there exist self-dual lattices in both spaces $(\mathcal{V}_1, \langle,\rangle_1), (\mathcal{V}_2, \langle,\rangle_2)$.

2.3. Local theta correspondence. Let $\widetilde{U(\mathcal{V}_i)}$ be the inverse image of $U(\mathcal{V}_i)$ in $\widetilde{\mathrm{Sp}(\mathcal{W})}$ under the map $\widetilde{\mathrm{Sp}(\mathcal{W})} \to \mathrm{Sp}(\mathcal{W})$. By restricting the Weil representation ω_ψ to $\widetilde{U(\mathcal{V}_1)} \cdot \widetilde{U(\mathcal{V}_2)}$, there is a correspondence between some irreducible admissible representations of $\widetilde{U(\mathcal{V}_1)}$ and those of $\widetilde{U(\mathcal{V}_2)}$. It is proved by Howe and Waldspurger that this correspondence is one-to-one and is called the *local theta correspondence*.

2.3.1. We know that the map $\widetilde{K} \to K$ splits for any compact subgroup K of $U(\mathcal{V}_i)$. By fixing an embedding of the Iwahori subgroup $I_i \to \widetilde{U(\mathcal{V}_i)}$ the definitions of minimal K-types and depths can be generalized to any irreducible admissible representation of the group $\widetilde{U(\mathcal{V}_i)}$.

2.4. Let $(U(\mathcal{V}_1), U(\mathcal{V}_2))$ be a reductive dual pair. Let Γ_{M} (resp. Γ'_{M}) be a fixed maximal good lattice in $\mathcal{V}_1$ (resp. $\mathcal{V}_2$). Let (M, N) be a pair of lattices in $\mathcal{V}_1$ such that

$$\begin{array}{ccccc} N\varpi_D & \subseteq & M & \subseteq & N \\ \cap\! \mid & & \cap\! \mid & & \cap\! \mid \\ \Gamma^*_{\mathrm{M}}\varpi_D & \subseteq & \Gamma_{\mathrm{M}} & \subseteq & \Gamma^*_{\mathrm{M}}. \end{array}$$

Define $J_{M,N} := \{\, g \in U(\mathcal{V}_1) \mid (g-1).N^* \subseteq M \,\}$, and $B_{M,N} := M^* \otimes_{\mathcal{O}_D} \Gamma'_{\mathrm{M}} + N^* \otimes_{\mathcal{O}_D} \Gamma'^*_{\mathrm{M}}$.

Theorem (Waldspurger). *Let A_0 be a self-dual lattice in $\mathcal{W}$ contained in $B_{M,N}$, then*

$$\mathcal{S}(A_0)^{J_{M,N}} = \omega_\psi(\mathcal{H}_2).(\mathcal{S}(A_0)_{B_{M,N}})^{J_{M,N}},$$

where $\mathcal{H}_2$ is the Hecke algebra of $\widetilde{U(\mathcal{V}_2)}$.

2.5. Suppose that we have small admissible lattice chains $\mathcal{L} = \{L_i\}_{i\in\mathbf{Z}}$, $\mathcal{L}' = \{L'_i\}_{i\in\mathbf{Z}}$ in $\mathcal{V}_1$, $\mathcal{V}_2$ with numerical invariants (m, m_0), (m', m'_0) respectively. Let d be a non-negative integer. We assume that $m = m'$ and $m + m_0 + m'_0 + d$ is even. Then define

$$B(\mathcal{L}, \mathcal{L}', d/m) = \sum_{i+j=(m-2-m_0-m'_0-d)/2} L_i \otimes_{\mathcal{O}_D} L'_j.$$

$B(\mathcal{L}, \mathcal{L}', d/m)$ is a lattice in $\mathcal{W}$ containing a self-dual lattice.

2.6. Theorem. *Assume that $d \geq 0$, $\gcd(d, m) = 1$, or $\gcd(d, m) = 2$ and $m_0 = 1$. Put $m'_0 = 0$ or 1 so that $m + m_0 + m'_0 + d$ is even. Let A_0 be a self-dual lattice contained in $B(\mathcal{L}, \mathcal{L}', d/m)$. Then*

$$\mathcal{S}(A_0)^{G_{\mathcal{L},(d/m)+}} = \omega_\psi(\mathcal{H}_2).\big(\textstyle\sum_{\mathcal{L}'} \mathcal{S}(A_0)_{B(\mathcal{L},\mathcal{L}',d/m)}\big),$$

where the sum is taken over all small admissible lattice chains $\mathcal{L}'$ with numerical invariant (m, m'_0).

