NONVANISHING OF GLOBAL THETA LIFTS

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In this summary we use the following definitions. Let F be a local field of characteristic zero or totally real number field, and let V be a nondegenerate symmetric bilinear space defined over F of *even* dimension:

$$\dim_F V = l.$$

Let n be a nonnegative integer, and let $\mathrm{Sp}(n)$ be the symplectic group of rank n. Let $\chi = \chi_V$ be the quadratic character of $F^\times$ if F is local, or of $\mathbb{A}^\times$ if F is a number field, defined by

$$\chi = \begin{cases} 1 \text{ if } \mathrm{disc}(V) = 1 \\ \omega_{F(\sqrt{\mathrm{disc}(V)})/F} \text{ if } \mathrm{disc}(V) \neq 1. \end{cases}$$

Here, $\omega_{F(\sqrt{\mathrm{disc}(V)})/F}$ is the quadratic character of $F^\times$ or $\mathbb{A}^\times$ associated to $F(\sqrt{\mathrm{disc}(V)})/F$ by class field theory. Let ω_n be the Weil representation of $\mathrm{O}(V(F)) \times \mathrm{Sp}(n, F)$ or $\mathrm{O}(V(\mathbb{A})) \times \mathrm{Sp}(n, \mathbb{A})$ if F is local or a number field, respectively.

Locally and globally, the Weil representation defines a correspondence between representations of our two isometry groups. Suppose F is local. Then $\sigma \in \mathrm{Irr}(\mathrm{O}(V(F)))$ and $\pi \in \mathrm{Irr}(\mathrm{Sp}(n, F))$ correspond if

$$\mathrm{Hom}_{\mathrm{O}(V(F)) \times \mathrm{Sp}(n,F)}(\omega_n, \sigma \otimes \pi) \neq 0.$$

In fact, except in the case when F is nonarchimedean of even residual characteristic it is known that if $\sigma \in \mathrm{Irr}(\mathrm{O}(V(F)))$ corresponds to $\pi \in \mathrm{Irr}(\mathrm{Sp}(n, F))$ then π is unique. For the purposes of exposition, we will assume that this statement, Howe duality, always holds. If $\sigma \in \mathrm{Irr}(\mathrm{O}(V(F)))$ then it is convenient to write $\Theta_n(\sigma) \neq 0$ if and only if σ occurs in the correspondence with $\mathrm{Irr}(\mathrm{Sp}(n, F))$. If F is a number field, and σ is an irreducible cuspidal automorphic representation of $\mathrm{O}(V(\mathbb{A}))$, then the theta lift $\Theta_n(\sigma)$ is defined and is a space of automorphic forms on $\mathrm{Sp}(n, \mathbb{A})$.

1. The main theorem. In this summary we address the following problem:

The nonvanishing problem. *If σ is an irreducible cuspidal automorphic representation of $\mathrm{O}(V(\mathbb{A}))$ then determine when $\Theta_n(\sigma) \neq 0$.*

We are especially interested in this problem if the underlying bilinear spaces have about the same dimension. In particular we are interested in the case $l = 2n = 4$, and later we will make some comments about the application of the main result to this case.

Before discussing approaches to this problem it is useful to recall some facts about theta lifts and Witt towers. Fix an irreducible cuspidal automorphic representation $\sigma = \otimes_v \sigma_v$ of $\mathrm{O}(V(\mathbb{A}))$. We have the following useful diagram:

$$\begin{array}{ccll}
 & & \Theta_l(\sigma) & \mathrm{Sp}(l) \\
 & & \vdots & \vdots \\
 & \nearrow & \Theta_{l/2+1}(\sigma) & \mathrm{Sp}(l/2+1) \\
\mathrm{O}(V),\ \sigma & \longrightarrow & \Theta_{l/2}(\sigma) & \mathrm{Sp}(l/2) \\
 & \searrow & \Theta_{l/2-1}(\sigma) & \mathrm{Sp}(l/2-1) \\
 & & \vdots & \vdots \\
 & & \Theta_0(\sigma) & \mathrm{Sp}(0)
\end{array}$$

The vertical sequence of symplectic groups is known as the **Witt tower**. Also, we write $\mathrm{O}(V)$ and $\mathrm{Sp}(l/2)$ on the same line since the underlying bilinear spaces for these groups have the same dimension. We can consider the sequence:

$$\Theta_0(\sigma), \Theta_1(\sigma), \Theta_2(\sigma), \ldots$$

Three facts are known: First, if $\Theta_k(\sigma) \neq 0$ then $\Theta_{k+1}(\sigma) \neq 0$. Second, $\Theta_l(\sigma) \neq 0$. And third, if $\Theta_k(\sigma) = 0$ and $\Theta_{k+1}(\sigma) \neq 0$, then $\Theta_{k+1}(\sigma)$, i.e., the **first occurrence** of σ, is cuspidal. Similar results hold locally.

