UNITARY REPRESENTATIONS WITH STRONGLY REGULAR INFINITESIMAL CHARACTER

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Outline

1. Abstract

We will sketch the proof of the following result, conjectured by Vogan and Zuckerman:

Theorem. *Let G be a reductive real Lie group. Suppose π is an irreducible Harish-Chandra module of G with strongly regular and real infinitesimal character. Then*

$$\pi \textit{ is unitary} \quad \Longleftrightarrow \quad \pi \cong A_{\mathfrak{q}}(\lambda) \quad \textit{(admissible)}.$$

2. Introduction

Let G be a reductive real Lie group. We would like to understand the set $\Pi_u(G)$ of equivalence classes of irreducible unitary representations of G. Our approach is motivated by the philosophy of the orbit method: there should be a close connection between $\Pi_u(G)$ and the coadjoint orbits of G. Roughly, we should be able to attach an element $\pi \in \Pi_u(G)$ to each coadjoint orbit $\mathcal{O}$ of G . The map $\mathcal{O} \mapsto \pi$ will not be quite a bijection.

For an element X of the complexified Lie algebra $\mathfrak{g}$ of G, let $\mathcal{O}_X$ denote the coadjoint orbit $Ad^*(G)\xi$, where ξ is the element of $\mathfrak{g}^*$ corresponding to X via the Killing form. Decompose X into semisimple and nilpotent parts $X = X_s + X_n$, with X_s further decomposable into hyperbolic and elliptic elements $X_s = X_h + X_e$. Let G_X be the isotropy group of X. Since the elements X_h, X_e, and X_n all commute, we have that

$$G_X = (G_{X_s})_{X_n} \subset G_{X_s} = (G_{X_h})_{X_e} \subset G_{X_h} \subset G,$$

and one can try to attach a representation to the orbit $\mathcal{O}_X$ by going in steps, from smaller to larger groups.

Example. Let $G = GL(2n, \mathbb{R})$. An elliptic orbit $\mathcal{O}_{X_e}$ corresponds to an element $Y \in \mathbb{R}^n$ of the form

$$Y = (\underbrace{a_1, \dots, a_1}_{d_1}, \underbrace{a_2, \dots, a_2}_{d_2}, \dots, \underbrace{a_r, \dots, a_r}_{d_r}, \underbrace{0, \dots, 0}_{m}),$$

with $a_1 > a_2 > \cdots > a_r > 0$ and $m = n - \sum_{i=1}^{r} d_i$. If $i(X_e) \leftrightarrow Y$ as above then

$$G_{X_e} \cong GL(d_1, \mathbb{C}) \times \cdots \times GL(d_r, \mathbb{C}) \times GL(2m, \mathbb{R}).$$

The idea is to construct π in three steps:
(1) Attach to $\mathcal{O}_{X_n}$ a unipotent representation of G_{X_s}: $\mathcal{O}_{X_n} \mapsto \pi_n$. This process is not completely understood.
(2) Assuming we have a representation π_n of G_{X_s}, use cohomological parabolic induction to attach $\mathcal{O}_{X_e} \mapsto \pi_e$, a representation of G_{X_h}, induced from π_n. This method is well understood.
(3) Use π_e and real parabolic induction to attach $\mathcal{O}_{X_h} \mapsto \pi_h$, a representation of G induced from π_e.

For step (2), suppose $X = X_e$. Recall that $ad(iX_e)$ has real eigenvalues. Let $\mathfrak{g} = \sum_{t \in \mathbb{R}} \mathfrak{g}_t$ be the corresponding eigenspace decomposition of $\mathfrak{g}$. Then the 0-eigenspace is $\mathfrak{g}_e = Lie(G_{X_e})$, and the Lie algebra $\mathfrak{q}_e = \sum_{t \geq 0} \mathfrak{g}_t = \mathfrak{g}_e + \mathfrak{u}_e$ (where $\mathfrak{u}_e = \sum_{t>0} \mathfrak{g}_t$) is a complex polarization at X. To obtain a representation of G, construct the Zuckerman derived functor module $\mathcal{R}^S_{\mathfrak{q}_e}(\pi_n)$. As a special case we assume that $\pi_n = \mathbb{C}_\lambda$, a one-dimensional representation of G_{X_e}.

