# Minimal Representations: Definition and Properties over Finite and Local Fields

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Let $G$ be a Chevalley group over a field $F$ which is simply laced i.e. its root system is of type $A$, $D$ or $E$. Denote by $B$ a fixed Borel subgroup and $W$ the associated Weyl group.

First consider when $F$ is a finite field with $q$ elements. Let 1 be the trivial representation of $B$. We then have the following isomorphisms:

$$\text{End}\ \text{Ind}_B^G(1) \cong \mathbb{C}[B\backslash G/B] \cong \mathbb{C}[W].$$

Associated with any character $\chi$ of $W$, there is a representation $\pi_\chi$ of $G$ which is invariant on $B$. This association is a one-to-one correspondence.

The following are some examples of this correspondence.

1) If $\chi$ is the trivial character of $W$, then $\pi_\chi$ is the trivial representation of $G$.

2) Let $\chi$ is the sign on $W$, i.e., $\chi(w) = \pm 1$ depending on whether $w$ can be written as a product of even numbered simple transpositions. Then $\pi_\chi$ is the Steinberg representation on $G$.

3) Let $\chi$ be the reflection representation of $W$ on the root space. Then $\pi_\chi$ is the smallest non-trivial representation of $G$. More precisely, let $\mathfrak{O}_{\min}$ be the smallest nontrivial nilpotent $G$-orbit. Then

$$\dim\Pi = q^{\frac{1}{2}\dim(\mathfrak{O}_{\min})} + \cdots .$$

We shall denote $\pi_\chi$ by $\Pi$ and call it the minimal representation.

Now we discuss the $p$-adic case. Let $I$ be the Iwahori subgroup and $\mathcal{H} = C_c^\infty(I\backslash G/I)$ be the Hecke algebra. In this case, $\Pi$ again corresponds to the reflection representation of $\mathcal{H}$ (see Lusztig's paper in Trans. AMS).

In what sense is $\Pi$ minimal? Let $(\pi, V)$ be any irreducible representation of the $p$-adic group $G$. Denote by $K_n$ the $n$-th principal congruence subgroup. Then from the germ expansion of the character of $(\pi, V)$:

$$\Theta_\pi = \sum_{\mathfrak{O}:\ \text{nilpotent orbits}} c_{\mathfrak{O}} \hat{\mu}_{\mathfrak{O}},$$

one obtains that (assuming the above expansion is normalized)

$$\dim V^{K_n} = cq^D + \cdots (\text{lower order terms}),$$

where

$$D = \frac{1}{2} \max_{c_{\mathfrak{O}} \neq 0} \dim \mathfrak{O}.$$

(Note that Mœglin and Waldspurger have shown that $c_{\mathfrak{O}}$ of leading orbits in the expansion are positive integers.) We have the following theorem:

**Theorem 1** (Kazhdan-Savin) *If* $\pi = \Pi$, *then* $D = \frac{1}{2}\dim \mathfrak{O}_{\min}$ *and* $c_{\mathfrak{O}_{\min}} = 1$.

In particular, we see that the growth of $K$-types for $\Pi$ is the smallest possible, justifying the name minimal.

We now present an example, over a finite field, how smallness of $\Pi$ can be used to obtain results on dual pair correspondences. Assume that $G = GL_{2n}$ over a finite field $F$ of $q$ elements. Let $P = MN$ be the Siegel parabolic subgroup so that $M \cong GL_n \times GL_n$, $N \cong M_n$.

Consider the restriction of $\Pi$ to $P$ (and $N$). Fix an additive character $\psi : F \to \mathbb{C}^\times$. Then we can obtain the additive characters of $N$ as follows:

$$\psi_A(x) = \psi(\mathrm{Tr}(Ax)), \quad A \in M_n(F).$$

Remark that these are all the irreducible representations of $N$. Let $\pi$ be any representation of $GL_{2n}$. Then

$$\pi|_N = \oplus \pi_A, \quad \pi_A = \{v \in \pi : \pi(x)v = \pi_A(x)v\}.$$

Note that $M$ permutes the summands. The $M$-orbits are parameterized by the rank of $A$. Let $\omega_k$ be the set of all $n \times n$ matrices of rank $k$, $k = 1, \cdots, n$. Then

$$|\omega_0| = 1,\ |\omega_1| = q^{2n-1} + \cdots,\ \cdots,\ |\omega_n| = q^{n^2} + \cdots.$$

