Jacquet modules and induced representations

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Scribe's note: This writeup is based on the speaker's original lecture notes.[1] Time constraints forced a number of items to be omitted during the talk. They have been included here.

I will not say much about the history of Jacquet modules, just a few brief remarks.

Jacquet modules do not make their appearance in representation theory until the end of the 1960's. Using them, Jacquet was able to obtain some important basic results in the representation theory of p-adic groups (e.g., his subrepresentation theorem and the equivalence of the different characterizations of cuspidal [Jc]). These results are still somewhat qualitative in nature. To use Jacquet modules as a calculational tool, more quantitative results are needed. A major step in this direction was taken by Casselman, who calculated the Jordan-Hölder series for the Jacquet module of an induced representation, among other things ([Cs]). A significant number of his results were obtained independently by Bernstein and Zelevinsky ([B-Z]).

Before going on to discuss how one can use Jacquet modules to study reducibility questions for induced representations, let me indicate some limitations to this approach. For example, consider $SL(2)$. If $\chi^2 = 1$ with $\chi \neq 1$, the representation $\mathrm{Ind}_{P_\emptyset}^{SL(2)}(\chi)$ is reducible. However, one cannot obtain this reducibility through a simple use of Jacquet modules. A similar situation occurs with $\mathrm{Ind}_{P_\emptyset}^{SL(2)}(1)$, except that it is irreducibility which cannot be shown through a simple use of Jacquet modules.

Let me make one simple remark about the examples above. In principle, one can use Frobenius reciprocity to determine the reducibility of parabolically induced representations when the inducing representation is unitary.

[1] The author is thankful to C. Jantzen for improving the style of the original notes.

However, in practice, one does not know the exact structure of the Jacquet modules, only the composition factors that arise. Even for the example $\mathrm{Ind}_P^{SL(2)}(\chi)$, $\chi^2 = 1$, $\chi \neq 1$, one needs other tools to determine reducibility. On the other hand, in the nonunitary case, even knowing the structure of the Jacquet modules is not necessarily enough to determine reducibility. For example, one knows the Jacquet module structure for $\mathrm{Ind}_P^{SL(2)}(|\cdot|^{\alpha})$, $\alpha \in \mathbb{R} \setminus \{0\}$, but this is not enough to give irreducibility for $\alpha \neq \pm 1$.

After this, one might wonder what the prospects are for using Jacquet modules in more complicated situations. Surprisingly, Jacquet modules are much more powerful in more complicated situations. The reason for this is simple: there are more standard parabolic subgroups. Therefore, one can compare information on Jacquet modules coming from different parabolic subgroups. For this approach to really be effective, one needs to direct one's attention to Jacquet modules with respect to large parabolic subgroups (which was not usually done in the early applications of this approach). We note that this approach is particularly convenient for classical groups because the Levi factors of their parabolic subgroups are direct products of general linear groups and smaller classical groups. And, we understand induced representations for general linear groups from the Bernstein-Zelevinsky theory ([B-Z],[Zl]).

My primary interest in the use of Jacquet modules was in understanding the non-cuspidal square-integrable representations and, more generally, the non-cuspidal tempered representations. The latter problem is clearly related to understanding reducibility of parabolic induction. But, the first problem is also very much related to understanding reducibility of parabolic induction, and the reducibility problems involved are even more complicated than in the tempered case. (The reason for this is that one must look for non-cuspidal square-integrable representations as subquotients of parabolically induced representations where the inducing representation is nonunitary.) In the long run, I am also motivated by the desire to develop tools for addressing the unitarizability problem for p-adic groups.

Clearly, we need some ideas on how to deal with reducibility questions for parabolically induced representations. As an illustration, I will give two simple, but useful, recipes for proving reducibility or irreducibility.

Let G be a connected reductive group over a local nonarchimedean field F

(later, when we start to deal with classical groups, we shall assume char$F \neq 2$). For a smooth representation π of G and a parabolic subgroup $P = MN$, we denote by $r_M^G(\pi)$ the normalized Jacquet module of π with respect to P (notation of Bernstein-Zelevinsky). We remind the reader that if V is the space of π, then $r_M^G(\pi)$ has

space: $V/V(N)$, where $V(N) = span\{\pi(n)v - v \mid n \in N, v \in V\}$

action: $(r_M^G(\pi))(m)[v + V(N)] = \delta_P^{-\frac{1}{2}}(m)\pi(m)v + V(N)$,

where δ_P denotes the modular function for P (since M normalizes N, this defines an action). For much of our discussion, we will essentially be dealing with semi-simplifications of representations. More precisely, we work in the Grothendieck group $R(G)$ of the category of smooth finite-length representations of G. Recall that two representations π_1, π_2 have $\pi_1 = \pi_2$ in $R(G)$ if $m(\rho, \pi_1) = m(\rho, \pi_2)$ for all smooth irreducible representations ρ, where $m(\rho, \pi_i)$ denotes the multiplicity of ρ in π_i. Then, r_M^G lifts to a mapping

$$r_M^G : R(G) \longrightarrow R(M).$$

We note that $R(G)$ admits a natural partial order: $\pi_1 \leq \pi_2$ if $m(\rho, \pi_1) \leq m(\rho, \pi_2)$ for all smooth irreducible ρ.

We now give criteria for reducibility and irreducibility of induced representations, For more details on these criteria, the reader is referred to section 3 of [Td3].

Reducibility criteria: **(RC)**

Suppose $P_0 = M_0 N_0$ and $P = MN$ are standard parabolic subgroups. Further, suppose σ is a smooth irreducible representation of M_0; π, Π smooth finite-length representations of G. Suppose that

1. $\mathrm{Ind}_{P_0}^G(\sigma) \leq \Pi$, $\pi \leq \Pi$.

