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Characters, characteristic cycles and nilpotent orbits

This talk is a survey of (some parts of) joint work with Wilfried Schmid. These notes are in very rough from. For a more detailed account see the announcement [SV1]. Two thirds of this work is written up in full detail and is available at: *http://www.math.harvard.edu/˜schmid.*

The Barbasch–Vogan conjecture. Let $G_{\mathbb{R}}$ be a linear connected semisimple Lie group. We denote by $G=(G_{\mathbb{R}})_c$ its complexification. Let $K_{\mathbb{R}}\subset G_{\mathbb{R}}$ be a maximal compact subgroup which is the fixed points of a Cartan involution θ, and denote by K its complexification. The corresponding Lie algebras will be denoted $\mathfrak{g}_{\mathbb{R}}, \mathfrak{g}, \mathfrak{k}_{\mathbb{R}}, \mathfrak{k}$. Let $\mathcal{N}\subset\mathfrak{g}\cong\mathfrak{g}^*$ be the nilpotent cone.

For an irreducible representation π of $G_{\mathbb{R}}$, two invariants were introduced some years ago. The first one is called the *Asymptotic Support* and was introduced in [BV]. Let Θ_π be the distribution character of π. The lift to the Lie algebra θ_π has an asymptotic expansion near zero. The leading term is of the form

$$\sum a_j\widehat{\mu}_j \tag{1}$$

where μ_j are the canonical measures of certain nilpotent orbits in $\mathcal{N}\cap i\mathfrak{g}_{\mathbb{R}}$ and a_j are constants in $\mathbb{C}$.

The second invariant was introduced in [V]. If V is the Harish–Chandra module attached to π, then choose a *good* K–invariant filtration $\dots\subset V_j\subset V_{j+1}\subset\dots$. Then the graded module $gr_{.}(V)$ is an $S(\mathfrak{g})$ module, so one can think of it as a K–equivariant coherent sheaf on $\mathfrak{g}$. It is supported on $\mathcal{N}\cap\mathfrak{p}$ ($\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ is the Cartan decomposition) and its support is K-invariant, i.e., it is a union of K-orbits $\mathcal{O}'_j$ on $\mathcal{N}\cap\mathfrak{p}$. Generically a coherent sheaf is a vector bundle on its support. Hence we can associate to π a linear combination

$$\sum b_j\mathcal{O}'_j \tag{2}$$

called the *Associated Cycle*. In this formula the constants b_j are positive integers, they are the ranks of the above vector bundle on $\mathcal{O}'_j$

It is natural to conjecture that the two expressions match via the *Kostant–Sekiguchi correspondence*, which is a bijection between nilpotent orbits on $\mathcal{N}\cap\mathfrak{p}$ and $\mathcal{N}\cap i\mathfrak{g}_{\mathbb{R}}$. This bijection is as follows:

$G_{\mathbb{R}}$–orbits $\mathcal{O}_j\subset i\mathfrak{g}_{\mathbb{R}}\cap\mathcal{N}$ are in 1–1 correspondence with θ–stable maps $\phi: sl(2,\mathbb{C})\to\mathfrak{g}$ with $\phi(sl(2,\mathbb{R}))\in\mathfrak{g}_{\mathbb{R}}$ such that $\phi(\begin{matrix}0&i\\0&0\end{matrix})\in\mathcal{O}_j$. Then the orbit $\mathcal{O}'_j$ is the K–orbit of $\phi(\begin{matrix}1&i\\i&-1\end{matrix})$. The statement of the following theorem is what is known as the Barbasch–Vogan conjecture.

Main Theorem. *In formulas (1) and (2), $\mathcal{O}_j\leftrightarrow\mathcal{O}'_j$ via the Kostant–Sekiguchi correspondence and $a_j=b_j$.*

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Corollary. *The constants a_j are non-negative integers.*

The idea of the proof is to give a geometric interpretation of the *Asymptotic Support* and the *Associated Cycle* and then relate the two geometrically. We will outline the various steps of the proof.