Considerable work has been done on the nonvanishing problem. One approach is the inner product formula [KR1]. For our problem, in the convergent range $2n \geq 2l$, the basic Rallis inner product formula gives the inner product of a theta lift of a cusp form with itself as the product of a nonzero special value of a partial L function attached to σ with a finite number of other nonzero local factors, which are related to the local theta correspondences; see [L]. For theta lifts in the other direction, from the symplectic group to the orthogonal group, there is the regularized inner product formula [KR1]. Here, a regularized inner product of the theta lift with itself is given as the residue of a product of a partial L function with some other local factors. This formula is valid in the range $2n + 2 \leq l$, so that the underlying bilinear spaces are allowed to have about the same dimension. There should be a regularized inner product formula for the case of theta lifts from the orthogonal group, though for either case, the local factors analogous to those of the basic identity still need to be understood. The inner product identity has applications besides the nonvanishing problem. There are other approaches to the nonvanishing problem. For example, in [GRS] it is shown that if σ is a generic irreducible cuspidal automorphic representation of the split $\mathrm{SO}(V)$, then the following holds. If the partial standard L-function $L^S(s, \sigma)$ has a pole at $s = 1$, then $\Theta_{l/2-1}(\sigma)$ is nonzero, cuspidal and generic, and if $L^S(s, \sigma)$ is holomorphic at $s = 1$ then $\Theta_{l/2}(\sigma)$ is nonzero, cuspidal and generic.

In this summary, generalizing work of Böcherer and Schulze-Pillot [BS-P] in the setting of classical modular forms, we describe a solution to this problem under some hypotheses. For our result we will make the following local hypotheses:

(L1) σ_v is tempered for all places v of F;

(L2) $\Theta_n(\sigma_v) \neq 0$ for all places v of F;

(L3) The first occurrence of σ_v is tempered for all places v of F.

We will make the following global hypotheses:

(G1) $2n \geq l$;

(G2) $\Theta_{n+1}(\sigma) \neq 0$;

(G3) If $n+1-l/2 = 1$, i.e., $2n = l$, then for all finite sets S of places including the infinite and ramified places, the partial standard L function $L^S(s, \sigma)$ does not have a zero at $s = n+1-l/2 = 1$.

In fact, in our result we will have to assume an additional hypothesis at the infinite places, but we expect to be able to remove this hypothesis. If $2n = l$, and v is nonarchimedean, then the last local hypothesis (L3) follows from (L1) and (L2); see Theorem 3. Notice that by the assumption (L1), $L^S(s, \sigma)$ does not vanish at $s = n+1-l/2$ if $n+1-l/2 > 1$, i.e., $2n > l$.

Before stating and proving the main result we need a lemma.

Lemma 1. [R4] *Suppose that F is local and nonarchimedean. Let $\sigma \in \mathrm{Irr}(\mathrm{O}(V(F)))$ be tempered, and assume that the first occurrence of σ is tempered. Let $\pi' \in \mathrm{Irr}(\mathrm{Sp}(n', F))$ correspond to σ, and assume that in the Witt tower π' is strictly past the first occurrence of σ and $2n' > l$. Then there exist $f \in \pi'$, $f^\vee \in \pi'^\vee$ and a standard section Φ of the degenerate principal series representation $I_{2n'}(s, \chi)$ of $\mathrm{Sp}(2n', F)$ such that the zeta integral*

$$Z(s, f, f^\vee, \Phi) = \int_{\mathrm{Sp}(n', F)} \Phi(i(g, 1), s) \langle \pi'(g) f, f^\vee \rangle \, dg$$

has a pole at $s = n' - l/2 - 1/2$.

A section $\Phi : \mathrm{Sp}(2n', F) \times \mathbb{C} \to \mathbb{C}$ of $I_{2n'}(s, \chi)$ is standard if the restriction of Φ to $\mathrm{Sp}(2n', \mathfrak{o})$ is independent of s. Later, we will make some remarks about the proof of the lemma.

