ON TRANSFER FACTORS

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Our goal is to describe the setting for a theory of general (twisted) endoscopy for reductive groups. Let G be a reductive group over a global field F, θ an automorphism of G, and ω a quasi-character on $G(\mathbb{A}_F)$ trivial on $G(F)$. Endoscopy for (G,θ,ω) concerns the representations π of $G(\mathbb{A}_F)$ such that $\pi\circ\theta$ is equivalent to $\omega\otimes\pi$. Then for each place v of F, we have $\pi_v\circ\theta_v\simeq\omega_v\otimes\pi_v$. In lines with Langlands' conjectures, these representations should appear in Langlands or Arthur packets Π such that $\Pi\circ\theta=\omega\otimes\Pi$. An approach to study these representations is by using (θ,ω)-twisted invariant harmonic analysis (Arthur-Selberg trace formula, (θ,ω)-twisted characters, (θ,ω)-twisted orbital integrals...).

¿From now on, we concentrate on the local setting: let G be a connected algebraic group defined over a local field F of characteristic 0, θ an automorphism of $G(F)$ and ω a quasi-character on $G(F)$.

If π is an irreducible representation of $G(F)$ such that $\pi\circ\theta\simeq\omega\otimes\pi$, then π should be the 'transfer' to $G(F)$ of a stable representation on some endoscopic group $H(F)$.

Remarks: • Things can't quite work this way in general. For example, the naturally defined endoscopic group H which appears in endoscopic data (see definition below) may have to be replaced by a z-extension H_1 of H (that is, a central extension by a torus, having some properties, see [K]). This is due to the lack of a suitable embedding of the L-group ${}^LH$ of H in the L-group ${}^LG$ of G. Nevertheless, the representation theory of $H_1(F)$ is essentially the same as that of $H(F)$.

• There may be a twist θ_H on H as well. This twist is simple in the sense that it is given by an inner automorphism of $H(F)$. In that case, we expect to transfer only certain representation of $H(F)$.

The first remark is relevant in most cases, but the second is not so significant.

Notes taken by David Renard.

Examples:

• Labesse-Langlands ([LL]) for $G = SL(2)$, $\theta = \mathrm{Id}$, ω trivial. In that case, H is G itself or an anisotropic torus T split over a quadratic extension of F. In the $H = G$ case, the representations are stable (transfer from G itself!). In the $H = T$ case, the transfer is from a quasi-character on T.

• The previous example is tied up with the problem of representations of $GL(2)$ such that $\pi \simeq \omega \otimes \pi$, where ω is the quasi-character $\omega = \lambda \circ \det$, $\lambda^2 = 1$. If λ is trivial, then the transfer is from $GL(2)$ to itself and everything is stable. If λ is not trivial, then the transfer is from a torus T as above.

• $G = GL(n)$, $\theta(g) = {}^t g^{-1}$.

• Cyclic base change: E/F is cyclic, $G' = \mathrm{Res}_F^E(G)$ ($G'(F) \simeq G(E)$), θ is given by the action of a generator of $\mathrm{Gal}(E/F)$ on G'.

We need some results of Steinberg ([St]) about automorphisms of root systems. We start with any (G, θ) and we attach to it a pair (G^*, θ^*) where G^* is a quasi-split inner form of G and θ^* is an automorphism of G^* having properties which make it convenient to study. Let us describe briefly how θ^* is defined. Let $\psi : G \to G^*$ an inner twist, and let $\mathbf{spl}_{G^*} = (B, T, \{X_\alpha\})$ be a F-splitting of G^*. This means that B is a Borel subgroup of G^* defined over F (G^* is quasi-split), $T \subset B$ is a maximal torus of G^* defined over F, and the X_α's are root vectors for the simple roots of T in G^* such that $\sigma \cdot X_\alpha = X_{\sigma\cdot\alpha}$ for all $\sigma \in \mathrm{Gal}(\overline{F}/F)$. Then there exists $g_\theta \in G^*(\overline{F})$ such that

$$\theta^* := \mathrm{Int}\, g_\theta \circ \psi \circ \theta \circ \psi^{-1}$$

preserves $\mathbf{spl}_{G^*}$. The automorphism θ^* of G^* depends only on the choice of $\mathbf{spl}_{G^*}$, and so $\sigma(\theta^*) = \theta^*$ for all $\sigma \in \mathrm{Gal}(\overline{F}/F)$ and θ^* is defined over F.

Root systems: we assume for expository purposes that G^* is simply connected. Then a result of Steinberg asserts that $G^1 := (G^*)^{\theta^*}$ is a connected reductive group and T^{θ^*}, B^{θ^*} are respectively a maximal torus and a Borel subgroup in G^1. The set of restricted roots is

$$R_{\mathrm{res}}(G^*, T) := \{\alpha_{\mathrm{res}} = \alpha_{|T^{\theta^*}},\ \alpha \in R(G^*, T)\}.$$

This is a root system in bijection with the θ^*-orbits in $R(G^*, T)$ and is generally nonreduced (for instance if G is of type A_{2n} and θ^* is given by a nontrivial automorphism of the Dynkin diagram). The set of indivisible roots of $R_{\mathrm{res}}(G^*, T)$ is $R(G^1, T^{\theta^*})$.