2. $r_M^G(\mathrm{Ind}_{P_0}^G(\sigma)) + r_M^G(\pi) \not\leq r_M^G(\Pi)$

3. $r_M^G(\mathrm{Ind}_{P_0}^G(\sigma)) \not\leq r_M^G(\pi)$.

Then, $\mathrm{Ind}_{P_0}^G(\sigma)$ is reducible.

Remarks:

1. One can almost always choose π and Π to be parabolically induced representations.

2. It is easy to get upper and lower estimates on the Jacquet modules of common irreducible subquotients of $\mathrm{Ind}_{P_0}^G(\sigma)$ and π; often they will give the Jacquet modules of the common irreducible subquotients exactly.

We now take up the question of showing irreducibility. First, fix a minimal parabolic subgroup $P_\emptyset$ of G. Then, we can (and do) choose Levi decompositions $P = MN$ of standard parabolic subgroups so that

$$r_{M_2}^{M_1} \circ r_{M_1}^G = r_{M_2}^G$$

for standard parabolics $P_1 \supset P_2$.

Suppose that σ is an irreducible representation of M_0 and $\mathrm{Ind}_{P_0}^G(\sigma)$ reduces. Write $\mathrm{Ind}_{P_0}^G(\sigma) = \pi_1 + \pi_2$, $\pi_1 > 0$, $\pi_2 > 0$, in the Grothendieck group $R(G)$. For any standard parabolic $P = MN$, let

$$T_{i,P} = r_M^G(\pi_i), \quad i = 1, 2$$

(viewed as element of $R(M)$). Then, the following must hold:

1. $T_{i,P} \geq 0$ and $T_{1,P} \neq 0$ if and only if $T_{2,P} \neq 0$

2. $T_{1,P} + T_{2,P} = r_M^G(\mathrm{Ind}_{P_0}^G(\sigma))$

3. $r_{M_2}^{M_1}(T_{i,P_1}) = T_{i,P_2}$ when $P_1 \supset P_2$.

Irreducibility criteria: **(IC)**

Let σ be an irreducible representation of M_0 and consider $\mathrm{Ind}_{P_0}^G(\sigma)$. If one can show there is no system of $T_{i,P}$'s as above ($i = 1, 2$; P running through the standard parabolics), then one has shown the irreducibility of $\mathrm{Ind}_{P_0}^G(\sigma)$.

Remark: When using this approach, it is usually possible to produce three standard parabolic subgroups P, P_1, P_2 with $P \subset P_1, P_2$, and $P \neq P_1, P_2$ for which one can already show that 1.-3. above cannot be satisfied.

Example: A simple example which often works is the following: assume $P = MN$, $P_1 = M_1N_1$, $P_2 = M_2N_2$ are standard parabolic subgroups with $P \subset P_1, P_2$ and $P \neq P_1, P_2$. Suppose there exists an irreducible subquotient τ_1 of $r^G_{M_1}(\mathrm{Ind}^G_{P_0}(\sigma))$ such that for any irreducible subquotient τ_2 of $r^G_{M_2}(\mathrm{Ind}^G_{P_0}(\sigma))$, we have

$$r^{M_1}_M(\tau_1) + r^{M_2}_M(\tau_2) \not\leq r^G_M(\mathrm{Ind}^G_{P_0}(\sigma)).$$

Then, $\mathrm{Ind}^G_{P_0}(\sigma)$ is irreducible.

To see that this holds, suppose $\mathrm{Ind}^G_{P_0}(\sigma) = \pi_1 + \pi_2$ as above. Without loss of generality, assume τ_1 is a subquotient for $r^G_{M_1}(\pi_1) = T_{1,P_1}$. Choose any irreducible subquotient τ_2 of $r^G_{M_2}(\pi_2) = T_{2,P_2}$. Then, by 2.,3.

$$r^G_M(\mathrm{Ind}^G_{P_0}(\sigma)) = T_{1,P} + T_{2,P} = r^{M_1}_M(T_{1,P_1}) + r^{M_2}_M(T_{2,P_2}) \geq r^{M_1}_M(\tau_1) + r^{M_2}_M(\tau_2),$$

contradicting the assumption $r^{M_1}_M(\tau_1) + r^{M_2}_M(\tau_2) \not\leq r^G_M(\mathrm{Ind}^G_{P_0}(\sigma))$.

We now give an example, to illustrate these ideas in a concrete situation.

Example: Let $G = GL_3(F)$. Let $P_\emptyset, P_1, P_2$ denote the three (proper) standard parabolic subgroups ($P_\emptyset$ minimal). These have Levi factors $A = F^\times \times F^\times \times F^\times$, $M_1 = GL_2(F) \times F^\times$, $M_2 = F^\times \times GL_2(F)$, respectively. We shall apply **(IC)** and **(RC)** to the induced representation $\rho = \mathrm{Ind}^G_{P_2}(\chi_1 \circ det_{GL(2)} \otimes \chi_2)$, where χ_1, χ_2 are characters (not necessarily unitary) of $F^\times$.

The semisimplified Jacquet modules for ρ may be calculated using Lemma 2.12 [B-Z] or the results in chapter 6 of [Cs]. We tabulate them below:

$$\begin{array}{ccccc}
\underline{r^G_{M_1}(\rho)} & & \underline{r^G_A(\rho)} & & \underline{r^G_{M_2}(\rho)} \\
|\cdot|^{-\frac{1}{2}}\chi_1 \otimes \sigma_1 & & |\cdot|^{-\frac{1}{2}}\chi_1 \otimes |\cdot|^{\frac{1}{2}}\chi_1 \otimes \chi_2 & & \chi_1 \circ det_{GL(2)} \otimes \chi_2 \\
 & & |\cdot|^{-\frac{1}{2}}\chi_1 \otimes \chi_2 \otimes |\cdot|^{\frac{1}{2}}\chi_1 & & \sigma_2 \otimes |\cdot|^{\frac{1}{2}}\chi_1 \\
\chi_2 \otimes \chi_1 \circ det_{GL(2)} & & \chi_2 \otimes |\cdot|^{-\frac{1}{2}}\chi_1 \otimes |\cdot|^{\frac{1}{2}}\chi_1 & &
\end{array}$$

