THE LANGLANDS CLASSIFICATION FOR UNITARY REPRESENTATIONS

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This is joint work with Susana Salamanca-Riba [SV].

Setting: G is a real reductive group; $\hat{G}_t \subset \hat{G}_u \subset \hat{G}_a$ are the tempered, unitary, and admissible duals of G; $P = MAN \subset G$ is a parabolic subgroup, $\tau \in \hat{M}_t$, $\nu \in \hat{A}_a$; $Re < \nu, \alpha >> 0$ for all weights α of $\mathfrak{a}$ on $\mathfrak{n}$.

Theorem 1. *(Langlands)*
(1) $Ind^G_{MAN}(\tau \otimes \nu \otimes 1) = I_P(\tau, \nu)$ *has a unique irreducible quotient* $J_P(\tau, \nu) \in \hat{G}_a$.
(2) $J_P(\tau, \nu) \simeq J_{P'}(\tau', \nu') \leftrightarrow (P, \tau, \nu) \sim (P', \tau', \nu')$ *(conjugacy by* G *on the right hand side)*
(3) Every irreducible admissible representation of G *is isomorphic to* $J_P(\tau, \nu)$ *for some* (P, τ, ν).

The tempered dual $\hat{G}_t$ has been classified, so this is a good reduction, and determines $\hat{G}_a$.

We now consider $\hat{G}_u \subset \hat{G}_h \subset \hat{G}_a$ where $\hat{G}_h$ is the *Hermitian* dual, i.e. irreducible representations which carry an invariant (not necessarily positive definite) Hermitian form.
Setting: $P = MAN$; $\rho \in \hat{M}_{h,Re}$; $\nu \in \hat{A}_u$ with $Im < \nu, \alpha >> 0$ for all weights α of $\mathfrak{a}$ on $\mathfrak{n}$. Here $M_{h,Re}$ is the Hermitian representations with *real* infinitesimal character (defined below). The following Theorem is essentially due to Harish-Chandra — see Knapp [K].

Theorem 2.
(1) $I_P(\rho, \nu)$ *is an irreducible Hermitian representation, and is unitary (respectively tempered) if and only if* ρ *is unitary (respectively tempered).*
(2) $I_P(\rho, \nu) \simeq I_{P'}(\rho', \nu') \leftrightarrow (P, \rho, \nu) \sim (P', \rho', \nu')$
(3) Every irreducible Hermitian representation of G *is isomorphic to* $J_P(\tau, \nu)$ *for some* (P, τ, ν).

Note:. This reduces the classification of $\hat{G}_u$ to the case of real infinitesimal character.

Fix a Cartan involution θ of G with fixed points K. Let $H = TA$ be a θ-stable Cartan subgroup with Lie algebra $\mathfrak{h}_0$, and complexified Lie algebra $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$. Let $\mathfrak{h}^* = \mathfrak{t}^* \oplus \mathfrak{a}^*$ be the duals. Then $\mathfrak{h}^* \supset \mathfrak{h}^*_{\mathbb{R}} \overset{def.}{=} i\mathfrak{t}^*_0 \oplus \mathfrak{a}^*_0$, and $\mathfrak{h}^* \supset i\mathfrak{h}^*_{\mathbb{R}} = \mathfrak{t}^*_0 \oplus i\mathfrak{a}^*_0$. We call $\mathfrak{h}_{\mathbb{R}}$ (respectively $i\mathfrak{h}^*_{\mathbb{R}}$) the canonical real (respectively imaginary) part of $\mathfrak{h}^*$.
For $\lambda \in \mathfrak{h}^*$ write $\lambda = Re(\lambda) + iIm(\lambda)$ with respect to this decomposition.

Note that $\mathfrak{h}^*_{\mathbb{R}}$ is invariant by $W = W(\mathfrak{g}, \mathfrak{h})$. Recall by the Harish-Chandra homomorphism $\mathfrak{z}(\mathfrak{g}) \simeq S(\mathfrak{h})^W$ and $Hom(\mathfrak{z}(\mathfrak{g}), \mathbb{C}) \simeq \mathfrak{h}^*/W$.

