PROPERTIES OF UNIPOTENT DISCRETE SERIES L-PACKETS FOR p-ADIC GROUPS

Mark Reeder

Boston College

1. Introduction. Let G be the rational points of a connected split group of adjoint type over a p-adic field F. A representation V of G is *unipotent* if there is a parahoric subgroup $P < G$, such that the restriction of V to P contains a cuspidal unipotent representation σ of the reductive quotient $\overline{P}$. By results of Moy and Prasad [MP], the pair (P, σ) is essentially unique. (Recall from Deligne-Lusztig theory that a representation of a finite reductive group $M = M(\overline{\mathbb{F}}_q)^{\mathrm{Fr}}$ is unipotent if it appears in one of the virtual M-modules $R_w := \sum_i (-1)^i H^i_c(\mathcal{B}_w, \bar{\mathbb{Q}}_\ell)$, where $\mathcal{B}_w$ is the variety of Borel subgroups B of $M(\overline{\mathbb{F}}_q)$ in relative position w, for some w in the Weyl group of $M(\overline{\mathbb{F}}_q)$. See [Car] for more details.)

In this talk we consider, for certain groups G, those unipotent representations which are square integrable, and we describe the structure of their L-packets in terms of formal degrees and K-types. The basic goal is to understand what properties are shared by representations in a single L-packet.

2. Unramified Langlands correspondence. We begin with Lusztig's recent classification of unipotent representations. Roughly speaking, it says that the unipotent representations of G are precisely those whose Langlands parameters are trivial on the inertia subgroup of the Weil group of F. Such a parameter amounts to an element $x \in \hat{G}$, the dual group of G, taken up to conjugacy. The precise statement of Lusztig's classification is

Theorem[Lusztig, [L1]**].** *The set* $\mathrm{Irr}_{unip}(G)$ *of irreducible unipotent representations of* G *is partitioned as*

$$\mathrm{Irr}_{unip}(G) = \coprod_{x \in \hat{G}/\mathrm{conj}} \Pi_x,$$

where the disjoint union is over conjugacy classes in $\hat{G}$, *and we have a parametrization*

$$\Pi_x = \{V_{x,\rho} : \ r \in \widehat{A(x)}\},$$

where $\widehat{A(x)}$ *is the set of irreducible characters of the finite group* $A(x) := \hat{G}_x / Z_{\hat{G}} \hat{G}^\circ_x$.

Here $\hat{G}_x$ is the centralizer in $\hat{G}$ of x, $\hat{G}^\circ_x$ is its identity component, and $Z_{\hat{G}}$ is the center of $\hat{G}$. The group $A(x)$ is either S_2^n, S_3, S_4, S_5, the latter three cases occuring only in exceptional groups.

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We have

$$\Pi_x = \coprod_{(P,\sigma)} \Pi_x(P,\sigma),$$

where $\Pi_x(P,\sigma)$ consists of those representations in Π_x containing (P,σ). Representations in Π_x may belong to different parabolically induced representations.

To describe properties of Π_x, it is helpful to give names to some objects attached to x. Let $x = su$ be the Jordan decomposition. By the Jacobson-Morozov theorem, there is a homomorphism

$$\phi_u : SL_2(\mathbf{C}) \longrightarrow \hat{G}$$

such that $u = \phi_u \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. We can arrange that ϕ_u maps the diagonal matrices into a maximal torus $\hat{T}$ containing s. Set

$$\tau = s\phi_u \begin{pmatrix} q^{-1/2} & 0 \\ 0 & q^{1/2} \end{pmatrix}.$$

By the definition of the dual group, we may view $\hat{T}$ as the set of unramified characters of a Borel subgroup B of the p-adic group G. Thus we have the (normalized) induced representation $\mathrm{Ind}_B^G(\tau)$. Some of the constituents of $\mathrm{Ind}_B^G(\tau)$ will appear in Π_x. More precisely, we have

Property 1. *The following are equivalent*

(1) $V_{x,\rho}$ *is a subquotient of* $\mathrm{Ind}_B^G \tau$.

(2) $V_{x,\rho} \in \Pi_x(\mathcal{I}, 1_{\mathcal{I}})$, *where* $\mathcal{I}$ *is an Iwahori subgroup and* $1_{\mathcal{I}}$ *is its trivial representation.*

(3) ρ *appears in the natural action of* $A(x)$ *on the homology* $H(\mathcal{B}^\tau \cap \mathcal{B}^u)$, *where* $\mathcal{B}$ *is the flag variety of* $\hat{G}$. *(Here superscripts denote fixed points.)*

The equivalence of (1) and (2) was proved by Borel [B] in the 70's, and that of (2) and (3) by Kazhdan-Lusztig [KL] in the 80's.