Examples.
(1) $X_e = 0$, $G_{X_e} = G$, and $\pi_n = \pi = \mathbb{C}_\lambda$.
(2) G_{X_e} is compact $\Leftrightarrow$ π is a discrete series representation.

Conversely, a 'backwards' map should involve the following steps:
(1) Given π, find a pair (M, π_M), where M is a real Levi subgroup of G and π_M is a unitary representation of M with real infinitesimal character such that $\pi \cong Ind_M^G(\pi_M)$.
(2) Given (M, π_M), find a pair (L, π_L), where $L = G_L$ is a Levi subgroup for a complex parabolic subalgebra $\mathfrak{q}$ (usually bigger than G_{X_s}), and π_L is a unitary representation of L such that $\pi_M \cong \mathcal{R}_{\mathfrak{q}}(\pi_L)$.
(3) Give a parametrization of all the pairs (L, π_L) in step (2).

We will be concerned here with a special case of step (2).

3. Representations with strongly regular real infinitesimal character

We use the usual notation: K is a maximal compact subgroup of G, $T \subset K$ a maximal compact torus; $\mathfrak{g}_0$ is the real Lie algebra of G, $\mathfrak{g} = (\mathfrak{g}_0)_{\mathbb{C}}$ the complexification, etc. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$, and $\mathfrak{h} = \mathfrak{t} + \mathfrak{a}$ a maximally compact Cartan subalgebra of $\mathfrak{g}$. Denote by $\Delta(\mathfrak{g}, \mathfrak{t})$ the roots of $\mathfrak{g}$ with respect to $\mathfrak{t}$, and define $\Delta(\mathfrak{g}, \mathfrak{h})$, $\Delta(\mathfrak{k}, \mathfrak{t})$, etc. analogously. As usual, fix a non-degenerate G- and

θ-invariant symmetric bilinear form $(,)$ on $\mathfrak{g}_0$, negative definite on $\mathfrak{k}$ and positive definite on $\mathfrak{p}$, and denote its complexification $(,)$ as well.

Let $(\pi, \mathcal{H})$ be an irreducible unitary representation of G, X_π its Harish-Chandra module, and let the infinitesimal character of π be given by the weight $\gamma \in \mathfrak{h}^*$. Fix a positive root system $\Delta^+(\gamma, \mathfrak{h}) \subset \{\alpha \in \Delta(\mathfrak{g}, \mathfrak{h}) : Re(\gamma, \alpha) \geq 0\}$, and let $\rho(\gamma)$ be one half the sum of the roots in $\Delta^+(\mathfrak{g}, \mathfrak{h})$. Recall (see Vogan's talk) that $\gamma \in \mathfrak{h}^*$ is called real if $\gamma \in \mathfrak{h}^*_{\mathbb{R}}$, where $\mathfrak{h}^*_{\mathbb{R}} = i\mathfrak{t}^*_0 + \mathfrak{a}^*_0 \subset \mathfrak{h}^*$.

Definition. The set $\Pi^{sr}_u(G)$ is the set of equivalence classes of irreducible unitarizable Harish-Chandra modules with infinitesimal character γ satisfying
(i) γ is real, and
(ii)$(\gamma - \rho(\gamma), \alpha) \geq 0$ for all positive roots α in Δ^+.
We say that γ is **strongly regular** if (ii) holds.

Examples.
(1) If $G = SL(2, \mathbb{R})$ then $\Pi^{sr}_u(G)$ consists of the discrete series and the trivial representation.
(2) For $G = SU(2,1)$, the set of strongly regular real infinitesimal characters forms a closed cone in the Weyl chamber of ρ, with vertex ρ.