where $\sigma_1 = \mathrm{Ind}_P^{GL(2)}(|\cdot|^{\frac{1}{2}}\chi_1 \otimes \chi_2)$ and $\sigma_2 = \mathrm{Ind}_P^{GL(2)}(|\cdot|^{-\frac{1}{2}}\chi_1 \otimes \chi_2)$ (here, P is the standard minimal parabolic subgroup of $GL(2)$). For a subquotient τ of $r_{M_1}^G$ (resp. $r_{M_2}^G$), the lines indicate which terms in $r_A^G(\rho)$ come from $r_A^{M_1}(\tau)$ (resp. $r_A^{M_2}(\tau)$).

We now apply **(IC)**, using the form of **(IC)** which appears in the preceding example. Assume $\chi_2 \neq |\cdot|^{\pm\frac{1}{2}}\chi_1, |\cdot|^{\pm\frac{3}{2}}\chi_2$. Take $\tau_1 = |\cdot|^{-\frac{1}{2}}\chi_1 \otimes (\mathrm{Ind}_P^{GL(2)}(|\cdot|^{\frac{1}{2}}\chi_1 \otimes \chi_2))$ (a subquotient of $r_{M_1}^G(\rho)$) and $\tau_2' = \chi_1 \circ det_{GL(2)} \otimes \chi_2$, $\tau_2'' = (\mathrm{Ind}_P^{GL(2)}(|\cdot|^{-\frac{1}{2}}\chi_1 \otimes \chi_2)) \otimes |\cdot|^{\frac{1}{2}}\chi_1$. For $\chi_2 \neq |\cdot|^{\pm\frac{1}{2}}\chi_1, |\cdot|^{\pm\frac{3}{2}}\chi_2$, these are all irreducible (in general, $\mathrm{Ind}_P^{GL(2)}(\chi \otimes \chi')$ is reducible if and only if $\chi' = |\cdot|^{\pm 1}\chi$). Then,

$$r_A^{M_1}(\tau_1) + r_A^{M_2}(\tau_2') > 2 \cdot |\cdot|^{-\frac{1}{2}}\chi_1 \otimes |\cdot|^{\frac{1}{2}}\chi_1 \otimes \chi_2$$

$$r_A^{M_1}(\tau_1) + r_A^{M_2}(\tau_2'') > 2 \cdot |\cdot|^{-\frac{1}{2}}\chi_1 \otimes \chi_2 \otimes |\cdot|^{\frac{1}{2}}\chi_1.$$

Therefore, it is easy to see that $r_A^{M_1}(\tau_1) + r_A^{M_2}(\tau_2) \not\leq r_A^G(\rho)$ for $\tau_2 = \tau_2'$ or τ_2''. By the preceding example, **(IC)** now implies irreducibility.

Next, we apply **(RC)** when $\chi_2 = |\cdot|^{\pm\frac{3}{2}}\chi_1$. Say $|\cdot|^{-\frac{3}{2}}\chi_1$, for concreteness. Take

$$\Pi = \mathrm{Ind}_{P_\emptyset}^G(|\cdot|^{-\frac{1}{2}}\chi_1 \otimes |\cdot|^{\frac{1}{2}}\chi_1 \otimes |\cdot|^{-\frac{3}{2}}\chi_1)$$

$$\pi = \mathrm{Ind}_{P_2}^G(((|det_{GL(2)}|^{-1}\chi_1 \circ det_{GL(2)}) \otimes |\cdot|^{\frac{1}{2}}\chi_1)$$

We have the following semisimplified Jacquet modules:

$$\begin{aligned}
r_A^G(\rho) &= |\cdot|^{-\frac{1}{2}}\chi_1 \otimes |\cdot|^{\frac{1}{2}}\chi_1 \otimes |\cdot|^{-\frac{3}{2}}\chi_1 + |\cdot|^{-\frac{1}{2}}\chi_1 \otimes |\cdot|^{-\frac{3}{2}}\chi_1 \otimes |\cdot|^{\frac{1}{2}}\chi_1 \\
&\quad + |\cdot|^{-\frac{3}{2}}\chi_1 \otimes |\cdot|^{-\frac{1}{2}}\chi_1 \otimes |\cdot|^{\frac{1}{2}}\chi_1
\end{aligned}$$

$$\begin{aligned}
r_A^G(\Pi) &= |\cdot|^{-\frac{1}{2}}\chi_1 \otimes |\cdot|^{\frac{1}{2}}\chi_1 \otimes |\cdot|^{-\frac{3}{2}}\chi_1 + |\cdot|^{-\frac{1}{2}}\chi_1 \otimes |\cdot|^{-\frac{3}{2}}\chi_1 \otimes |\cdot|^{\frac{1}{2}}\chi_1 \\
&\quad + |\cdot|^{\frac{1}{2}}\chi_1 \otimes |\cdot|^{-\frac{1}{2}}\chi_1 \otimes |\cdot|^{-\frac{3}{2}}\chi_1 + |\cdot|^{\frac{1}{2}}\chi_1 \otimes |\cdot|^{-\frac{3}{2}}\chi_1 \otimes |\cdot|^{-\frac{1}{2}}\chi_1 \\
&\quad + |\cdot|^{-\frac{3}{2}}\chi_1 \otimes |\cdot|^{-\frac{1}{2}}\chi_1 \otimes |\cdot|^{\frac{1}{2}}\chi_1 + |\cdot|^{-\frac{3}{2}}\chi_1 \otimes |\cdot|^{\frac{1}{2}}\chi_1 \otimes |\cdot|^{-\frac{1}{2}}\chi_1
\end{aligned}$$

$$\begin{aligned}
r_A^G(\pi) &= |\cdot|^{-\frac{3}{2}}\chi_1 \otimes |\cdot|^{-\frac{1}{2}}\chi_1 \otimes |\cdot|^{\frac{1}{2}}\chi_1 + |\cdot|^{-\frac{3}{2}}\chi_1 \otimes |\cdot|^{\frac{1}{2}}\chi_1 \otimes |\cdot|^{-\frac{1}{2}}\chi_1 \\
&\quad + |\cdot|^{\frac{1}{2}}\chi_1 \otimes |\cdot|^{-\frac{3}{2}}\chi_1 \otimes |\cdot|^{-\frac{1}{2}}\chi_1.
\end{aligned}$$