Property 2. *The following are equivalent*

(1) $V_{x,\rho}$ *has a Whittaker model*

(2) $V_{x,\rho}$ *contains the Steinberg representation of* $G(\mathbb{F}_q)$, *pulled back to a hyperspecial maximal compact subgroup of* G.

(3) $\rho = 1$ *and* u *belongs to the dense* $\hat{G}_\tau$*-orbit on the set of* $v \in \hat{G}$ *satisfying* $\tau v \tau^{-1} = v^q$.

The second condition in (3) is automatic if it is known in advance that $V_{x,\rho}$ is tempered. The fact that $V_{x,1}$ is then the unique generic representation in Π_x is an expected feature of any tempered L-packet.

The equivalence (1)=(2) was proved for $\mathcal{I}$-spherical representations by Barbasch-Moy [BM], Li [Li], and myself [R4], and [R3] contains the equivalence with (3). The validity of Property (3) for all unipotent representations was established recently, [R5], by showing that $\Pi_x(P,\sigma)$ contains no generic representation unless $(P,\sigma) = (\mathcal{I}, 1_{\mathcal{I}})$. This is well-known for the reductive quotients, and a little Bruhat-Tits theory is used to lift non-genericity to the p-adic group.

Let $\mathrm{Irr}^2_{unip}(G)$ be the set of square integrable unipotent representations of G, up to isomorphism.

Expected Property 3. *The following are equivalent.*

(1) $\Pi_x \subset \mathrm{Irr}^2_{unip}(G)$
(2) $V_{x,1} \in \mathrm{Irr}^2_{unip}(G)$
(3) *x is not contained in a proper Levi subgroup of G.*

There are only finitely many conjugacy classes of $x \in \hat{G}$ satisfying (3). The equivalence of (2) and (3) was proved by Kazhdan-Lusztig. I have not traced through the bijection $(x, \rho) \longrightarrow V_{x,\rho}$ explicitly enough to verify (1)=(2) in general. However, for specific groups G, one can enumerate $\mathrm{Irr}^2_{unip}(G)$ by counting the square integrable representations of the Hecke algebras $\mathcal{H}(P, \sigma)$ for each parahoric P and cuspidal unipotent σ on $\bar{P}$. Then other properties of square integrable L-packets, to be discussed shortly, enable one to assign an (x, ρ) to a representation in $\mathrm{Irr}^2_{unip}(G)$, in such a way that Property 3 holds. This bijection (now between finite sets) is no doubt the restriction of Lusztig's bijection.

Example: Take $G = E_8$. We have $\# \mathrm{Irr}^2_{unip}(G) = 98$, partitioned into 31 L-packets Π_x. We will describe one of them. Take P with $\overline{P}$ having derived group of type E_6. The finite group $E_6(\mathbb{F}_q)$ has two cuspidal unipotent representations, $\sigma_\theta = E_6[\theta]$ (notation as in Carter), where θ is one of the two nontrivial cube roots of unity.

Consider the Hecke algebra $\mathcal{H}(P, \sigma)$, consisting of locally constant compactly supported functions $f : G \longrightarrow \mathrm{End}(\sigma)$ satisfying $f(p_1 g p_2) = \check{\sigma}(p_1) f(g) \check{\sigma}(p_2)$, with convolution multiplication. The structure of $\mathcal{H}(P, \sigma)$ was determined by Morris [M], and in this case $\mathcal{H}(P, \sigma_\theta)$ is an affine Hecke algebra of type G_2, with generators T_s, T_r, T_0, the usual braid relations for affine Weyl group G_2, and the additional relations $(T_a - q^{c(a)}(T_a + 1) = 0$, where $a \in \{s, r, 0\}$, $c(s) = 9$, $c(r) = c(0) = 1$. This is not the Hecke algebra of Iwahori-spherical functions on any p-adic group.

Given an admissible G-module V containing σ upon restriction to P, we consider the $\mathcal{H}(P, \sigma)$-module $V^\sigma := \mathrm{Hom}_P(\sigma, V)$. This is a functor which induces a bijection between irreducible V's containing (P, σ) and irreducible $\mathcal{H}(P, \sigma)$-modules. There is a notion of square-integrability for abstract Hecke modules [Mat] and this bijection preserves square-integrability. In the present case, $\mathcal{H}(P, \sigma_\theta)$ has exactly four L^2 modules, hence E_8 has exactly four L^2 representations containing σ_θ upon restriction to the E_6-parahoric. We describe one of them. The $\mathcal{H}(P, \sigma_\theta)$-module is one-dimensional, and each of the generators T_a acts by -1. Let $V_{x,\theta}$ denote the corresponding E_8-module, where $x = su \in E_8(\mathbf{C}) = \hat{G}$ is given as follows. The unipotent element u is $E_8(b_5)$ according to the Bala-Carter classification. The centralizer of ϕ_u is S_3, and s is a three-cycle! in S^3. The promised L-packet is