We begin with the diagram (see [KSd]):

$$\begin{array}{ccc} \{G_{\mathbb{R}}\text{–representations}\} & \longleftrightarrow & \{\text{Harish–Chandra modules}\} \\ \uparrow & & \uparrow \\ D_{G_{\mathbb{R}}}(X)_{-\lambda} & \xleftrightarrow[{[MUV]}]{\gamma} & D_K(X)_{-\lambda} \end{array} \tag{3}$$

Here X is the flag manifold, $\lambda \in H^q(X,\mathbb{C}) \cong \mathfrak{h}^*$ determines an infinitesimal character, $D_{G_{\mathbb{R}}}(X)_{-\lambda}$ is the derived category of $G_{\mathbb{R}}$–equivariant sheaves on X with twist $-\lambda$, and $D_K(X)_{-\lambda}$ is the analogous category of K–equivariant sheaves. The upward arrows are given as follows:

$$\begin{aligned} D_{G_{\mathbb{R}}}(X)_{-\lambda} &\longrightarrow \{\text{virtual representations of } G_{\mathbb{R}}\} \\ \mathcal{F} &\mapsto \chi(Ext^*(\mathbb{D}\mathcal{F}, \mathcal{O}_X^{an}(\lambda)), \\ D_K(X)_{-\lambda} &\longrightarrow \{\text{virtual Harish–Chandra modules}\} \\ \mathcal{F} &\mapsto \chi(Ext^*(\mathbb{D}\mathcal{F}, \mathcal{O}_X^{alg}(\lambda))). \end{aligned} \tag{4}$$

The map γ comes from [MUV]:

$$\begin{array}{ccc} G_{\mathbb{R}} & \longrightarrow & G_{\mathbb{R}}/K_{\mathbb{R}} \times X \\ {\scriptstyle a}\big\downarrow & & {\scriptstyle p}\big\downarrow \\ X & & X \end{array} . \tag{5}$$

The map a is $a(g,x) = g^{-1} \cdot x$, and the map p is projection on the second factor. Given a sheaf $\mathcal{F} \in D_K(X)_{-\lambda}$, there is a unique $\tilde{\mathcal{F}}$ on $G_{\mathbb{R}}/K_{\mathbb{R}} \times X$ such that $a^!\mathcal{F} = p^!\tilde{\mathcal{F}}$. Then define $\gamma(\mathcal{F}) := Rp_!(\tilde{\mathcal{F}}) \in D_{G_{\mathbb{R}}}(X)_{-\lambda}$.

We embed (3) into a larger commutative diagram:

$$\begin{tikzcd} \{G_{\mathbb{R}}\ \mathit{representations}\} \arrow[r, leftrightarrow] \arrow[ddd, bend right=60, "BV"'] & \{HC\ \mathit{modules}\} \arrow[ddd, bend left=60, "V"] \\ D_{G_{\mathbb{R}}}(X)_{-\lambda} \arrow[u] \arrow[r, leftrightarrow, "\gamma"] \arrow[d, "CC"'] & D_K(X)_{-\lambda} \arrow[u] \arrow[d, "CC"] \\ \overline{H}_{top}(T^*_{G_{\mathbb{R}}}X) \arrow[d, "\mathrm{gr}\mu^*_\lambda"'] & \overline{H}_{top}(T^*_K X) \arrow[l, "\Phi\ \text{ induced by } \gamma"'] \arrow[d, "\mathrm{gr}\mu^*_\lambda"] \\ \{\text{cycles on } i\mathfrak{g}_{\mathbb{R}} \cap \mathcal{N}\} \arrow[r, leftrightarrow, "\phi \text{ induced by } \Phi", "K-S"'] & \{\text{cycles on } \mathfrak{p} \cap \mathcal{N}\} \end{tikzcd} \tag{6}$$

The rest of the talk is devoted to explaining the terms in this diagram and indicating why the diagram commutes. We begin by explaining the notion of a characteristic cycle

$$CC : D_c^b(X) \to \{\text{Lagrangian cycles on } T^*X\}, \tag{7}$$

due to Kashiwara (see, [KSa]). Here, temporarily, X denotes any (oriented) algebraic manifold and $D_c^b(X)$ denotes the semi-algebraically constructible bounded derived category of sheaves on X. The following "axioms" uniquely determine, and allow one to geomterically compute $CC(\mathcal{F})$:

(a) $CC(\mathbb{C}_X) = [X]$.
(b) CC descends to the Grothendieck group of $D_c^b(X)$
(c) CC is locally defined
(d) if $j : U \to X$ is open, then

$$CC(Rj_*\mathcal{F}) = \lim_{s\to 0^+} CC(\mathcal{F}) + sd(\log f),$$

where f is any defining equation of the boundary of U.