4. Admissible $A_{\mathfrak{q}}(\lambda)$ modules

We now describe the modules which will parametrize the strongly regular set. Fix a positive system $\Delta^+(\mathfrak{k}, \mathfrak{t})$ and a weight $\xi \in i\mathfrak{t}^*_0$, and let $\mathfrak{q}(\xi)$ be the θ-stable parabolic subalgebra of $\mathfrak{g}$ defined by ξ, i.e.,

$$\mathfrak{q}(\xi) = \mathfrak{g}(\xi) + \mathfrak{u}(\xi) \quad \text{with}$$
$$\mathfrak{g}(\xi) = \mathfrak{h} + \sum_{(\alpha,\xi)=0} \mathfrak{g}_\alpha \quad \text{and}$$
$$\mathfrak{u}(\xi) = \sum_{(\alpha,\xi)>0} \mathfrak{g}_\alpha.$$

Example. If $G = GL(2n, \mathbb{R})$ and $\xi \leftrightarrow Y \in \mathbb{R}^n$ as in the introduction, with

$$Y = (\underbrace{a_1, \ldots, a_1}_{d_1}, \underbrace{a_2, \ldots, a_2}_{d_2}, \ldots, \underbrace{a_r, \ldots, a_r}_{d_r}, \underbrace{0, \ldots, 0}_{m}),$$

then

$$\mathfrak{g}(\xi) \cong \mathfrak{gl}(d_1, \mathbb{C}) \times \cdots \times \mathfrak{gl}(d_r, \mathbb{C}) \times \mathfrak{gl}(2m, \mathbb{R}).$$

If $G(\xi)$ is the Levi subgroup of G corresponding to $\mathfrak{g}(\xi)$, let $\lambda : \mathfrak{g}(\xi) \longrightarrow \mathbb{C}_\lambda$ be a linear functional.

Definition. λ and $\mathfrak{q}(\xi)$ are **admissible** if
(i) ξ is dominant for $\Delta^+(\mathfrak{k}, \mathfrak{t})$,
(ii) λ is the differential of a unitary character of $G(\xi)$, and
(iii) $(\alpha, \lambda \restriction_{\mathfrak{t}}) \geq 0$ for all $\alpha \in \Delta(\mathfrak{u}(\xi), \mathfrak{t})$.

Given such a pair $(\mathfrak{g}(\xi), \lambda)$, construct the Zuckerman module

$$A_{\mathfrak{q}(\xi)}(\lambda) = \mathcal{R}^S_{\mathfrak{q}(\xi)}(\mathbb{C}_\lambda)$$

with $S = dim(\mathfrak{u}(\xi) \cap \mathfrak{k})$ and so that this module has infinitesimal character $\lambda + \rho_{\mathfrak{q}(\xi)}$.

Examples. Let $\gamma = \rho$ and $\lambda = 0$, and write $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$.
(1) If $\mathfrak{l} = \mathfrak{q} = \mathfrak{g}$ then $A_\mathfrak{q}$ is the trivial representation of G.
(2) We have that $\mathfrak{l} \subset \mathfrak{k} \Leftrightarrow A_\mathfrak{q}(0)$ is a discrete series representation.

Theorem. *(Vogan, Speh) Let $\mathfrak{q} = \mathfrak{l} + \mathfrak{u} \subset \mathfrak{g}$ be a θ-stable subalgebra, and suppose $\lambda : \mathfrak{l} \longrightarrow \mathbb{C}$ is admissible. Then $A_\mathfrak{q}(\lambda)$ is irreducible, nonzero, and unitarizable.*

The proof of irreducibility and nonvanishing is due to Speh and Vogan, unitarizability was proved by Vogan.

5. Classification theorem and proof

Now we sketch the proof of our *Main Classification Theorem:*

Theorem A.

$$\Pi_u^{sr}(G) \cong \{admissible\ A_\mathfrak{q}(\lambda)\ modules\}.$$

Because of the results of Speh and Vogan, this theorem will follow from

Theorem B. *(SR) If $\pi \in \Pi_u^{sr}(G)$ then π is isomorphic to an $A_\mathfrak{q}(\lambda)$ module.*

Proof. (outline) We use the following two facts:

Fact 1. *(Zuckerman) Every $A_\mathfrak{q}(\lambda)$ module has a unique lowest K-type (LKT) of highest weight*

$$\mu(\mathfrak{q}, \lambda) = \lambda \restriction_\mathfrak{t} + 2\rho(\mathfrak{u} \cap \mathfrak{p}).$$

Examples. Let $G = SU(1,1)$ and $\gamma = \rho$.
(a) If $\mathfrak{q} = \mathfrak{l} = \mathfrak{g}$ then $\mu(\mathfrak{q}, 0) = (0,0)$ (trivial representation).
(b) If $\mathfrak{q} = \mathfrak{t} + \mathfrak{n}$ then $\mu(\mathfrak{q}, 0) = (1,-1)$ (holomorphic discrete series).
(c) If $\bar{\mathfrak{q}} = \mathfrak{t} + \bar{\mathfrak{n}}$ then $\mu(\bar{\mathfrak{q}}, 0) = (-1,1)$ (antiholomorphic discrete series).