$$\Pi_x = \{V_{x,1}, V_{x,\theta}, V_{x,\bar{\theta}}\}.$$

2. Formal degrees. The following is our

Main Result. *For $G = G_2, F_4, E_6$ or classical of rank ≤ 4, there is a bijection (unique up to choosing roots of unity)*

$$(x,\rho) \mapsto V_{x,\rho} : \{(x,\rho) :\ x \notin \text{ proper Levi of } \hat{G},\ \rho \in \widehat{A(x)}\} \longrightarrow \mathrm{Irr}^2_{unip}(G),$$

such that the formal degree, with volume of Iwahori equal to one, is given by

$$\deg V_{x,\rho} = \frac{\rho(1)}{\#A(x)\#Z_{\hat{G}}} \frac{q^\nu \prod' \alpha(\tau) - 1}{\prod' q\alpha(\tau) - 1}, \tag{1}$$

where ν is the number of positive roots of $\hat{G}$, and $\prod'$ denotes the product of nonzero terms over all roots of $\hat{G}$.

Remarks

i) We have $\deg V_{x,\rho} = \rho(1) V_{1,\rho}$. Assuming certain conjectures about L-packets and stable distributions, Shahidi has shown that the formal degrees in an L-packet should be proportional. Our formula indicates that the proportionality constant should be $\rho(1)$.

ii) Let $\mathcal{B}^u$ be the variety of Borel subgroups of $\hat{G}$ containing u. One can show that, as a power series in q, the right side of (1) is

$$\deg V_{x,\rho} = \frac{\rho(1)\det(I - Ads|_{\mathfrak{m}_u})}{\#A(x)\#Z_{\hat{G}}} q^{\dim \mathcal{B}^u} + \text{higher powers},$$

where $\mathfrak{m}_u$ is the Lie algebra of the centralizer of the image of ϕ_u. The determinant is nonzero when su is not in a proper Levi subgroup of $\hat{G}$. For exceptional groups, those u which are part of L^2 Langlands parameters are determined by their $\dim \mathcal{B}^u$. Thus, the formal degree of a L^2-module V, as a rational function in q, determines the unipotent part of the Langlands parameter of V. A similar phenomenon was observed for finite Hecke algebras by Lusztig [L4], and predicted by him for affine Hecke algebras with equal parameters.

Recall that the value at 0 of the L-function $L(x)$ of the Langlands parameter corresponding to x is

$$L(x,0) = \det(I - Ad(\tau)|_{\mathfrak{g}^u})^{-1}.$$

If we grade $\mathfrak{g}^u = \mathfrak{g}_0 + \mathfrak{g}_1 + \ldots$, we see that $\det(I - Ads|_{\mathfrak{m}_u})^{-1}$ is the factor of $L(x,0)$ coming from $\mathfrak{g}_0$.

iii) Our formulation of (1) was inspired by the following result of Heckman and Opdam [HO], valid for any split G. They assert that

$$\deg(V_{x,1}) = c\frac{q^\nu \prod' \alpha(\tau) - 1}{\prod' q\alpha(\tau) - 1},$$

where c is a nonzero rational number, independent of q, but otherwise unknown. As with previous formal degree calculations by Shalika, Borel and myself, this relies on the existence of a nice K-type, namely the Steinberg, which exists for $\rho = 1$ by property 2. Formula (1) indicates (and shows, in the cases for which (1) is proved) that

$$c = \frac{1}{\#A(x)\#Z_{\hat{G}}}.$$

iv) Since the conference, formula (1) has been verified for all non-Iwahori-spherical unipotent L^2 representations of E_8.

4. How to compute formal degrees We verify (1) by computing each formal degree separately. The main tool, that avoids the dependence on nice K-types, is a theorem of Schneider and Stühler, which says, when applied to unipotent representations, that

$$\deg V = \sum_c (-1)^{\dim c} \frac{\dim V^{U_c}}{vol(P_c^+)}, \tag{2}$$

where the sum is over G-orbits of facets in the building, P_c^+ is the stabilizer of facet c, and U_c is the pro-unipotent radical of P_c^+. There are about $2^{\ell+1}$ such facets, where ℓ is the rank of G. Using the duality theory for finite reductive groups developed by Alvis, Curtis and Kawanaka, one can reduce to a sum of about $\ell + 1$ terms (essentially a sum over vertices). Moreover, if V contains some cuspidal unipotent (σ, P), then we only get a contribution from facets c containing the facet of P in their closure. (Note: Schneider and Stühler require $char F = 0$ in their general formula, but at least for unipotent representations, formal degrees can be expressed in terms of Hecke algebra formal degrees. It follows that formula (2) holds for any F.)