If $\pi_A \neq 0$ for some $A$ in $\omega_k$, then one can estimate

$$\dim\pi \geq |\omega_k| \dim\pi_A.$$

Applying the above argument to the minimal representation $\Pi$ we see that

$$\Pi|_N = \Pi^N \oplus \bigoplus_{A\in\omega_1} \Pi_A, \ \dim\Pi_A = 1, \ \text{if } \mathrm{rank}A = 1.$$

(We can do a similar thing in the $p$-adic case using the work of Mœglin and Waldspurger.) We summarize:

**Theorem 2** *We have*

$$\Pi|_P = \Pi^N \oplus \mathbb{C}[\omega_1],$$

*where the action of $P = MN$ on $f \in \mathbb{C}[\omega_1]$ is given by*

$$\begin{cases} \Pi(x) f(A) = \psi_A(x) f(A), & x \in N; \\ \Pi(m) f(A) = f(mA), & m \in M. \end{cases}$$

Next, we define a dual pair, $GL_2 \times GL_n$, in $GL_{2n}$ by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \longmapsto \begin{bmatrix} aI_n & bI_n \\ cI_n & dI_n \end{bmatrix}, \quad \text{for} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL_2;$$
$$g \longmapsto \begin{bmatrix} g & 0 \\ 0 & g \end{bmatrix}, \quad \text{for} \quad g \in GL_n.$$

We consider the restriction

$$\Pi|_{GL_2\times GL_n} = \oplus\left(\pi \otimes \theta(\pi)\right),$$

where $\pi$ is over all irreducible representations of $GL_2$ ($\theta$ could be 0 or reducible). This can be further analyzed. But we turn to an application to symmetric spaces.

Let $U = GL_2 \cap N$ which is isomorphic to $F$. Abusing notation, we define $\psi(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}) = \psi(u)$ which is a character of $U$. Put

$$\Pi^{U,\psi} = \{v \in \Pi \mid \Pi(u)v = \psi(u)v\}.$$

Using the above restriction of $\Pi$ to $GL_2 \times GL_n$, we see that

$$\Pi|^{U,\psi}_{GL_2\times GL_n} = \oplus\pi^{U,\psi} \otimes \theta(\pi),$$

and since $\dim \pi^{U,\psi} \leq 1$,

$$\Pi^{U,\psi} = \oplus\theta(\pi),$$

where $\pi$ runs over all generic representations of $GL_2$. On the other hand, by the previous theorem,

$$\Pi^{U,\psi} = \left(\pi^N\right)^{U,\psi} \oplus \mathbb{C}[\omega_1]^{U,\psi}.$$

Of course, the first summand is 0 by the previous theorem. The second summand is equal to $\mathbb{C}[\Omega]$ where

$$\Omega = \{A \in \omega_1 |\ \psi_A|_U = \psi\} = \{A \in \omega_1 | \mathrm{Tr} A = 1\},$$

which is a rank-one symmetric space $GL_n/GL_{n-1} \times GL_1$. Finally, we conclude

**Proposition 1**

$$\mathbb{C}[GL_n/GL_{n-1} \times GL_1] = \oplus\theta(\pi),$$

*where $\pi$ runs over the generic representations of $GL_2$.*

In particular we see that the decomposition of $\mathbb{C}[GL_n/GL_{n-1} \times GL_1]$ is a consequence of the dual pair correspondence. All rank-one symmetric spaces can be treated in this way. We can then give the following list. ($A_1$ means the group of type $A_1$, etc.)

$$G \supseteq A_1 \times H \qquad \text{dual pair.}$$
$$\Pi^{U,\psi} = \mathbb{C}[H/K] = \oplus\theta(\pi).$$

| $G$ | $H$ | $K$ |
|---|---|---|
| $GL_{2n}$ | $GL_n$ | $GL_{n-1} \times GL_1$ |
| $Sp_{2n}$ | $SO_n$ | $SO_{n-1}$ |
| $D_{2n}$ | $Sp_{2n}$ | $Sp_{2n} \times Sp_{2n-2}$ |
| $E_7$ | $F_4$ | $B_4$ |
| $SO_{2n}$ | $SO_{2n-6}$ | $SO_{2n-4}$ |

For the second pair, see Howe's paper in Journal of Functional Analysis for details.

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