3. Moment maps and minimal K-types

3.1. Trace forms. Let $(U(\mathcal{V}_1), U(\mathcal{V}_2))$ be a reductive dual pair. Let $\mathfrak{B}_1$ (resp. $\mathfrak{B}_2$) denote the trace form of the Lie algebra $\mathfrak{g}$ (resp. $\mathfrak{g}'$) of $U(\mathcal{V}_1)$ (resp. $U(\mathcal{V}_2)$) i.e., $\mathfrak{B}_i(X, Y) = \mathrm{Trd}_{D/F}(\mathrm{tr}(XY))$. For $x, y \in \mathcal{V}_i$, define $c_{x,y}\colon \mathcal{V}_i \to \mathcal{V}_i$ by $c_{x,y}(v) = x\langle y, v\rangle_i - \epsilon_i y\langle x, v\rangle_i$. Then $c_{x,y} \in \mathfrak{g}$ (or $\mathfrak{g}'$). It is not difficult to check that

$$\mathfrak{B}_i(c_{x,y}, X) = \mathrm{Trd}_{D/F}(\langle -X.y, x\rangle_i + \epsilon_i\langle X.x, y\rangle_i),$$

where X in $\mathfrak{g}$ (resp. $\mathfrak{g}'$).

3.1.1. Moment maps. As usual let $\mathcal{W} = \mathcal{V}_1 \otimes_D \mathcal{V}_2$ and $\mathfrak{g}^*$ (resp. $\mathfrak{g}'^*$) be the dual space of the Lie algrbra $\mathfrak{g}$ (resp. $\mathfrak{g}'$). We have the moment maps $\mathfrak{M}\colon \mathcal{W} \to \mathfrak{g}^*$ and $\mathfrak{M}'\colon \mathcal{W} \to \mathfrak{g}'^*$. Now we identify $\mathfrak{g}$ (resp. $\mathfrak{g}'$) with $\mathfrak{g}^*$ (resp. $\mathfrak{g}'^*$) by the trace form $\mathfrak{B}_1$ (resp. $\mathfrak{B}_2$). Therefore we may regard $c_{x,y}$ as an element in $\mathfrak{g}^*$ (resp. $\mathfrak{g}'^*$) for $x, y \in \mathcal{V}_1$ (resp. $\mathcal{V}_2$). If $w = \sum_i a_i \otimes b_i \in \mathcal{W}$, where $a_i \in \mathcal{V}_1$, $b_i \in \mathcal{V}_2$, then $\mathfrak{M}(w) = \sum_{i,j} c_{a_i, a_j\langle b_j, b_i\rangle_2}$ and $\mathfrak{M}'(w) = \sum_{i,j} c_{b_i, b_j\langle a_j, a_i\rangle_1}$. We can prove that if $w \in B(\mathcal{L}, \mathcal{L}', d/m)$ then $\mathfrak{M}(w) \in \mathfrak{g}^*_{\mathcal{L},-d/m}$. Let $\bar{\mathfrak{M}}(w) = \mathfrak{M}(w) + \mathfrak{g}^*_{\mathcal{L},(-d/m)+} \in \mathfrak{g}^*_{\mathcal{L},-d/m}/\mathfrak{g}^*_{\mathcal{L},(-d/m)+}$.