Thus, the main problem is to compute V^{U_c}. First assume V is Iwahori-spherical. Here all facets contribute to the formal degree, but there are additional tools from algebraic geometry at our disposal. It is enough to let $q \to 1$ and determine $[V]_{q=1}$ as a module over the affine Weyl group $\widetilde{W}$. Then V^{U_c} is found by restricting to the appropriate parahoric subgroup of $\widetilde{W}$.

The following result was proved by Lusztig for $s = 1$, and without the tildes, but is easily extended to the version stated here. Some notation: Let $\widetilde{W}_s$ be the full inverse image in $\widetilde{W}$ of the stabilizer W_s of s in W. We can view s as a character of $\widetilde{W}_s$. Let $\mathcal{B}_s^u$ be the fixed points of u in the flag manifold of $\hat{G}_s$. And let $H(\mathcal{B}_s^u)_\rho$ be the ρ-isotypic component in homology, with grading ignored, viewed as the Springer representation of W_s, and with translations acting by cap-product with the Chern classes of their corresponding line bundles.

Proposition. *Assume $V_{x,\rho}$ has Iwahori fixed vectors. Then*

$$[V_{x,\rho}]_{q=1} = sgn \otimes \mathrm{Ind}_{\widetilde{W}_s}^{\widetilde{W}} \big(s \otimes H(\mathcal{B}_x^u)_\rho\big).$$

The W_s-modules $H(\mathcal{B}_s^u)_\rho$ were determined by the Green polynomial calculations of Beynon-Spaltenstein [BS] and Shoji [Sh], and the decomposition of the induced representations is greatly facilitated by the tables of Alvis [A].

This result does not apply to non-Iwahori spherical representations, but in compensation, these involve smaller Hecke algebras (at most rank 4, for exceptional G), and we can use the elementary theory of weight diagrams, as developed in [R2].

5. Leading K-types. The above calculations lead to the following (as yet imprecise) observation, namely a relation between Langlands parameters, and Lusztig's families of unipotent representations.

First, some recollections on truncation (see [L3]). Let W_0 be a finite Coxeter group, and let R denote its reflection representation. Let $\mathbb{H} = \oplus \mathbb{H}_d$ denote the W_0-harmonic polynomials on R. For any W_0 module U, let $b(U)$ be the minimum d for which $\langle U, \mathbb{H}_d \rangle \neq 0$. Define

$$J_0(U) := \sum_{\substack{\psi \in \hat{W}_0 \\ b(\psi) = b(U)}} \langle \psi, U \rangle \psi.$$

Now suppose V is a unipotent representation of our p-adic group G, containing the cuspidal unipotent pair (P, σ). Let W_0 be the finite Coxeter group underlying $\mathcal{H}(P, \sigma)$. Then we define

$$J(V) := J_0\big([V^\sigma]_{q=1}\big),$$

where $[V^\sigma]_{q=1}$ is viewed as a W_0-module by restricting from the affine Weyl group underlying $\mathcal{H}(P, \sigma)$. We call $J(V)$ the "leading K-type" of V, even though it could, at first glance, be reducible.

However, we have observed: *For the groups and representations mentioned above, $J(V)$ is irreducible, and almost determines the L-packet of V.*

To explain this, recall that Lusztig has partitioned the unipotent representations of $G(\mathbb{F}_q)$ into families $\mathcal{F}_u$, one family for each special unipotent class $[u]$ in $\hat{G}$. Furthermore, he showed there is a canonical quotient Γ_u of $A(u)$, with the following property: Let $M(u)$ be the set of conjugate pairs (s, ρ), where $s \in \Gamma_u$ and ρ is a representation of the centralizer of s. Then we have a parametrization

$$\mathcal{F}_u := \{\chi^u_{s,\rho} : \ (s, \rho) \in M(u)\},$$

with certain nice properties, and which Lusztig has explicitly calculated [L3].

The connection with leading K-types is as follows. It is only verified in the examples for which (1) is proved, $E8$, and a few others. To keep things simple, assume

(i) The canonical quotient $A(u) \longrightarrow \Gamma_u$ is an isomorphism (this rarely fails, for exceptional groups). Identify $A(u) = \Gamma(u)$.

(ii) The unipotent element u does not live in a proper Levi subgroup of $\hat{G}$. (Again, this holds with few exceptions, if u is part of an L^2 parameter). Then we can identify $A(su)$ with the centralizer of s in Γ_u.

Then we have

$$J(V_{su,\rho}) = \chi^u_{s,\rho}.$$

Thus, the Langlands parameter can often be read off from Lusztig's parameter for the leading K-type.

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Chestnut Hill, MA 02167

E-mail address: `reederma@bc.edu`