Therefore, one can immediately check that applying **(RC)** with $M = A$ gives reducibility. The case $\chi_2 = |\cdot|^{\frac{3}{2}}\chi_1$ is similar. (We note that for both $\chi_2 = |\cdot|^{\pm\frac{3}{2}}\chi_1$, ρ contains a one-dimensional subquotient), A final word: the case $\chi_2 = |\cdot|^{\pm\frac{1}{2}}\chi_1$ cannot be resolved using **(IC)** or **(RC)** (it is irreducible–see [Zl]). We shall encounter this phenomenon again later.

To present some applications of these ideas in the case of classical groups, and say a few words about their proofs, we first need to introduce some notation.

First, let me recall some of the notation of Bernstein-Zelevinsky for general linear groups ([B-Z]). For representations π_1, π_2 of $GL(N_1), GL(n_2)$, they let $\pi_1 \times \pi_2$ denote the representation obtained by inducing $\pi_2 \otimes \pi_2$ from the parabolic subgroup

$$P_{(n_1,n_2)} = M_{(n_1,n_2)} N_{(n_1,n_2)} = \left\{ \left(\begin{array}{c|c} GL(n_1) & * \\ \hline 0 & GL(n_2) \end{array} \right) \right\}.$$

Note that $M_{(n_1,n_2)} \cong GL(n_1) \times GL(n_2)$.

They let $R_n = R(GL(n))$ (Grothendieck group) and set

$$R = \bigoplus_{n \geq 0} R_n.$$

Then, $\times$ lifts to R in a natural way, giving a multiplication $\times : R \times R \longrightarrow R$, which extends bilinearly to give $\times : R \otimes R \longrightarrow R$. They also introduced a comultiplication

$$m^* : R \longrightarrow R \otimes R$$

defined on irreducible representations π by

$$m^*(\pi) = \sum_{k=0}^{n} s.s.(r_{M_{(k,n-k)}}^{GL(n)}(\pi)) \in R \otimes R,$$

then extended it additively.

Bernstein and Zelevinsky showed that with this multiplication and comultiplication, R has the structure of a Hopf algebra. But, they did not really use this structure in their treatment of p-adic general linear groups.

Next, let me recall the definitions of some classical groups. Assume $\mathrm{char} F \neq 2$. Let I_n denote the $n \times n$ identity matrix; J_n the $n \times n$ matrix with $j_{a,b} = 1$ if $a + b = n + 1$, $j_{a,b} = 0$ if not. If ^{t} denotes transpose and ${}^\tau$ transpose with respect to the other diagonal (antidiagonal), then

$$Sp(n) = Sp(n,F) = \left\{ g \in GL(2n,F) \middle| \ {}^t g \begin{pmatrix} 0 & J_n \\ -J_n & 0 \end{pmatrix} g = \begin{pmatrix} 0 & J_n \\ -J_n & 0 \end{pmatrix} \right\},$$

$$SO(2n+1) = SO(2n+1,F) = \{ g \in SL(2n+1,F) | \ {}^\tau g g = I_{2n+1} \}.$$

In what follows, we use S_n to denote $Sp(n)$ or $SO(2n+1)$. (The similarity in structures allows us to treat both families simultaneously.)

We now introduce some notation as in [Td1]. Recall that the maximal standard parabolic subgroups of S_n have the form

$$P_{(k)} = M_{(k)} N_{(k)} = \left\{ \left(\begin{array}{c|c|c} g & * & * \\ \hline 0 & h & * \\ \hline 0 & 0 & {}^\tau g^{-1} \end{array} \right) \in S_n \ \middle| \ g \in GL(k), \ h \in S_{n-k} \right\}.$$

Note that $M_{(k)} \cong GL(k) \times S_{n-k}$ in the obvious way. If π is a representation of $GL(k)$ and σ is a representation of S_{n-k}, we let $\pi \rtimes \sigma$ denote the representation of S_n obtained by inducing $\pi \otimes \sigma$ from $P_{(k)}$. (We use $\rtimes$ to make it clear that we are multiplying a representation of $GL(k)$ and a representation of S_{n-k}; any similarity to the notation for semidirect product is purely coincidental.)

Now, we set

$$R(S) = \bigoplus_{n \geq 0} R(S_n),$$

a direct sum of Grothendieck groups. We lift $\rtimes$ to $\rtimes : R \times R(S) \longrightarrow R(S)$, and $R(S)$ is an R-module in this way. By bilinearity, we may extend $\rtimes$ to $R \otimes R(S)$. We now define a comodule structure: if σ is an irreducible representation of S_n, set

$$\mu^*(\sigma) = \sum_{k=0}^{n} s.s.(r_{M_{(k)}}^{S_n}(\sigma)) \in R \otimes R(S).$$

We extend μ^* additively to $\mu^* : R(S) \longrightarrow R \otimes R(S)$.

If π is an admissible representation of $GL(p)$, σ an irreducible cuspidal representation of S_k, and τ a subquotient of $\pi \rtimes \sigma$, then we set

$$s_{GL}(\tau) = r_{M_{(p)}}^{S_{k+p}}(\tau).$$

This is often a very important Jacquet module; very convenient for calculations since it essentially requires only the GL-theory.