Fact 2. *(Vogan, Zuckerman) If $X \in \Pi_u^{sr}(G)$, has LKT $\mu = \mu(\mathfrak{q}, \lambda)$ for λ and $\mathfrak{q}$ admissible, and if $\lambda = 0$ on $[\mathfrak{l}, \mathfrak{l}]$, then $X \cong A_\mathfrak{q}(\lambda)$.*

Using these two results, Theorem B follows from the proof of the following

Proposition C. *If X has strongly regular real infinitesimal character and LKT μ, then either*
(a) X is not unitarizable, or
(b) there are admissible λ and $\mathfrak{q}$ with $\mu = \mu(\mathfrak{q}, \lambda)$, in which case $X \cong A_\mathfrak{q}(\lambda)$.

In order to find suitable $\mathfrak{q}$ and λ for a given μ, we define the following unitary data. (This machinery was developed with David Vogan.) Given a dominant weight $\mu \in i\mathfrak{t}_0^*$, set $2\rho_c = 2\rho(\mathfrak{k})$ and choose a set of positive roots $\Delta^+(\mathfrak{g}, \mathfrak{t})$, with respect to which $\mu + 2\rho_c$ is dominant. Set $2\rho = \sum_{\alpha \in \Delta^+(\mathfrak{g},\mathfrak{t})} \alpha$, and

$$\lambda_u = P(\mu + 2\rho_c - 2\rho) \in i\mathfrak{t}_0^*,$$

where P is the projection onto the convex cone

$$C = \{\chi \in i\mathfrak{t}_0^* : (\chi, \alpha) \geq 0,\ \text{for all positive } \alpha\}.$$

Now let $\mathfrak{q}_u = \mathfrak{q}_u(\lambda_u) = \mathfrak{g}_u + \mathfrak{u}_u$, and $G_u = G(\lambda_u)$.

To motivate this definition, recall that the highest weight of the LKT of a discrete series X with Harish-Chandra parameter λ is given by the Blattner parameter $\mu = \lambda + \rho - \rho_c$. We can recover λ by finding a suitable ρ and solving for λ:

$$\lambda = \mu + 2\rho_c - \rho.$$

For his classification of admissible representations, Vogan used this idea to attach to the highest weight μ of a LKT of an admissible Harish-Chandra module the dominant weight $\lambda_a(\mu)$, which (using an observation of Carmona) may be computed as follows:

$$\lambda_a(\mu) = P(\mu + 2\rho_c - \rho).$$

(The projection is necessary because for the LKT of an arbitrary admissible Harish-Chandra module, the weight $\mu + 2\rho_c - \rho$ is not always dominant.)

Example. Let $G = Sp(4,\mathbb{R})$, $\mu = (a,2)$, $a > 0$. Then $\mu + 2\rho_c - 2\rho = (a-3,-1)$.
If $a = 3$ then $\lambda_u = (0,0)$, and $G_u = G$.
If $a > 3$ then $\lambda_u = (a-3,0)$, and $G_u \cong U(1) \times Sp(2,\mathbb{R})$.

This leads to the following

Definition. The weight $\mu \in i\mathfrak{t}_0^*$ is **unitarily small** if $G_u = G$. A K-type is unitarily small if its highest weight is.

Examples.
1. The trivial K-type is unitarily small.
2. For $G = SL(2,\mathbb{R})$, the unitarily small weights are $0, \pm 1, \pm 2$. Notice that this set contains the LKT's for the first two discrete series representations of G.
3. For $G = Sp(4,\mathbb{R})$, the unitarily small K-types are those with highest weights of the form (a,b) with $3 \geq a \geq b \geq -3$ and $a - b \leq 4$.

Now we prove the proposition for unitarily small K-types.

Lemma. *(Vogan, SR) The K-type with highest weight μ is unitarily small if and only if μ is of the form*

$$\mu = \lambda_u + \sum_{\beta \in \Delta(\mathfrak{p},\mathfrak{t})} c_\beta \beta$$

with $0 \leq c_\beta \leq 1$.