Theorem 2. *Assume the above hypotheses, and additionally assume Lemma 1 holds in the archimedean case. Then $\Theta_n(\sigma) \neq 0$.*

Proof. We follow the idea due to Böcherer and Schulze-Pillot [BS-P]. Suppose $\Theta_n(\sigma) = 0$. Write $n' = n+1$. Then $\Theta_{n'}(\sigma)$ is cuspidal. Let π' be an irreducible constitutent of $\Theta_{n'}(\sigma)$. We will obtain a contradiction by computing the order of $L^S(s, \pi', \chi)$ at $s = n' - l/2$ in two different ways. Let S be the set of all ramified places, including the archimedean places. On the one hand, we have

$$L^S(s, \pi', \chi) = L^S(s, \sigma) \zeta_F^S(s - (n' - l/2)) \zeta_F^S(s - (n' - l/2 - 1)) \prod_{k=-(n'-l/2)}^{n'-l/2-2} \zeta_F^S(s-k).$$

By hypothesis, $L^S(s, \sigma)$ does not have a zero at $s = n' - l/2$. Also, $\zeta_F^S(s - (n' - l/2))$ has a zero of order $|S| - 1$ at $s = n' - l/2$, $\zeta_F^S(s - (n' - l/2 - 1))$ has a simple pole at

$s = n' - l/2$ and the remaining factors have neither a pole or zero at $s = n' - l/2$. Thus, altogether, $L^S(s, \pi', \chi)$ has a zero of order at most $|S| - 2$ at $s = n' - l/2$. On the other hand, for any factorizable vectors $f \in \pi$, $f^\vee \in \pi^\vee$ and factorizable standard section Φ that are unramified outside of S,

$$L^S(s, \pi', \chi) \prod_{v \in S} Z(s - 1/2, f_v, f_v^\vee, \Phi_v) = Z^*(s - 1/2, f, f^\vee, \Phi).$$

See [KR1]. Here, $Z^*(s - 1/2, f, f^\vee, \Phi)$ is a certain normalized global zeta integral analogous to the local zeta integral of the lemma. By the lemma and the assumption, we can pick f_v, $f_v^\vee$ and Φ_v for $v \in S$ such that $Z(s - 1/2, f_v, f_v^\vee, \Phi_v)$ has a pole at $s = n' - l/2$ for $v \in S$. By a deep result of [KR1], the global zeta integral $Z^*(s - 1/2, f, f^\vee, \Phi)$ has at most a simple pole at $s = n' - l/2$. It follows that $L^S(s, \pi', \chi)$ has a zero of order at least $|S| - 1$ at $s = n' - l/2$, a contradiction. $\square$

2. Tempered representations and the local correspondence. We will now consider a result which often makes the last local assumption (L3) unnecessary. In this section, F is a local nonarchimedean field of characteristic zero.

Theorem 3. [R3] *Suppose that $\sigma \in \mathrm{Irr}(\mathrm{O}(V(F)))$ is tempered, and $\pi \in \mathrm{Irr}(\mathrm{Sp}(n, F))$ corresponds to σ.*

(1) *If $2n \leq l$, then π is tempered;*
(2) *If $2n > l$, π is the first occurrence of σ, and π is not tempered, then*

$$\pi = L(\delta_1 \otimes \cdots \otimes \delta_t \otimes \tau),$$

where

$$\delta_1 \cong \mathrm{Ind}_{P_{p_1, \ldots, p_s}}^{\mathrm{Gl}(n_1, F)} (\mathbf{St}_{p_1} \otimes \cdots \otimes \mathbf{St}_{p_s}) \otimes \chi(\det) |\det|^{n - l/2 - c/2},$$

with $\min(p_1, \ldots, p_s) \geq n - l/2 + 1$ *and c an integer such that* $\min(p_1, \ldots, p_s) - 1 \geq c > 0$.

Here, $\mathbf{St}_k$ is the Steinberg representation of $\mathrm{Gl}(k, F)$, and $L(\delta_1 \otimes \cdots \otimes \delta_t \otimes \tau)$ is the Langlands quotient form of π, where the induction is from the parabolic subgroup of $\mathrm{Sp}(n, F)$ with Levi factor $\mathrm{Gl}(n_1, F) \times \cdots \times \mathrm{Gl}(n_t, F) \times \mathrm{Sp}(n_0, F)$, $n = n_1 + \cdots + n_t + n_0$, where $\delta_i \in \mathrm{Irr}(\mathrm{Gl}(n_i, F))$ are essentially tempered with

$$e(\delta_1) > \cdots > e(\delta_t) > 0,$$

and $\tau \in \mathrm{Irr}(\mathrm{Sp}(n_0, F))$ is tempered. The following diagram illustrates Theorem 3.

$$\begin{array}{lcl} & & \vdots \\ & \nearrow & \mathrm{Sp}(l/2+1) \\ \mathrm{O}(V),\ \sigma \text{ tempered} & \longrightarrow & \left.\begin{array}{l}\mathrm{Sp}(l/2)\\ \mathrm{Sp}(l/2-1)\\ \vdots \\ \mathrm{Sp}(0)\end{array}\right\} \begin{array}{l}\text{corresponding}\\ \text{representation}\\ \text{is tempered}\end{array} \\ & \searrow & \end{array}$$

Theorem 3 follows by arguments similar to those of [K]. The key point is the inductive nature of the Weil representation. Part (1) of Theorem 3 shows that the last local hypothesis (L3) can be removed in the interesting case $l = 2n$, if v is nonarchimedean. It would be very interesting to determine whether π as in (2) of Theorem 3 actually exist.