For admissible finite-length representations π of $GL(k)$ and σ of S_q, we would like a simple formula for $\mu^*(\pi \rtimes \sigma)$. This would enable us to calculate factors in a Jordan-Hölder series for Jacquet modules of $\pi \rtimes \sigma$. To this end, let $s : R \otimes R \longrightarrow R \otimes R$ denote transposition: $s(\sum x_i \otimes y_i) = \sum y_i \otimes x_i$. If $\tilde{}$ denotes contragredient and $m : R \otimes R \longrightarrow R$ multiplication (i.e., $\times$ lifted to $R \otimes R$), set

$$M_S^* = (m \otimes 1) \circ (\tilde{} \otimes m^*) \circ s \circ m^* : R \longrightarrow R \otimes R.$$

Note that $R \otimes R(S)$ is an $R \otimes R$-module in the natural way: specifically, $(\pi_1 \otimes \pi_2) \rtimes (\pi_3 \otimes \sigma) = (\pi_1 \times \pi_3) \otimes (\pi_2 \rtimes \sigma)$. We have the following formula ([Td2]):

Theorem: For admissible finite-length representations π of $GL(p)$ and σ of S_q, we have

$$\mu^*(\pi \rtimes \sigma) = M_S^*(\pi) \rtimes \mu^*(\sigma).$$

We shall need some additional notation for general linear groups (cf. [B-Z],[Zl]). Let $\nu = |det|_F$ on $GL(n)$, where $|\cdot|_F$ denotes the modulus character of F. For an irreducible cuspidal representation ρ of $GL(p)$, the set

$$\{\rho, \nu\rho, \ldots, \nu^k\rho\} = [\rho, \nu^k\rho]$$

is called a segment in the set of (equivalence classes of) irreducible cuspidal representations of general linear groups. We will frequently denote such segments by Δ. For such a segment, we have

$$\delta(\Delta) \hookrightarrow \nu^k\rho \times \nu^{k-1}\rho \times \ldots \times \nu\rho \times \rho \longrightarrow \zeta(\Delta),$$

with $\delta(\Delta)$ the unique irreducible subrepresentation and $\zeta(\Delta)$ the unique irreducible quotient. Then, $\delta(\Delta)$ is an essentially square-integrable representation (i.e, after a twist by a character of the group, the matrix coefficients are square-integrable mod the center of the group), and Bernstein showed that every essentially square-integrable representation of a general linear group arises in this way. Moreover, we note that

$$m^*(\delta([\rho, \nu^k\rho])) = \sum_{i=-1}^{k} \delta([\nu^{i+1}\rho, \nu^k\rho]) \otimes \delta([\rho, \nu^i\rho]).$$

We now review the Langlands classification for general linear groups. Any irreducible essentially square-integrable representation δ of a general linear group can be written $\delta = \nu^{e(\delta)}\delta^u$, with $e(\delta) \in \mathbb{R}$ and δ^u unitarizable. For irreducible essentially square-integrable representations $\delta_1, \ldots, \delta_k$ of general linear groups, choose a permutation p of $1, \ldots, k$ such that

$$e(\delta_{p(1)}) \geq \ldots \geq e(\delta_{p(k)}).$$

Then, $\delta_{p(1)} \times \ldots \times \delta_{p(k)}$ has a unique irreducible quotient, which we denote by $L(\delta_1, \ldots, \delta_k)$. This is (part of) the Langlands classification for general linear groups.

Now, we start to analyze induced representations of the classical groups S_n. Our goal is to study reducibility questions, with an eye toward finding square-integrable subquotients.

The first case to consider is the following: let ρ be an irreducible unitarizable cuspidal representation of $GL(n)$, σ a similar representation of S_m, and $\alpha \in \mathbb{R}$. If $\nu^\alpha\rho \rtimes \sigma$ reduces, then $\rho \cong \tilde{\rho}$. We cannot say anything about α using only Jacquet modules. This is essentially the same situation as with $SL(2)$ (discussed above), and the same basic problem arises. However, we know from Harish-Chandra's work that for any such ρ, σ with $\rho \cong \tilde{\rho}$, there is a unique $\alpha_0 \geq 0$ such that $\nu^{\alpha_0}\rho \rtimes \sigma$ reduces and $\nu^\alpha\rho \rtimes \sigma$ is irreducible for all $\alpha \in \mathbb{R} \setminus \{\pm\alpha_0\}$.

For a given pair (ρ, σ) as above, the key piece of information which we build on is the value of α_0 (rather than the actual representations ρ, σ). However, I should point out that the question of determining α_0 for a given pair (ρ, σ) is a difficult one.

What is known about α_0? If ρ is a character of $F^\times = GL(1)$ and σ is the trivial representation of S_0 (the trivial group), then the values of α_0 have been known for decades. The first case other than $SL(2) = Sp(1)$ or $SO(3)$ was settled by J.-L. Waldspurger in the '80's ([Wl]). F. Shahidi made great progress toward understanding α_0. He showed that if σ is generic and $\operatorname{char} F = 0$, then

$$\alpha_0 \in \left\{0, \frac{1}{2}, 1\right\}$$

([Sh2]). I shall call this type of reducibility "generic reducibility," whether or not σ is actually generic. Shahidi also determined α_0 explicitly for some cases with σ the trivial representation of S_0. Further results in this case were recently obtained by F. Murnaghan and J. Repka ([M-R]).

In the case of nongeneric σ, M. Reeder ([Rd2]) and C. Mœglin ([Mg]) independently showed that one could have nongeneric reducibility, i.e., $\alpha_0 \notin \{0, \frac{1}{2}, 1\}$. Further, Mœglin formulated a conjecture expressing α_0 in terms of the local Langlands correspondence. However, a discussion of this conjecture would be too great a digression.