Example. For $G = Sp(4,\mathbb{R})$, $(3,2) = 2e_1 + (e_1 + e_2) + e_2$.

Lemma. *(Kumaresan, Vogan, Zuckerman) Assume that $\pi \in \Pi_u^{sr}(G)$ has a LKT of the form*

$$\chi = \sum_{\beta \in \Delta(\mathfrak{p},\mathfrak{t})} c_\beta \beta$$

with $0 \leq c_\beta \leq 1$. Then there is a parabolic subalgebra $\mathfrak{q}' = \mathfrak{l}' + \mathfrak{u}' \subset \mathfrak{g}$ such that $\chi = 2\rho(\mathfrak{u} \cap \mathfrak{p})$.

Since for μ unitarily small we have that $\lambda_u = 0$ on $[\mathfrak{g},\mathfrak{g}] \supset [\mathfrak{l}',\mathfrak{l}']$, the conditions of Fact 2 are satisfied, and the proposition is proved for this case. For general μ, we prove the following

Proposition. *(Vogan, SR) For any weight* $\mu \in i\mathfrak{t}_0^*$*, there is a positive set* $(\Delta^+)(\mathfrak{g}_u)$ *such that* μ *is of the form*

$$\mu = \lambda_u + 2\rho(\mathfrak{u}_u \cap \mathfrak{p}) + \sum_{\beta \in (\Delta^+)(\mathfrak{g}_u \cap \mathfrak{p})} c_\beta \beta$$

with $0 \leq c_\beta \leq 1$.

The result now follows from the following proposition, which we prove by extending Kumaresan's lemma to this situation.

Proposition. *If* X *is hermitian, with strongly regular infinitesimal character and LKT* μ*, then either* X *is not unitary or there is a parabolic subalgebra* $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ *such that*

$$\mu = \lambda_u + 2\rho(\mathfrak{u} \cap \mathfrak{p}).$$

6. Motivation (for our choice of G_u)

The strongly regular case is a good easy example to illustrate why λ_u might be the right parameter to choose for the classification of the unitary dual of G. Here is one of the reasons for this choice.

We follow an idea of Barbasch and Vogan: given a weight μ which is dominant for $\mathfrak{k}$, we try to find a Levi subgroup $L \subset G$ and a representation of $L \cap K$ with highest weight μ^L such that

$$\{X \in \Pi_u^{rs}(G) \text{ with LKT } \mu\} \leftrightarrow \left\{X^L \in \Pi_u^{rs}(L) \text{ with LKT } \mu^L\right\},$$

and if X^L is not unitary then the signature change survives in the representation of G under the bottom layer map (defined below).

A K-type μ does not determine $\mathfrak{q}$ or λ uniquely. Recall that for an admissible $A_\mathfrak{q}(\lambda)$, we have conditions of the form $(\lambda, \alpha) \geq 0$ on $\mathfrak{u}$ and $\lambda \perp \Delta(\mathfrak{l})$. To achieve uniqueness, we try to find a maximal algebra $\mathfrak{q}$ by adding to $\mathfrak{l}$ root spaces associated to roots α with $(\lambda, \alpha) = 0$. We know that under the given regularity conditions, we have

Lemma. *(Vogan)*

$$A_\mathfrak{q}(\lambda) \cong A_{\mathfrak{q}'}(\lambda') \quad \Leftrightarrow \quad \mathfrak{u} \cap \mathfrak{p} = \mathfrak{u}' \cap \mathfrak{p} \quad \textit{and} \quad \lambda = \lambda' \textit{ on } L \cap L'.$$

Using this lemma and the conditions for admissibility on λ and $\mathfrak{q}$, we set

$$\mu' = \mu + \sum_{\substack{(\alpha, \mu) > 0 \\ \alpha \in \Delta(\mathfrak{k}, \mathfrak{t})}} \alpha \qquad \text{and} \qquad \mathfrak{q}_{VZ}(\mu) = \mathfrak{q}(\mu').$$

We claim that μ is the LKT of an $A_\mathfrak{q}(\lambda)$ module if and only if

$$(*) \qquad (\mu', \alpha) \geq (2\rho(\mathfrak{u}_{VZ}), \alpha) \quad \text{for all} \quad \alpha \in \Delta(\mathfrak{u}_{VZ}).$$

In that case set $\lambda = \mu' - 2\rho(\mathfrak{u}_{VZ})$. Then $\mu = \mu(\mathfrak{q}_{VZ}, \lambda)$. (In fact, μ an $A_\mathfrak{q}(\lambda)$ LKT *implies* that $\mu' = \lambda + 2\rho(\mathfrak{u}_{VZ})$.)