Twisted conjugacy classes: We call $\delta^* \in G^*$ θ-semi-simple if $\operatorname{Int}\delta^* \circ \theta^*$ preserves a pair (B, T) in G^* (where B and T are respectively a θ^*-stable Borel subgroup and T a θ^*-stable maximal torus in G^* with $T \subset B$). Let us denote by $\operatorname{Cent}_{\theta^*}(\delta^*, G^*)$ the centralizer of $\operatorname{Int}\delta^* \circ \theta^*$ in G^*. This is a reductive group with maximal torus T^{θ^*} (we still assume G^* to be simply connected) and $R(\operatorname{Cent}_{\theta^*}(\delta^*, G^*), T^{\theta^*}) \subset R_{\mathrm{res}}(G^*, T)$.

The map:

$$\begin{aligned} m : G(\overline{F}) &\to G^*(\overline{F}) \\ \delta &\mapsto \psi(\delta) g_\theta \end{aligned}$$

induces a map from θ-twisted conjugacy classes in $G(\overline{F})$ to θ^*-twisted conjugacy classes in $G^*(\overline{F})$:

$$m(g\delta\theta(g)^{-1}) = \psi(g) m(\delta) \theta^*(\psi(g))^{-1}.$$

This map does not respect in general the Galois action, and this introduce the complications mentioned in the second remark above. Nevertheless, this does not happen in most interesting examples and we shall ignore this difficulty, by assuming that

$$m : \mathrm{Cl}(G(\overline{F}), \theta) \to \mathrm{Cl}(G^*(\overline{F}), \theta^*)$$

is defined over F.

Constructions for transfer: Let us recall some notations about L-groups. Let H be a connected reductive algebraic group defined over F . The group H is identified with the group of its points over $\overline{F}$. We will denote by σ_H the action of $\mathrm{Gal}(\overline{F}/F)$ on H.

Following [KS], by a pair in H, we mean a couple (B, T) where B is a Borel subgroup of H and T a maximal torus in B, and by a splitting of H, we mean a triple $\mathbf{spl}_H = (B, T, \{X\})$ where (B, T) is a pair in H and $\{X\}$ a collection of non-zero root vectors, one for each simple root of T in B. The group H is quasi-split if and only if it has a F-splitting, i.e. one preserved by σ_H. It will be convenient to construct L-groups as follows. Suppose that H is quasi-split and let $\mathbf{spl}_H = (B, T, \{X\})$ be a F-splitting. Let $\hat{H}$ be the dual group of H and $\mathbf{spl}_{\hat{H}} = (\mathcal{B}, \mathcal{T}, \{\mathcal{X}\})$ a splitting of $\hat{H}$. The dual of the based root datum $\Psi(B, T)$ obtained from (B, T) is isomorphic to the based root datum $\Psi(\mathcal{B}, \mathcal{T})$ obtained from $(\mathcal{B}, \mathcal{T})$. The action σ_H induces an action on $\Psi(B, T)$, which tranfers to an action $\sigma_{\hat{H}}$ on $\Psi(\mathcal{B}, \mathcal{T})$ by this isomorphism. Let us denote also by $\sigma_{\hat{H}}$ the action on $\hat{H}$ which induces $\sigma_{\hat{H}}$ on $\Psi(\mathcal{B}, \mathcal{T})$ and preserves the splitting $\mathbf{spl}_{\hat{H}}$. Let W_F be the Weil group of F (see [Bo]). As usual, we define the L-group ${}^L H$ of H to be the semi-direct product $\hat{H} \rtimes W_F$, where the action ρ_H of $W_{\mathbb{R}}$ on

$\hat{H}$ factors through $\mathrm{Gal}(\overline{F}/F)$, with $\rho_H(1\times\gamma)$ acting by $\sigma_{\hat{H}}(\gamma)$ for all $\gamma\in\mathrm{Gal}(\overline{F}/F)$. Note that such a realization of the L-group depends on the choice of the splitting $\mathbf{spl}_{\hat{H}}$ but not of the splitting $\mathbf{spl}_H$. If H is not quasi-split, we choose a quasi-split inner form H^* of H, and a L-group for H will be one constructed from H^*. Suppose we have a L-group data for G given by a splitting $\mathbf{spl}_{\hat{G}}=(\mathcal{B},\mathcal{T},\{\mathcal{X}\})$. Then θ induces bijections of the based root data $\Psi(B^*,T^*)$ and $\Psi(\mathcal{B},\mathcal{T})$. We denote by $\hat{\theta}$ the automorphism of $\hat{G}$ which induces this bijection of $\Psi(\mathcal{B},\mathcal{T})$ and preserves $\mathbf{spl}_{\hat{G}}$. We denote by ${}^L\theta$ the automorphism $\hat{\theta}\rtimes 1_{W_F}$ of ${}^LG$.