It is generally expected that

$$\alpha_0 \in \frac{1}{2}\mathbb{Z}.$$

This would follow, e.g., from Shahidi's conjecture on the existence of a generic representation in each L^2 L-packet [Sh1].

In the case of general linear groups, we have the following fact from Bernstein-Zelevinsky.

<u>Fact</u>: Let δ_i, $i = 1, 2$, be an irreducible essentially square-integrable representation of $GL(n_i)$. If $\delta_1 \times \delta_2$ reduces and δ_1 or δ_2 is cuspidal, then $\delta_1 \times \delta_2$ contains an essentially square-integrable subquotient. Starting with cuspidal representations, each essentially square-integrable representation arises this way in a finite number of steps.

This is one of the reasons I studied reducibility questions for representations of the form $\delta \rtimes \sigma$ with δ essentially square-integrable and σ cuspidal or vice-versa.

Since there are already several papers treating generic reducibilities, I shall talk mainly about non-generic reducibilities. Let us note that even in the non-generic setting, non-generic reducibilities should not occur too often (this follows from the Mœglin conjecture). The study of what happens for the case of non-generic reducibilities is still in its infancy, so the results I will discuss are simpler. In fact, the existence of such reducibilities seems to date from last year; I first learned of them at the end of last year from M. Reeder ([Rd1]).

Let ρ be an irreducible, unitarizable cuspidal representation of $GL(p)$; σ an irreducible cuspidal representation of S_q. Suppose that $\nu^\alpha\rho \rtimes \sigma$ reduces for some $\alpha > 1$, $\alpha \in \frac{1}{2}\mathbb{Z}$ (i.e., non-generic reducibility). Then, $\nu^\alpha\rho \rtimes \sigma$ contains a unique square-integrable subquotient, which we denote by $\delta(\nu^\alpha\rho, \sigma)$.

Let ρ_0 be an irreducible, unitarizable cuspidal representation of $GL(p_0)$. Question: when does

$$(*) \qquad \nu^\beta\rho_0 \rtimes \delta(\nu^\alpha\rho, \sigma)$$

reduce? To answer this, we consider two fundamentally different cases.

Case 1: $\rho \not\cong \rho_0$

This case is easy. Using **(RC)** and **(IC)**, one can show that

$$(*) \text{ reduces } \iff \nu^\beta\rho_0 \rtimes \sigma \text{ reduces.}$$

Further, if $(*)$ reduces and $\beta \neq 0$, then $\nu^\beta\rho_0 \rtimes \delta(\nu^\alpha\rho, \sigma)$ contains a unique square-integrable subquotient. If $(*)$ reduces and $\beta = 0$, there is no square-integrable subquotient.

Case 2: $\rho \cong \rho_0$

First, from **(IC)**, we have that $(*)$ is irreducible for $\beta \in \mathbb{R} \setminus \{\pm(\alpha - 1), \pm\alpha, \pm(\alpha+1)\}$. Using **(RC)**, we see that $(*)$ is reducible for $\beta = \pm(\alpha - 1), \pm(\alpha+1)$. Later, we shall say more about what happens at the reducibility points.

Neither **(IC)** nor **(RC)** can be applied to decide what happens with

$$\nu^\alpha\rho \rtimes \delta(\nu^\alpha\rho, \sigma).$$

(This is similar to what happened earlier with our $GL(3)$ example when $\chi_2 = |\cdot|^{\pm\frac{1}{2}}\chi_1$.) It is important to determine what happens here. Failure to solve this will have a ripple effect: as the rank of the classical group increases, there will be an increasing number of induced representations which cannot be fully analyzed by Jacquet module methods.

Fact: If $\alpha \in \frac{1}{2}\mathbb{Z}$ with $\alpha \geq 1$, then $\nu^\alpha \rtimes \delta(\nu^\alpha, \sigma)$ is irreducible.

Sketch of proof: We shall illustrate the proof with the case $\alpha = 1$. Suppose that $\Pi = \nu\rho \rtimes \delta(\nu\rho, \sigma)$ reduces. Since

$$\text{length}(s_{GL}(\Pi)) = 2$$

and

$$\nu\rho \times \nu\rho \otimes \sigma \leq s_{GL}(\Pi),$$

there exists an irreducible subquotient π of Π such that

$$s_{GL}(\pi) = \nu\rho \times \nu\rho \otimes \sigma.$$

This implies

$$(**) \qquad \delta([\nu^{-1}\rho, \nu\rho]) \otimes \sigma \not\leq s_{GL}(\rho \rtimes \pi).$$

Further, one easily gets that

$$\rho \rtimes \pi \leq \nu^{-1}\rho \times \rho \times \nu\rho \rtimes \sigma$$

and

$$\delta([\nu^{-1}\rho, \nu\rho]) \rtimes \sigma \leq \nu^{-1}\rho \times \rho \times \nu\rho \rtimes \sigma.$$

The multiplicity of $\nu\rho \times \delta([\rho, \nu\rho]) \otimes \sigma$ in s_{GL} of all three of these representations is 2. Therefore, $\rho \rtimes \pi$ and $\delta([\nu^{-1}\rho, \nu\rho]) \rtimes \sigma$ must have a common irreducible subquotient τ (since the Jacquet functor is exact). Thus, $(**)$ implies $\delta([\nu^{-1}\rho, \nu\rho]) \otimes \sigma \not\leq s_{GL}(\tau)$. On the other hand, since $\delta([\nu^{-1}\rho, \rho])$ is unitary, Frobenius reciprocity tells us that $\delta([\nu^{-1}\rho, \nu\rho]) \otimes \sigma \leq s_{GL}(\tau)$, a contradiction. Therefore, Π is irreducible.