3.2. Let $\mathcal{L} = \{L_i\}_{i\in\mathbf{Z}}$ be a regular small admissible lattice chain in $\mathcal{V}_1$ and (π_1, V_1) be an irreducible admissible representation of $\widetilde{U(\mathcal{V}_1)}$ with non-trivial $G_{\mathcal{L},(d/m)+}$-fixed vectors for some $d > 0$. Suppose that there is a vector $w \in B(\mathcal{L}, \mathcal{L}', d/m)$ such that $\mathcal{S}(A_0)_w$ has non-trivial image under the map $\Pi\colon \mathcal{S}(A_0) \to V_1 \otimes_{\mathbf{C}} V_2$. Now we have an action of $G_{\mathcal{L},d/m}$ on the one-dimensional space $\Pi(\mathcal{S}(A_0)_w)$. Then we have a character denoted by ψ_w of $G_{\mathcal{L},d/m}/G_{\mathcal{L},(d/m)+}$ on $\Pi(\mathcal{S}(A_0)_w)$. Because $(G_{\mathcal{L},d/m}/G_{\mathcal{L},(d/m)+})^\wedge \simeq \mathfrak{g}^*_{\mathcal{L},-d/m}/\mathfrak{g}^*_{\mathcal{L},(-d/m)+}$, there exists an element $X \in \mathfrak{g}^*_{\mathcal{L},-d/m}$ such that the coset $X + \mathfrak{g}^*_{\mathcal{L},(-d/m)+}$ corresponds to the character ψ_w of the quotient group $G_{\mathcal{L},d/m}/G_{\mathcal{L},(d/m)+}$ on $\Pi(\mathcal{S}(A_0)_w)$.

3.2.1. Now this character ψ_w is given by $g \mapsto \psi(\frac{1}{2}\langle\langle(g-1).w, w\rangle\rangle)$, where $g \in G_{\mathcal{L},d/m}$. Assume that $w = \sum_i \gamma_i \otimes \eta_i$, where $\gamma_i \in \mathcal{V}_1$, $\eta_i \in \mathcal{V}_2$, then we have

$$\psi(\frac{1}{2}\langle\langle(g-1).(\sum_i \gamma_i \otimes \eta_i), \sum_i \gamma_i \otimes \eta_i\rangle\rangle) = \psi(\mathfrak{B}_1(\sum_{i,j} -\frac{1}{4}c_{\gamma_i,\gamma_j\langle\eta_j,\eta_i\rangle_2}, g-1)).$$

Therefore the element X in subsection 3.2 can be chosen to be $-\frac{1}{4}\sum_{i,j} c_{\gamma_i,\gamma_j\langle\eta_j,\eta_i\rangle_2}$ i.e., $-\frac{1}{4}\mathfrak{M}(w)$.

3.3. For a given small admissible lattice chain $\mathcal{L}$ of period m in $\mathcal{V}$, let $\mathbf{v}_i = L_i/L_{i+1}^\sharp$. Identify $L_i/L_{i+1}^\sharp$ with $L_{i+m}/L_{i+1+m}^\sharp$ via multiplication by ϖ_D. Let $\mathbf{v} = \bigoplus_{i=1}^m \mathbf{v}_i$, then $\mathbf{v}$ is a space over the residue field of D and graded by $\mathbf{Z}/m\mathbf{Z}$. Define $\mathbf{v}'_{i'} = L'_{i'}/L'^\sharp_{i'+1}$, $\mathbf{v}' = \bigoplus_{i=1}^m \mathbf{v}'_i$ similarly. Now we know that $\bar{\mathfrak{M}}(w)$ is a graded homomorphism of $\mathbf{v}$ of degree $-d$ i.e., $\bar{\mathfrak{M}}(w)(\mathbf{v}_k) \subseteq \mathbf{v}_{k-d}$ for all k.

Proposition. *$\bar{\mathfrak{M}}(w)$ is a nilpotent endomorphism of $\mathbf{v}$ if and only if $\bar{\mathfrak{M}}'(w)$ is a nilpotent endomorphism of $\mathbf{v}'$.*

3.4. If $\mathcal{L}'$ is a small admissible lattice chain but not a regular lattice chain, then some $\mathbf{v}'_i$ is trivial. If we also assume that $\gcd(d,m) = 1$, then it is clear that $\bar{\mathfrak{M}}'(w)$ is a nilpotent endomorphism of $\mathbf{v}'$. Therefore we have the following proposition.

Proposition. *Suppose that $w \in B(\mathcal{L}, \mathcal{L}', d/m)$ for some regular small admissible lattice chain $\mathcal{L}$ and a small admissible lattice chain $\mathcal{L}'$. Assume that $\gcd(d,m) = 1$, and the residue class $-\frac{1}{4}\mathfrak{M}(w) + \mathfrak{g}^*_{\mathcal{L},(-d/m)+}$ contains no nilpotent element, then any two lattices in $\mathcal{L}'$ are distinct i.e., $\mathcal{L}'$ is a regular small admissible lattice chain.*

Therefore we need only to take care of the case $\gcd(d,m) = 2$ by Theorem 1.8.