We call the tuple $(H,\mathcal{H},s,\xi)$ an endoscopic data for (G,θ) if:

1. H is a quasi-split connected reductive algebraic group defined over F,
2. $\mathcal{H}$ is a split extension of W_F by $\hat{H}$ such that the L-action of W_F on $\hat{H}$ defined by this extension coincide with ρ_H,
3. s is a $\hat{\theta}$-semi-simple element in $\hat{G}$.
4. $\xi:\mathcal{H}\to {}^LG$ is a L-homomorphism satisfying:
   (a) $\mathrm{Int}\, s\circ {}^L\theta\circ\xi = a\cdot\xi$ where a is a 1-cocycle of W_F in $Z(\hat{G})$ corresponding to the quasi-character ω in the Langlands classification (see [Bo]).
   (b) ξ maps isomorphically $\hat{H}$ into $\mathrm{Cent}_{\hat{\theta}}(s,\hat{G})_0$

The complications mentioned in the first remark above arise in the case where $\mathcal{H}$ is not isomorphic to ${}^LH$. This forces us to use a z-extension H_1 of H in order to transfer orbital integrals from $G(F)$ to $H_1(F)$. As we said, this is a quite general phenomenon (for example, if $F=\mathbb{R}$ and θ comes from a base change $\mathbb{C}/\mathbb{R}$, we have to consider such $\mathcal{H}$ not isomorphic to ${}^LH$). For the sake of simplicity, we assume here that $\mathcal{H}\simeq {}^LH$. Then the lifting of characters should be from $H(F)$ to $G(F)$ and is obtained as the transpose of the matching between certain linear combinations of (θ,ω)-orbital integrals on $G(F)$ and stable orbital integrals on $H(F)$. Up to a normalizing factor, the orbital integral of a function $f\in\mathcal{C}_c^\infty(G(F))$ in a θ-strongly regular element $\delta\in G(F)$ is

$$\int_{G(F)/G(F)^\delta} f(g\delta\theta(g)^{-1})\,d\dot{g}$$

where $d\dot{g}$ is an invariant measure on the quotient. Transfer factors should appear as weights in these combinations. The matching of orbital integrals rely upon geometric correspondences between $G(F)$ and $H(F)$ that we shall now described.

Geometric correspondences. There is a canonical map defined over $\mathbb{F}$:

$$\mathcal{A}_{H/G} : \mathrm{Cl}_{ss}(H) \to \mathrm{Cl}_{ss}(G, \theta)$$

between semi-simple conjugacy classes in H and θ-semi-simple twisted conjugacy classes in G ([KS] 3.3.A).

We consider the most regular elements (strongly θ-regular) of G, i.e. those for which $\mathrm{Cent}_\theta(\delta, G)$ is abelian. If $\gamma \in H$, we denote by $\mathcal{O}_\gamma$ its conjugacy class in $G(\overline{F})$.

Definition. An element $\gamma \in H$ is strongly G-regular if $\mathcal{A}_{H/G}(\mathcal{O}_\gamma)$ is a stronly θ-regular θ-conjugacy class in G.

Strongly G-regular implies strongly regular ([KS], 3.3.C)

We say that $\gamma \in H(F)$ is a norm of $\delta \in G(F)$ if $\mathcal{O}_\gamma$ maps to the θ-conjugacy class of δ (in $G(\overline{F})$). If such a δ does not exist, we say that γ is not a norm.

We define the stable θ-conjugacy class of a strongly θ-regular element of $G(F)$ to be the intersection of its θ-conjugacy class in $G(\overline{F})$ with $G(F)$. This is an union of a finite number of θ-conjugacy class in $G(F)$. If a strongly G-regular $\gamma \in H(F)$ is a norm, then it is the norm of exactly one stable θ-conjugacy class in $G(F)$.

Transfer factors: Let $\gamma \in H(F)$ be strongly G-regular, and $\delta \in G(F)$ strongly θ-regular. If γ is not a norm of δ, then we set $\Delta(\gamma, \delta) = 0$. If γ is a norm of δ, $\Delta(\gamma, \delta) = \Delta(\gamma', \delta')$ if γ' is stably conjugate to γ and δ' is θ-conjugate to δ. The result we want is the following: for every function $f \in \mathcal{C}_c^\infty(G(F))$ there exists a function $f^H \in \mathcal{C}_c^\infty(H(F))$ such that f and f^H have Δ matching orbital integrals, i.e.

$$\Phi^{st}(\gamma, f^H) = \sum_\delta \Delta(\gamma, \delta)\Phi(\delta, f)$$

where $\Phi^{st}(., f^H)$ is the stable orbital integral of f^H and $\Phi(., f)$ is the twisted orbital integral of f. The sum is taken over a system of representative of θ-conjugacy classes in $G(F)$ in the stable θ-conjugacy class of δ.

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