LIFTING AUTOMORPHIC FORMS FROM G_2 TO PGSp(6) USING THE MINIMAL REPRESENTATION OF E_7

GORDAN SAVIN

1. Introduction

This talk concerns the construction of certain automorphic forms. It contains joint work with B. Gross.

Let us begin with some questions.

Question: Give a cusp form on GL_2.
Answer: Ramanujan's Δ function.

Question: Give a cusp form on GL_3.
Answer: The Gelbart-Jacquet lift of Δ from GL_2

Question: Give a cusp form on GL_n.
Best Approaches: Jacquet-Langlands, base change, etc.

The moral of the story is that, although they are known to exist, it is not easy to construct cusp forms explicitly. However, once we have a cusp form on one group, it is sometimes possible to construct cusp forms on other groups if we have some explicit version of Langlands' functoriality.

2. A canonical form on G_2

First we construct the global form of G_2 that we will use. Let $\mathbb{O}$ be the octonion algebra of Cayley numbers. Coxeter has shown that $\mathbb{O}$ may be defined over $\mathbb{Z}$. Hence, we can get an algebra over any ring R by setting $\mathbb{O}(R) = \mathbb{O} \otimes_{\mathbb{Z}} R$. Set $G = \mathrm{Aut}\,\mathbb{O}$. This is an algebraic group defined over $\mathbb{Z}$. As is well known,

- $G(\mathbb{R})$ is the compact real form of G_2, and
- for finite p, $G(\mathbb{Q}_p)$ is split G_2. (Indeed, this is the only form of G_2 over a p-adic field.)

Note that, since G is compact at $\mathbb{R}$, $G(\mathbb{Q})\backslash G(\mathbb{A})$ is compact.

B. Gross has proved the following facts about G.

Date: Notes by Karl E. Rumelhart.

1. $G(\mathbb{Z}) \cong G_2(2)$. This notation refers to the Chevalley group over $\mathbb{F}_2$. (Note that it is obvious that $G(\mathbb{Z})$ is finite because it is a discrete subgroup of the compact group $G(\mathbb{R})$.)
2. Let $K_p = G(\mathbb{Z}_p)$ be the hyperspecial maximal compact subgroup of $G(\mathbb{Q}_p)$, and let $K_p(1)$ be the first congruence subgroup. We have $K_p/K_p(1) \cong G_2(p)$. Define

$$K = \prod_p K_p, \quad \text{and} \quad K' = K_2(1) \prod_{p>2} K_p.$$

Then it is true that

$$G(\mathbb{A}) = G(\mathbb{Q}) \times G(\mathbb{R}) \times K'.$$

3. Let W be the irreducible representation of $G(\mathbb{R})$ with highest weight ρ. Note that, as G_2 has six positive roots, the dimension of W is $2^6 = 64$. Then the following is true

$$W|_{G(\mathbb{Z})\cong G_2(2)} = \mathrm{st}.$$

Here st is the Steinberg reresentation of the finite Chevalley group $G_2(2)$.

Now we construct an automorphic form on G. It should be viewed as an analogue of Ramanujan's Δ. (Later we will lift is to PGSp_6 using the minimal representation of E_7.)

Proposition 1. *There exists a unique automorphic representation* $\pi = \bigotimes_v \pi_v \subseteq L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$ *which satisfies*

1. $\pi_\infty = W$,
2. $\pi_2 = St$, *the p-adic Steinberg representation, and*
3. *for* $p > 2$, π_p *is spherical.*

Proof. Any such π has K' invariants. Furthermore, using Gross' decomposition of $G(\mathbb{A})$ and the Peter-Weyl theorem,

$$\begin{aligned} L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))^{K'} &= L^2(G(\mathbb{Q})\backslash G(\mathbb{A})/K') \\ &= L^2(G(\mathbb{R})) \\ &= \hat{\bigoplus}_{E\in \check{G}(\mathbb{R})} E \otimes E. \end{aligned}$$

This is a representation of $K/K' \times G(\mathbb{R}) \cong G_2(2) \times G(\mathbb{R})$, and the part corresponding to π must be $W \otimes W$. Restricting from $G(\mathbb{R})$ to $G(\mathbb{Z})$ gives a representation of $G_2(2) \cong G(\mathbb{Z})$ which is st. Since the Steinberg representation of $G_2(\mathbb{Q}_2)$ is the only irreducible representation with minimal K-type st, the proposition follows. $\square$

3. Lifting to PGSp_6

Set $G' = \mathrm{PGSp}_6$. We have the following inclusion of dual groups.

$$\hat{G} = G_2(\mathbb{C}) \overset{i}{\hookrightarrow} \hat{G}' = \mathrm{Spin}_7(\mathbb{C}).$$

By the Langlands philosophy, there should be a corresponding lift of automorphic forms. Of course, understanding this is a difficult problem and, even when we have good local control, it is not clear how to realize the lift automorphically. Our idea is use dual pairs.

Let H be the group of adjoint type over $\mathbb{Q}$ with the following properties: $H^0(\mathbb{R}) = E_{7,3}$ (this is the group that acts on the exceptional tube domain), and $H(\mathbb{Q}_p)$ is split. H has been studied by Satake. Its restricted root system over $\mathbb{Q}$ is of type C_3. The long root spaces are 1-dimensional and the short root spaces are 8-dimensional. In fact, the short root spaces can be identified with $\mathbb{O}$. Morally, one can think of H as $\mathrm{PGSp}_6(\mathbb{O})$. It is now easy to see how $G \times G'$ embeds as a dual pair in H. Since $\mathbb{Q} \hookrightarrow \mathbb{O}(\mathbb{Q})$, $G' = \mathrm{PGSp}_6(\mathbb{Q}) \hookrightarrow \mathrm{PGSp}_6(\mathbb{O}) = H$. Moreover, since $\mathbb{Q} = \mathbb{O}^G$, it is clear that the centralizer of G is isomorphic to G'.