3.4.1. In this subsection we assume that $\gcd(d,m) = 2$. The idea is: if $\mathcal{L}'$ is not a lattice chain, then we can find regular lattice chains $\mathcal{M}, \mathcal{M}'$ of period $m/2$ such that $B(\mathcal{L}, \mathcal{L}', d/m) \subseteq B(\mathcal{M}, \mathcal{M}', \frac{d}{2}/\frac{m}{2})$. Therefore we can reduce the problem to situation in Subsection 3.4.

3.5. Theorem. *Let $(G, G') = (U(\mathcal{V}_1), U(\mathcal{V}_2))$ be a reductive dual pair. Let (π_1, V_1) (resp. (π_2, V_2)) be an irreducible admissible representation of $\widetilde{U(\mathcal{V}_1)}$ (resp. $\widetilde{U(\mathcal{V}_2)}$) such that V_1 and V_2 are paired by the local theta correspondence. Then the depth of V_1 is equal to the depth of V_2.*

Now the proof of Theorem 3.5 becomes clear. At least one of the local Dynkin diagrams of $U(\mathcal{V}_1)$, $U(\mathcal{V}_2)$ does not have ramified ends. We assume the local Dynkin diagram of $U(\mathcal{V}_1)$ satisfies this condition, so we can apply Theorem 1.5.1 and Proposition 1.7.2 to describe the minimal K-types of irreducible admissible representations of $\widetilde{U(\mathcal{V}_1)}$. If we assume that $V_1^{G_{\mathcal{L},(d/m)+}} \neq \{0\}$, then there is a regular small admissible lattice chain $\mathcal{L}'$ of period m or $m/2$ (the latter happens only when m is even) in $\mathcal{V}_2$ such that $V_2^{G_{\mathcal{L}',(d/m)+}}$ (or $V_2^{G_{\mathcal{L}',(\frac{d}{2}/\frac{m}{2})+}}$) is not trivial. Finally results in subsection 3.2.1 and 3.3 enforce the depth of π_1 has to be equal to the depth of π_2.

3.6. Therefore we have the following theorem immediately from Theorem 3.5 and the stable range of the local theta correspondence. This theorem complete our analysis on minimal K-types for all p-adic classical groups.

Theorem. *Let (π, V) be an irreducible admissible representation of a p-adic classical group G, then (π, V) has a minimal K-type $(G_{x,r}, \chi)$ where x is a generalized barycenter in the Bruhat-Tits building of G.*

3.7. The depths for classical groups. Let ℓ_G be the maximal period of all small admissible lattice chains in $\mathcal{V}$, then the depth of an irreducible admissible representation of a classical group G will be of the form s/t, where $t \leq \ell_G$. The number ℓ_G can be computed easily for any G, for example $\ell_G = 2n$ for $G = \mathrm{Sp}_{2n}(F)$.

Corollary. *If (π, V) is an irreducible admissible representation of a classical group G of depth s/ℓ_G such that $\gcd(s, \ell_G) = 1$, then π is supercuspidal.*

Of course, this corollary is well known when $G = \mathrm{GL}_n(F)$, in which case ℓ_G is just equal to n.

3.8. Cases for depth zero. The following corollary connects a simple relation between the local theta correspondence over a p-adic filed and the local theta correspondence over its residue field. Here we will write $G_{L,r}$ instead of $G_{\mathcal{L},r}$ if $\mathcal{L}$ is a regular self-dual lattice chain of period 1 or 2 determined by a single lattice L. It is not difficult to see: if a representation (π, V) of $U(\mathcal{V})$ is of depth zero, then $V^{G_{L,0+}} \neq \{0\}$ for some good lattice L in $\mathcal{V}$.