Example. Let $G = Sp(4,\mathbb{R})$ and $\mu = (a,2)$ with $a > 2$. Then $\mu' = (a+1,1)$, and $\Delta(\mathfrak{u}_{VZ}) = \{2e_1, e_1+e_2, e_1-e_2, 2e_2\}$. We have $(\mu', 2e_2) = 2 < 4 = (2\rho(\mathfrak{u}_{VZ}), 2e_2)$, and μ is *not* an $A_{\mathfrak{q}}(\lambda)$ LKT.

The idea is to use the above claim to prove that such a representation is not unitary: if $(*)$ does not hold for a given K-type μ, set

$$\mathfrak{g}_r(\mu) = \mathfrak{l}_{VZ} + \sum_{\substack{\alpha \in \mathfrak{u}_{VZ} \\ (\mu',\alpha) \leq (\mu', 2\rho(\mathfrak{u}_{VZ}))}} \mathfrak{g}_{\pm\alpha} \supseteq \mathfrak{l}_{VZ},$$

$$\mathfrak{u}_r(\mu) = \sum_{\substack{\alpha \in \mathfrak{u}_{VZ} \\ (\mu',\alpha) > (\mu', 2\rho(\mathfrak{u}_{VZ}))}} \mathfrak{g}_{\alpha} \subseteq \mathfrak{u}_{VZ},$$

and

$$\mathfrak{q}_r = \mathfrak{g}_r + \mathfrak{u}_r.$$

Then show that an (irreducible) representation X with such a K-type can be imbedded into a Zuckerman module induced from a hermitian Harish-Chandra module of the group $G_r(\mu)$. Do this so that there is a correspondence (via the bottom layer map) between the LKT's of the irreducible module X and the LKT's of the inducing representation of $G_r(\mu)$.

Explicitly, the bottom layer map is

$$\mathcal{R}^K_{\mathfrak{q}}\left((X^L)_{L\cap K}\right) : \mu^L \longrightarrow \begin{cases} \mu^L \otimes \Lambda^{top}\mathfrak{u}\cap\mathfrak{p} & \text{if this is } \mathfrak{k} \text{ dominant,} \\ 0 & \text{otherwise.} \end{cases}$$

The group L in this case will be $G_r(\mu)$.

Then, using Parthasarathy's Dirac operator inequality (see Vogan's talk), prove nonunitarity for the representation of $G_r(\mu)$ by exhibiting two lowest K-types with the property that the form is indefinite on their sum. Then verify that this indefiniteness prevails when we take the corresponding K-types in X, via the bottom layer map.

In the example above, $\mathfrak{g}_r(\mu) \cong \mathfrak{t} + \mathfrak{sp}(2,\mathbb{C})$. In that case, the LKT for the representation of $G_r(\mu)$ corresponding to the K-type $(a,2)$ is $(a-3,1)$. The number 1 on the $\mathfrak{sp}(2)$ factor helps us use Parthasarathy's Dirac operator inequality to show that the hermitian form on the representation of $G_r(\mu)$ is indefinite on the sum of the K-types with highest weights $(a-3,1)$ and one of the form $(a-3,1)\pm 2e_2$. In other words, the K-types we need to consider for the possible change of signature of the hermitian form are $(a-3,3)$, $(a-3,1)$, and $(a-3,-1)$.

Now, via the bottom layer map, these weights correspond to the weights $(a,4)$, $(a,2)$, and $(a,0)$. However, the weight $(a,4)$ is not even dominant if $a=3$. Therefore, we need to find a bigger parabolic subalgebra attached to the LKT of the representation. Note that if $a=3$, then $\mathfrak{q}_u(\mu) = \mathfrak{g}$, and there is no reduction in this case.