Let Π_v be the minimal representation of H_v. It is a result of Kim that

$$\bigotimes_v \Pi_v \subseteq L^2(H(\mathbb{Q})\backslash H(\mathbb{A})).$$

This representation should be viewed as an analogue of the Weil representation. In particular, it can be used to provide explicit liftings.

We turn to results on correspondences. There are two types of local results (real and p-adic) and global results.

Real. Since $G(\mathbb{R})$ is compact, we can write

$$\Pi_\infty = \bigoplus_{E \in \check{G}(\mathbb{R})} E \otimes \Theta(E).$$

It can be shown that $\Theta(E)$ is always holomorphic discrete series and that the correspondence $E \mapsto \Theta(E)$ is a Langlands' lift.

p-adic. For π a representation of $G(\mathbb{Q}_p)$, define $\Theta(\pi) = \{\pi' | \pi \otimes \pi' \text{ is a quotient of } \Pi_p\}$. Also, we will write π_S for the spherical representation of G with Satake parameter S. Similarly, $\pi_{S'}$ is the spherical representation of G' with Satake parameter S'. Then, with Magaard, we have obtained the following results.

1. If π_S is tempered, then $\Theta(\pi_S) = \{\pi_{i(S)}\}$. (Recall that $i\colon \hat{G} \hookrightarrow \hat{G}'$.)
2. If $\pi_{S'}$ is a quotient of Π_p, then $S' = i(S)$ for some S.
3. $\Theta(\mathrm{St}) = \{\mathrm{St}'\}$. Here $\mathrm{St}, \mathrm{St}'$ are the Steinberg representations of G, G' respectively. Note that this result is predicted by the local Langlands correspondence since the inclusion i takes regular nilpotent elements to regular nilpotent elements.

Global. (These results are motivated by work of Ginzburg, Rallis and Soudry.) For π any form on G, define $\Theta(\pi)$ by the usual theta integral. There are two types of results about $\Theta(\pi)$: cuspidality and non-vanishing.

CUSPIDALITY. *$\Theta(\pi)$ is cuspidal if at least one p-adic component is generic, that is, has a Whittaker model.* Note that St is generic.

NON-VANISHING. Let $P = MN$ be the Siegel parabolic of G'. We have $M \cong \mathrm{GL}_3$ and $N \cong 3 \times 3$ symmetric matrices. The action of M on N is given by $gA = (\det g)^{-1} g A g^T$. Our idea for proving non-vanishing of the theta lift is to study Fourier coefficeints along N. To show that a lift is non-zero, it is enough to show that one of its Fourier coefficients is non-zero. We will do this by relating it to a certain period functional on G. To define the period we will need the following result on M-orbits in N.

Proposition 2. *M orbits of rank three elements in N are parametrized by quaternion algebras over $\mathbb{Q}$.*

Proof. A matrix A in N defines a quadratic form by $q(x) = x^T A x$. Thus, with A we can associate the even Clifford algebra $D = \mathrm{Cliff}^+(q)$. By the general formula, $\dim D = 2^{3-1} = 4$. $\square$

Fix a non-trivial additive character $\Psi\colon \mathbb{A}/\mathbb{Q} \to \mathbb{C}^*$. For each $A \in N(\mathbb{Q})$ set $\Psi_A(X) = \Psi(\mathrm{trace}(AX))$. Now, for σ' any automorphic representation on G', we can associate the "Ath Fourier coefficient", $\lambda'_A\colon \sigma' \to \mathbb{C}$ given by

$$\lambda'_A(f') = \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} f'(X)\psi_A(X)dX.$$

On the other hand, by the proposition there is a quaternion algebra, D, corresponding to A. Assume that D is definite at ∞ so that $D \hookrightarrow \mathbb{O}$. Let C be the centralizer of D in G. C is isomorphic to the norm one elements in D. Now if π is any automorphic representation of G, we can define a period functional $\lambda_D\colon \pi \to \mathbb{C}$ by

$$\lambda_D(f) = \int_{C(\mathbb{Q})\backslash C(\mathbb{A})} f(x)dx.$$

We have the following result.

Theorem 1. *Let π be an automorphic representation of G. Then the functional $\lambda_D \neq 0$ on π if and only if the functional $\lambda'_A \neq 0$ on $\Theta(\pi)$.*

Using the theorem, we can prove that the canonical form on G constructed above has a non-zero lift. Indeed, just take A to be the 3×3 identity matrix. The corresponding quaternion algebra is Hamilton's quaternions. It is easy to see that in this case the period functional is non-zero. Thus, the lift has a non-zero Fourier coefficient. In particular, the lift does not vanish.

We summarize our results in the following corollary.

Corollary 1. *There exists a cusp form π' on G' such that*

1. *π'_∞ is holomorphic discrete series with infinitesimal character $2\rho'$.*
2. *$\pi'_2 = St'$.*
3. *For $p > 2$, π'_p is spherical with the Satake parameter coming from G_2.*