3. Poles of local zeta integrals. We again assume that F is a local nonarchimedean field of characteristic zero. In this section we consider the two ingredients that go into the proof of Lemma 1. First, if $\sigma \in \mathrm{Irr}(\mathrm{O}(V(F)))$ is tempered and the first occurrence of σ is tempered, then the Langlands quotient data for lifts of σ further up the Witt tower can be determined.

Theorem 4. [R3] *Suppose that $\sigma \in \mathrm{Irr}(\mathrm{O}(V(F)))$ is pre-unitary. Let n and n' be positive integers such that $2n' \geq 2n \geq l$. Let $\pi \in \mathrm{Irr}(\mathrm{Sp}(n,F))$ and $\pi' \in \mathrm{Irr}(\mathrm{Sp}(n',F))$. If π is tempered and σ corresponds to π and π', then*

$$\pi' = L(\chi|\ |^{n'-l/2} \otimes \chi|\ |^{n'-1-l/2} \otimes \cdots \otimes \chi|\ |^{n+1-l/2} \otimes \pi).$$

Theorem 4 also follows by arguments similar to those of [K]; in addition some results about degenerate prinicipal series representations and the theta correspondence from [KR2] are used. We have the following diagram.

$$\begin{array}{llll}
 & & \vdots & \vdots \\
 & & L(\chi|\ |^{n+2-l/2} \otimes \chi|\ |^{n+1-l/2} \otimes \pi) & \mathrm{Sp}(n+2) \\
 & & L(\chi|\ |^{n+1-l/2} \otimes \pi) & \mathrm{Sp}(n+1) \\
 & & \pi & \mathrm{Sp}(n) \\
 & \nearrow & & \vdots \\
\mathrm{O}(V),\ \sigma & & & \mathrm{Sp}(l/2)
\end{array}$$

Lemma 1 now follows from Theorem 4 and the inductive properties of the zeta integrals [R4].

Applications. Assume $2n = l = 4$. This case is interesting because many of the hypotheses of Theorem 2 are understood, and so Theorem 2 should lead to the construction of many interesting cuspidal automorphic representations of $\mathrm{Sp}(2,\mathbb{A})$ or $\mathrm{GSp}(2,\mathbb{A})$. It turns out to be convenient to work with similitudes instead of isometries [R1]; because of the nature of the theta correspondence for similitudes, Theorem 2 should be sufficient to settle the question of nonvanishing. We expect to be able to characterize the σ for which $\Theta_n(\sigma) \neq 0$ in terms of distinguished representations, as in the local nonarchimedean case [R2].

References

[B-PS] S. Böcherer and R. Schulze-Pillot, *Siegel modular forms and theta series attached to quaternion algebras*, Nagoya Math. J. **121** (1991), 35–96.

[GRS] D. Ginzburg, S. Rallis and D. Soudry, *Periods, poles of L–functions and symplectic-orthogonal theta lifts*, preprint.

[K] S.S. Kudla, *On the local theta correspondence*, Invent. math. **83** (1986), 229–255.
[KR1] S.S. Kudla and S. Rallis, *A regularized Siegel-Weil formula: The first term identity*, Annals of Math. **140** (1994), 1–80.
[KR2] S.S. Kudla and S. Rallis, *Ramified degenerate principal series representations for* $\mathrm{Sp}(n)$, Israel J. of Math. **78** (1992), 209–256.
[L] J.-S. Li, *Non-vanishing theorems for the cohomology of certain arithmetic quotients*, J. Reine Angew. Math. **428** (1992), 177–217.
[R1] B. Roberts, *The theta correspondence for similitudes*, Israel J. of Math. **94** (1996), 285–317.
[R2] B. Roberts, *The nonarchimedean theta correspondence for* $\mathrm{GSp}(2)$ *and* $\mathrm{GO}(4)$, Trans. Amer. Math. Soc. (to appear).
[R3] B. Roberts, *Tempered representations and the theta correspondence*, preprint (submitted for publication).
[R4] B. Roberts, *Poles of local* L*–functions and the theta correspondence*, in preparation.

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