For $\alpha > 1$, assuming reducibility, one introduces π in a similar way as above and uses $\delta([\nu^{-(\alpha-1)}\rho, \nu^{\alpha-1}\rho]) \rtimes \pi$, $\delta([\nu^{-\alpha}\rho, \nu^{\alpha}\rho]) \rtimes \sigma$ and $\nu^{\alpha}\rho \times \nu^{-\alpha}\rho \times \delta([\nu^{-(\alpha-1)}\rho, \nu^{\alpha-1}\rho]) \rtimes \sigma$ to produce a similar contradiction (cf. [Td3]). $\square$

One might well ask how many problems we will encounter in applying these methods to more complicated situations. In principle, after settling this delicate case and one similar one, we can settle reducibility for a large family of induced representations, essentially just using **(IC)** and **(RC)**. For example:

Examples:

1. Let χ be a character of $F^\times$ and $St_{Sp(n)}$ the Steinberg representation of Sp_n. Then, for $n \geq 1$, $\chi \rtimes St_{Sp(n)}$ is reducible if and only if $\chi = \nu^{\pm(n+1)}$ or $\chi^2 = 1_{F^\times}$.

2. Suppose that σ is cuspidal generic and Δ a segment. Then, $\delta(\Delta) \rtimes \sigma$ is reducible if and only if $\tau \rtimes \sigma$ is reducible for some $\tau \in \Delta$. (Note that the assumption σ generic is just to constrain the kinds of reducibility which can occur.)

One can apply a generalized Zelevinsky involution ([Ab], [S-S]) to obtain reducibility results for degenerate principal series for the Siegel parabolic and the $F^\times \times Sp_{n-1}$ parabolic. Actually, C. Jantzen settled the reducibility of any degenerate principal series of these groups, and in the maximal parabolic case, obtained lengths, Langlands data of the irreducible subquotients, and determined the lattice of subrepresentations ([Jn1], [[Jn2]). Actually, his results are a bit more general: rather than just using one-dimensional representations, segment representations are used.

Let me now return to that representation $\nu^\beta\rho \rtimes \delta(\nu^\alpha\rho, \sigma)$, $\alpha \geq 1$, that we considered earlier. We have two pairs of reducibility points

$$\beta = \pm(\alpha+1), \pm(\alpha-1).$$

At $\beta = \pm(\alpha+1)$, we get a square-integrable subquotient. This square-integrable subquotient belongs to a family of square-integrable representations which are closely related to the Steinberg representations. In particular, for $\ell \geq 1$, the representation $\delta([\nu^\alpha\rho, \nu^{\alpha+\ell}\rho]) \rtimes \sigma$ contains a unique square-integrable subrepresentation, which we denote by $\delta([\nu^\alpha\rho, \nu^{\alpha+\ell}\rho], \sigma)$. If $G = Sp_n$, ρ is the trivial representation of $F^\times = GL(1)$, and σ the trivial representation of $Sp(0)$ (so that $\alpha = 1$), then $\delta([\nu\rho, \nu^n\rho], \sigma)$ is the Steinberg representation of Sp_n. We note that

$$\mu^*(\delta([\nu^\alpha\rho, \nu^{\alpha+\ell}\rho], \sigma)) = \sum_{i=-1}^{\ell} \delta([\nu^{\alpha+i+1}\rho, \nu^{\alpha+\ell}\rho]) \otimes \delta([\nu^\alpha\rho, \nu^{\alpha+i}\rho], \sigma).$$

The reducibility at $\beta = \pm(\alpha-1)$ starts a second family of more unusual square-integrable representations. Let $k \in \mathbb{Z}$ be such that $0 < \alpha - k < k$. Then,

$$\zeta([\nu^{\alpha-k}\rho, \nu^\alpha\rho]) \rtimes \sigma$$

contains a unique irreducible subrepresentation, which we denote by

$$\delta([\nu^{\alpha-k}\rho, \nu^{\alpha}\rho], \sigma).$$

We note that

$$\mu^*(\delta([\nu^{\alpha-k}\rho, \nu^{\alpha}\rho], \sigma)) = \sum_{i=0}^{k+1} \zeta([\nu^{\alpha-k}\rho, \nu^{\alpha-i}\rho]) \otimes \delta([\nu^{\alpha-i+1}\rho, \nu^{\alpha}\rho], \sigma).$$

These representation are square-integrable. Notice that the presence of ζ's in the Jacquet modules is in marked contrast to the square-integrable representations we have encountered above.

If we continue this approach, the next place we might look for square-integrable representations is among the subquotients of

$$\nu^{\beta}\rho \rtimes \delta([\nu^{\alpha}\rho, \nu^{\alpha+1}\rho], \sigma)$$

when β is a reducibility point. Now, the reducibility points are $\beta = \pm(\alpha+2)$ and $\beta = \pm(\alpha-1)$. When $\beta = \pm(\alpha+2)$, one of the subquotients is the square-integrable representation $\delta([\nu^{\alpha}\rho, \nu^{\alpha+2}\rho, \sigma])$, which we have already encountered.

When $\beta = \pm(\alpha-1)$, there is also a square-integrable subquotient. This subquotient is the beginning of the following family of square-integrable representations: for $k, \ell \in \mathbb{Z}$ such that $0 < \alpha - k \leq \alpha \leq \alpha + \ell$, the representation

$$\delta([\nu^{\alpha+1}\rho, \nu^{\alpha+\ell}\rho]) \times \zeta([\nu^{\alpha-k}\rho, \nu^{\alpha}\rho]) \rtimes \sigma$$

contains a unique irreducible subrepresentation, which we denote by

$$\delta([\nu^{\alpha-k}\rho, \nu^{\alpha+\ell}\rho], \sigma).$$

This representation is square-integrable, and we have

$$s_{GL}(\delta([\nu^{\alpha-k}\rho, \nu^{\alpha+\ell}\rho, \sigma)) = L(\nu^{\alpha-k}\rho, \nu^{\alpha-k+1}\rho, \ldots, \nu^{\alpha-1}\rho, \delta([\nu^{\alpha}\rho, \nu^{\alpha+\ell}\rho])) \otimes \sigma.$$

Note that for $k = 0$ or $\ell = 0$, this reduces to one of the representations we have already considered.