Definition. An infinitesimal character λ, i.e. an element of $Hom(\mathfrak{z}(\mathfrak{g}), \mathbb{C})$, is *real* if it corresponds to an element of $\mathfrak{h}^*_{\mathbb{R}}$ under the Harish-Chandra homomorphism.

For example the infinitesimal character of the discrete series representation are real (contained in $i\mathfrak{t}^*_o$), and the infinitesimal character of the (tempered) minimal principal series are imaginary (in $i\mathfrak{a}^*_0$).

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Cohomological Parabolic Induction

This produces unitary representations from the centralizer of a compact torus.
Idea: This should produce all representation of real infinitesimal character from representations of infinitesimal character 0 (although this is not strictly true) or from representations with imaginary infinitesimal character.

The work of Salamanca-Riba and Vogan is concerned with trying to find a version of this idea which is true.

Start with $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u} \subset \mathfrak{g}$, $\overline{\mathfrak{q}} \cap \mathfrak{q} = \mathfrak{l}$, and $L = Norm_G(\mathfrak{q})$. (For notation and more details on this construction, see [K].) For example if $G = U(p, q)$ then $L = \prod U(p_i, q_i)$. Let $S = dim(\mathfrak{u} \cap \mathfrak{k})$, and consider the functor $\mathcal{L}^S_{\mathfrak{q}}$, which takes $(\mathfrak{l}, L \cap K)$-modules to $(\mathfrak{g}, K)$-modules. It takes infinitesimal character γ_L for L to infinitesimal character $\gamma_L + \rho(\mathfrak{u})$ for $\mathfrak{g}$. If π_L has lowest $L \cap K$-type μ_L then $\mathcal{L}^S_{\mathfrak{q}}(\pi_L)$ may contain the K-type $\mu_L + 2\rho(\mathfrak{u} \cap \mathfrak{p})$ (note that $2\rho(\mathfrak{u} \cap \mathfrak{p})$ is the weight of the one-dimensional representation $\Lambda^{top}(\mathfrak{u} \cap \mathfrak{p})$.)

Now suppose $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ is θ-stable, and τ_L is a unitary representation of L. Let T_L be the compact part of the center of L (analogous to the A in $P = MAN$). Let $\lambda_L \in \mathfrak{t}^*_\ell$ be the character of τ_L on T_L. Assume:

(1) $< \lambda_L + \rho(\mathfrak{u}), \alpha >> 0$ for all $\alpha \in \Delta(\mathfrak{u})$ (here $\rho(\mathfrak{u})$ is one half the sum of the weights of T_L on $\mathfrak{u}$).
(2) (infinitesimal character of τ_L) restricted to $[\ell, \ell]$ is purely imaginary.

The following theorem is a version of Theorem 2.

Theorem 3.

(1) $\mathcal{L}^S_{\mathfrak{q}}(\tau_L)$ *is irreducible and tempered*
(2) $\mathcal{L}^S_{\mathfrak{q}}(\tau_L) \simeq \mathcal{L}^S_{\mathfrak{q}'}(\tau'_{L'})$ *if and only if* $(L, \tau) \sim (L', \tau')$.
(3) *Every irreducible tempered representation appears this way.*

Note. The conditions imply τ_L is tempered.

Note. Theorem 2 (reducing to real infinitesimal character) together with Theorem 3 (reducing to imaginary infinitesimal character) reduces the classification of tempered representations to those with infinitesimal character 0. This forces G to be quasisplit, containing a Borel subgroup $B = MAN$, and a representation $Ind^G_B(\delta \otimes 1 \otimes 1)$. Here δ is a representation of M/M_0, and the reducibility of these induced representations is known by work of Knapp and Stein. This gives a complete classification of tempered representations (equivalent to [KZ], but not by the same method).