For the fact that the last axiom holds in the real algebraic setting see [SV2].

Let us return to our situation and to the diagram (6). Let $\mu : T^*X \to \mathcal{N}$ be the moment map. Then $\mu^{-1}(\mathfrak{p}) = T_K^*X$ the union of conormal bundles to the orbits of K on X. Similarly, $\mu^{-1}(i\mathfrak{g}_{\mathbb{R}}) = T_{G_{\mathbb{R}}}^*(X)$. Restricted to the categories $D_{G_{\mathbb{R}}}(X)_{-\lambda}$ and $D_K(X)_{-\lambda}$ the map CC takes values as indicated in diagram (6). The map $\mathrm{gr}\mu_\lambda^*$ in diagram (6) is defined by integrating e^λ over the fibers of the map μ_λ and taking the "leading term". To compute the map Φ induced by γ and the map ϕ induced by Φ we proceed as follows.

Let $f : X \to Y$ be proper proper map between algebraic manifolds. Then there is a diagram

$$\begin{array}{ccccc} T^*X & \xleftarrow{df} & X\times_Y T^*Y & \xrightarrow{\rho} & T^*Y \\ \downarrow & & \downarrow & & \downarrow \\ X & = & X & \longrightarrow & Y \end{array} \tag{8}$$

Under these hypotheses Kashiwara-Schapira give the following formula:

$$CC(Rf_*(\mathcal{F})) = \rho_* df^* CC(\mathcal{F}). \tag{9}$$

The definition of Φ involves understanding the effect of the non-proper pushforward $Rp_!$ on characteristic cycles. For this we embed $G_{\mathbb{R}}/K_{\mathbb{R}}$ in its closure to make p proper and use (d) with the defining function $e^{-<x,x>}$ for the boundary of $G_{\mathbb{R}}/K_{\mathbb{R}}$, here $< \, , \, >$ denotes the Killing form. The function $e^{-<x,x>}$ vanishes to infinite order on the boundary and hence is not real algebraic. Its use is justified by results of van den Dries-MacIntyre-Marker, see [DM] and [SV2].

We get the following formula for ϕ:

$$\phi(C) = \lim_{s\to 0+} \{sAd(exp(Rez))z \mid z \in C\}. \tag{10}$$

Theorem. *ϕ coincides with the Kostant–Sekiguchi correspondence.*

It remains to show that the vertical arrows give the *Asymptotic Support* and the *Associated Cycle*, respectively. For the *Associated Cycle* this follows from a result of Chang [C]: One observes that the associated cycle is defined essentially in the same way as a characteristic cycle of a $\mathcal{D}$-module. The argument for the *Asymptotic Support* is more subtle. One uses the following result. Let $\theta(\mathcal{F})$ be the character (on the Lie algebra) of the representation corresponding to the sheaf $\mathcal{F} \in D_{G_{\mathbb{R}}}(X)_{-\lambda}$. For λ regular, there is a map $\mu_\lambda : T^*X \to \Omega_\lambda = G \cdot \lambda$ given by $\mu_\lambda = \mu + v_\lambda$. Here μ is the moment map from before. To define v_λ let U be a compact form of G which is compatible with K and $G_{\mathbb{R}}$. Choose a point $x_0 \in X$. Then $v_\lambda : X \to \Omega_\lambda$ is given by $v_\lambda(ux_0) = u \cdot \lambda$. We have the following generalization of "Kirillov's universal character formula":

Theorem. *For any $\phi \in C_c^\infty(\mathfrak{g}_{\mathbb{R}})$,*

$$\theta(\mathcal{F})(\phi) = \frac{1}{(2\pi i)^n} \int_{\mu_\lambda(CC(\mathcal{F}))} \widehat{\phi}\sigma_\lambda^n.$$

Remarks.

(1) The map μ_λ as well as the idea that such an integral formula should hold for some cycle, without specifying it to be $\mu_\lambda(CC(\mathcal{F}))$, is due to Rossman. Our contribution is proving the formula and specifying the cycle to be $\mu_\lambda(CC(\mathcal{F}))$.

(2) In the case when $\mathcal{F}$ corresponds to a tempered representation, the formula in the theorem is equivalent to Kirillov's formula.

References

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