If we continue with this strategy, starting with a representation $\delta([\nu^{\alpha-k}\rho, \nu^{\alpha+\ell}\rho], \sigma)$, we usually just get new members of the same family. But, there are two interesting directions one can go. I shall just indicate what happens in these directions with a pair of examples.

Examples:

1. Suppose that $\nu^{\frac{3}{2}}\rho \rtimes \sigma$ reduces ([Rd2]). Then, $\nu^{\beta}\rho \rtimes \delta([\nu^{\frac{1}{2}}\rho, \nu^{\frac{3}{2}}\rho], \sigma)$ reduces for
$$\beta \in \left\{\pm\frac{5}{2}, \pm\frac{1}{2}\right\}.$$
   For $\beta = \pm\frac{5}{2}$, the square-integrable representation $\delta([\nu^{\frac{1}{2}}\rho, \nu^{\frac{5}{2}}\rho], \sigma)$ is a subquotient.

   When $\beta = \frac{1}{2}$, the induced representations has length two. One of the irreducible subquotients is the Langlands quotient, hence not square-integrable, or even tempered. The other is tempered but not square-integrable. We may ask how this tempered representation arises. In fact, this tempered representation is a subquotient of
$$\delta([\nu^{-\frac{1}{2}}\rho, \nu^{\frac{1}{2}}\rho]) \rtimes \delta(\nu^{\frac{3}{2}}\rho, \sigma).$$
   So, the prospects of finding interesting new square-integrable representations in this direction do not seem that good. But, there is another step to check, which I have not yet done.

2. Suppose that $\nu^{3}\rho \rtimes \sigma$ reduces ([Mg]). Then,
$$\nu^{\beta}\rho \rtimes \delta([\nu^{2}\rho, \nu^{4}\rho], \sigma)$$
   is reducible if and only if
$$\beta \in \{\pm 1, \pm 3, \pm 5\}.$$
   For $\beta = \pm 1, \pm 5$, we get square-integrable representations from the family we have already considered.

   Let us look at $\beta = 3$. Take
$$\pi = \delta([\nu^{2}\rho, \nu^{3}\rho]) \rtimes \delta([\nu^{3}\rho, \nu^{4}\rho], \sigma)$$
$$\Pi = \nu^{2}\rho \times \nu^{3}\rho \rtimes \delta([\nu^{3}\rho, \nu^{4}\rho], \sigma)$$
   and apply **(RC)**. Then, there is a unique common irreducible subquotient τ_d of π and $\nu^{3}\rho \rtimes \delta([\nu^{2}\rho, \nu^{4}\rho], \sigma)$. Moreover,
$$s_{GL}(\tau_d) = L(\delta([\nu^{2}\rho, \nu^{3}\rho]), \delta([\nu^{3}\rho, \nu^{4}\rho])) \otimes \sigma.$$

The Casselman square-integrability criteria implies that τ_d is square-integrable.

A detailed analysis of reducibility points, square-integrable subquotients, etc., obtained by going in this direction is left for the near future; I hope, this fall.

Now, using [Jn3], we can combine the square-integrable representations obtained earlier for generic reducibilities ([Td4], [Td5]) with the ones we have just described in the case of non-generic reducibilities to obtain new square-integrable representations.

A further question which one may ask is roughly the following: given a fixed pair (ρ, σ) with non-generic reducibility, can one have in this setting more than one square-integrable representation with the same infinitesimal character? The answer may be quite easy, but I have not worked on it yet.

For generic reducibilities, the answer is "yes." Actually, one can have as many as one wants. Let me close with a simple example showing this. The example is for $SO(7)$, with Iwahori-fixed vectors.

First, using **(RC)**, it is easy to show that

$$\delta([\nu^{-\frac{1}{2}}, \nu^{\frac{1}{2}}]) \rtimes 1 = T_1 \oplus T_2$$

with T_1, T_2 irreducible tempered. Now,

$$\delta([\nu^{-\frac{1}{2}}, \nu^{\frac{3}{2}}]) \rtimes 1 \hookrightarrow \nu^{\frac{3}{2}} \times \delta([\nu^{-\frac{1}{2}}, \nu^{\frac{1}{2}}]) \rtimes 1 = \nu^{\frac{3}{2}} \rtimes T_1 \oplus \nu^{\frac{3}{2}} \rtimes T_2.$$

The multiplicity of $\delta([\nu^{-\frac{1}{2}}, \nu^{\frac{3}{2}}]) \otimes 1$ in $s_{GL}(\nu^{\frac{3}{2}} \times \delta([\nu^{-\frac{1}{2}}, \nu^{\frac{1}{2}}]) \rtimes 1)$ is two; in $s_{GL}(\nu^{\frac{3}{2}} \rtimes T_1)$ and $s_{GL}(\nu^{\frac{3}{2}} \rtimes T_2)$, one. From this, one can conclude that $\delta([\nu^{-\frac{1}{2}}, \nu^{\frac{3}{2}}]) \rtimes 1$ has two irreducible subrepresentations, which we denote by $\delta([\nu^{-\frac{1}{2}}, \nu^{\frac{3}{2}}], 1)_{T_i}$, $i = 1, 2$. With a little more effort, one can show that they are both square-integrable.

We note that the two square-integrable subrepresentations have the same infinitesimal character. This is just a very special case of a much more general situation ([Td5]).

It is also an interesting example from another point of view. It shows that the standard module $\delta([\nu^{-\frac{1}{2}}, \nu^{\frac{3}{2}}]) \rtimes 1$, which is induced from a generic representation, has a non-generic irreducible subrepresentation. Therefore, the standard module cannot have an injective Whittaker model. This cannot happen for general linear groups by the results of Jacquet-Shalika ([J-S]).

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