3.8.1. Let L (resp. L') be a good lattice in $\mathcal{V}_1$ (resp. $\mathcal{V}_2$). Let $B = B(L, L') = L^* \otimes L' + L \otimes L'^*$. It is easy to check that $B\varpi_F \subseteq B^* \subseteq B$. Then B/B^* can be identified as a vector space over $\mathbf{f}$ with a non-degenerated skew-symmetric bilinear form. Hence we can consider the finite symplectic group $\mathrm{Sp}(B/B^*)$ and its Weil representation. Let K_B be the stabilizer of B in $\mathrm{Sp}(\mathcal{W})$ and $K'_B := \{\, g \in K_B \mid (g-1).B \subseteq B^* \,\}$. Now we have a close relation between the Weil representation of (the metaplectic cover of) a p-adic symplectic group and the Weil representation of a finite symplectic group as follows.

Proposition. *Let $\mathcal{S}$ be the Weil representation of $\widetilde{\mathrm{Sp}(\mathcal{W})}$, then the action of $K_B/K'_B \simeq \mathrm{Sp}(B/B^*)$ on $\mathcal{S}^{K'_B}$ is the Weil representation.*

3.8.2. Proposition. *The pair $(G_L/G_{L,0^+}, G'_{L'}/G'_{L',0^+})$ is a reductive dual pair in the finite symplectic group $\mathrm{Sp}(B/B^*)$ defined over the residue field $\mathbf{f}$ of F.*

3.8.3. Corollary. *Let $(G, G') = (U(\mathcal{V}_1), U(\mathcal{V}_2))$ be a reductive dual pair. Let (π_1, V_1) be an irreducible smooth representation of $\widetilde{U(\mathcal{V}_1)}$ and (π_2, V_2) be an irreducible smooth representation of $\widetilde{U(\mathcal{V}_2)}$ such that two representations correspond in the local theta correspondence. Then $V_1^{G_{L,0+}} \neq \{0\}$ for some good lattice L in $\mathcal{V}_1$ if and only if there is a good lattice L' in $\mathcal{V}_2$ such that $V_2^{G'_{L',0+}} \neq \{0\}$. In particular, $\mathrm{depth}(\pi_1) = 0$ if and only if $\mathrm{depth}(\pi_2) = 0$. Moreover if ϕ_1 is an irreducible representation of $G_L/G_{L,0^+}$ which* *occurs in* $V_1^{G_{L,0+}}$*, then there is an irreducible representation* ϕ_2 *of* $G'_{L'}/G'_{L',0+}$ *which occurs in* $V_2^{G'_{L',0+}}$ *such that* ϕ_1 *and* ϕ_2 *correspond in the correspondence for the dual pair* $(G_L/G_{L,0+}, G'_{L'}/G'_{L',0+})$.

3.8.4. Corollary. *Let* $(U(\mathcal{V}_1), U(\mathcal{V}_2))$ *be a reductive dual pair. Let* I_i *be the Iwahori subgroup of* $U(\mathcal{V}_i)$*. And Let* (π_i, V_i) *be an irreducible admissible representation of* $\widetilde{U(\mathcal{V}_i)}$*,* $i = 1, 2$*. Suppose that* π_1, π_2 *are paired by local theta correspondence, then* $V_1^{I_1} \neq \{0\}$ *if and only if* $V_2^{I_2} \neq \{0\}$.

Corollary 3.8.4 has been proved by A.-M. Aubert for unramified reductive dual pairs.

4. References

[MVW] C. Mœglin, M.-F. Vigneras and J.-L. Waldspurger, *Correspondances de Howe sur un corps p-adigues,* Lect. Note. Math. **1291**, Springer-Verlag, Berlin-Heidelberg-New York, 1987.

[MP1] A. Moy and G. Prasad, Unrefined minimal K-types for p-adic groups, Invent. Math. **116** (1994), 393–408.

[MP2] A. Moy and G. Prasad, Jacquet functors and unrefined minimal K-types, Comment. Math. Helv. **71** (1996), 98–121.

[P] S.-Y. Pan, local theta correspondence and unrefined minimal K-types, preprint, 1997.

[PY] S.-Y. Pan and J.-K. Yu, Unrefined minimal K-types for p-adic classical groups, preprint, 1997.

[W] J.-L. Waldspurger, Demonstration d'une conjecture de duality de Howe dans le case p-adiques, $p \neq 2$, in *Festschrift in honor of I. Piatetski-Shapiro*, Israel Math. Conf. Vol. **2** (1990), 267–324.

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