We try to do something similar for unitary representations.

Let $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$, π_L, λ_L on T_L be as before. We now strenthen the positivity conditions: assume

$$\boxed{< \lambda_L, \alpha >> 0 \quad \text{for all weights } \alpha \text{ of } \mathfrak{t} \text{ in } \mathfrak{u}}$$

This is stronger than the condition $< \lambda_L + \rho(\mathfrak{u}), \alpha >> 0$ from before. We replace the condition of purely imaginary infinitesimal character (i.e. $Re(\lambda) = 0$) with:

$$Re(\text{infinitesimal character of } \pi_L \text{ on}[\ell, \ell]) \in \text{convex hull of } W(\ell, \mathfrak{h}) \cdot \rho_L$$

This allows the real part of λ to be small but non-zero. For example it allows for the trivial representation of L.

Theorem.

(1) $\mathcal{L}^S_{\mathfrak{q}}(\pi_L)$ *is irreducible and unitary*

(2) $\mathcal{L}^S_{\mathfrak{q}}(\pi_L) \simeq \mathcal{L}^S_{\mathfrak{q}'}(\pi'_{L'})$ *if and only if* $(L, \pi_L) \sim (L', \pi'_{L'})$

Main Conjecture:. *Every unitary representation occurs this way.*

This theorem and conjecture, together with the Harish-Chandra/Knapp result, reduce the problem of computing $\hat{G}_u$ to the case of

$$\text{real infinitesimal character } \lambda \in \text{the convex hull of } \rho$$

This is always the hardest part. One advantage of this conjecture is that it comes with an idea of a proof: Parthasarathy's Dirac inequality, together with a conjectural strengthening of it would imply the conjecture above.

Recall $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ comes with a non-degenerate symmetric form which is positive definite on $\mathfrak{p}_0$, negative definite on $\mathfrak{k}_0$ and $\mathfrak{k}_0 \perp \mathfrak{p}_0$. The adjoint action defines a map $Ad : K \to O(\mathfrak{p}_0)$. Suppose for simplicity that G is connected. Then the spin cover of $SO(\mathfrak{p}_0)$ pulls back to a two-fold cover $\tilde{K}$ of K. Composing the map $\tilde{K} \to Spin(\mathfrak{p}_0)$ with the spin representation, gives an action of $\tilde{K}$ on the spin module $Cliff(\mathfrak{p}_0)$, the Clifford algebra of $\mathfrak{p}_0$. Recall $Cliff(\mathfrak{p}_0)$ has a positive definite $Spin(\mathfrak{p}_0)$-invariant Hermition form.

Now if V is irreducible and unitary, the positive definite invariant Hermitian form gives a positive definite invariant Hermition form on $V \otimes S$, and we construct the Dirac operator $D : V \otimes S \to V \otimes S$. Explicitly

$$D(v \otimes s) = \sum_{i=1}^{n} X_i v \otimes X_i s$$

Here $\{X_i\}$ is an orthogonal basis of $\mathfrak{p}_0$, and the action on v is the $\mathfrak{g}$-action, the action on s is via the Clifford algebra action.

Now suppose $\tilde{u}$ is the highest weight of an irreducible $\tilde{K}$-representation appearing in $V \otimes S$. Then

Theorem. *(Parthasarathy)*

If V has infinitesimal character character $\gamma \in \mathfrak{h}^*$, then

$$< \tilde{u} + \rho_c, \tilde{u} + \rho_c > \geq < \gamma, \gamma >$$

Conjecture. *The inequality in the previous Theorem may be replaced by:*

$$Re(\gamma) \in \textit{ the convex hull of } W(\mathfrak{g}, \mathfrak{h}) \cdot (\tilde{u} + \rho_c)$$

Note. This conjecture implies the main conjecture.

References

[K] A. Knapp, *Representation Theory of Real Semisimple Groups: an Overview Based on Examples*, Princeton University Press, Princeton, New Jersey, 1986.

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