Representations of $p$-adic groups

Jessica Fintzen

University of Cambridge, Duke University and IAS

November 2020
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Suppose $G$ splits over a tame extension of $F$ and $p \nmid |W|$, then Yu’s construction yields all supercuspidal representations.
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<table>
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<th>type</th>
<th>$A_n (n \geq 1)$</th>
<th>$B_n, C_n (n \geq 2)$</th>
<th>$D_n (n \geq 3)$</th>
<th>$E_6$</th>
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<td>W</td>
<td>$</td>
<td>$(n + 1)!$</td>
<td>$2^n \cdot n!$</td>
</tr>
</tbody>
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<tr>
<th>type</th>
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<th>$E_8$</th>
<th>$F_4$</th>
<th>$G_2$</th>
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<tbody>
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<td>W</td>
<td>$</td>
<td>$2^{10} \cdot 3^4 \cdot 5 \cdot 7$</td>
<td>$2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$</td>
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</tbody>
</table>

**Diagram:**

- **Depth**
  - $0$ prime $p$
  - $p$ large
  - $p$ very large

- **Exhaustion**
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Suppose $G$ splits over a tame extension of $F$ and $p \nmid |W|$, then Yu’s construction yields all supercuspidal representations.

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A construction analogous to Yu’s construction yields all cuspidal $\overline{F}_\ell$-representations if $p \nmid |W|$ (and $G$ is tame).
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1. Construct a representation $\rho_K$ of a compact (mod center) subgroup $K \subset G$ (e.g. $K = \text{SL}_n(\mathbb{Z}_p)$ inside $G = \text{SL}_n(\mathbb{Q}_p)$).
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2. Build a representation of $G$ from the representation $\rho_K$ (keyword: compact-induction).
Example of a supercuspidal representation

\[ G = \text{SL}_2(F), \]
Example of a supercuspidal representation

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\[ \simeq \begin{pmatrix} 0 & \mathbb{F}_q \\ \mathbb{F}_q & 0 \end{pmatrix} \]
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Supercuspidal representation:

\[ \text{c-ind}_{K}^{G} \rho_K = \left\{ f : G \to \mathbb{C} \middle| f(kg) = \rho_K(k)f(g) \ \forall g \in G, \ k \in K \right\} \]
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\[ G\text{-action: } g.f(\star) = f(\star \cdot g) \]
Yu’s construction and my exhaustion result

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\[ \rho_K : \left( \begin{array}{cc} 1 + p & p \\ \mathcal{O} & 1 + p \end{array} \right) \rightarrow \left( \begin{array}{cc} 1 + p & p \\ \mathcal{O} & 1 + p \end{array} \right) / \left( \begin{array}{cc} 1 + p & p^2 \\ \mathcal{O} & 1 + p \end{array} \right) \]
\[ \sim \left( \begin{array}{cc} 0 & F_q \\ F_q & 0 \end{array} \right) \rightarrow F_q \rightarrow \mathbb{C}^* \]
\[ \left( \begin{array}{cc} 0 & a \\ b & 0 \end{array} \right) \mapsto a + b \]

Supercuspidal representation:

\[ \text{c-ind}_{\rho_K}^G = \left\{ f : G \rightarrow \mathbb{C} \left| \begin{array}{c} f(kg) = \rho_K(k)f(g) \ \forall g \in G, k \in K \\ f \ \text{compactly supported} \end{array} \right\} \right\} \]

\[ G \text{-action: } g.f(*) = f(* \cdot g) \]
Yu’s construction and my exhaustion result

\[ G = SL_2(F), \]
\[ x \in B(G), r = 0.5, \]
character \( \rho_K \)

(Reeder–)Yu

Example of a supercuspidal representation

\[ G = SL_2(F), K = \left( \begin{array}{cc} 1 + p & p \\ \mathcal{O} & 1 + p \end{array} \right) \times \{ \pm 1 \} \]
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\[ G_{x,0.5} \]
\[ G_{x,0.5} / G_{x,0.5+} \]
\[ \rho_K : \left( \begin{array}{cc} 1 + p & p \\ \mathcal{O} & 1 + p \end{array} \right) 
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\[ \simeq \left( \begin{array}{cc} 0 & \mathbb{F}_q \\ \mathbb{F}_q & 0 \end{array} \right) \rightarrow \mathbb{F}_q \rightarrow \mathbb{C}^* \]
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Supercuspidal representation:

\[ c\text{-}\text{ind}_{K\rho_K}^G(f : G \rightarrow \mathbb{C} \mid f(kg) = \rho_K(k)f(g) \ \forall g \in G, k \in K \text{ } f \text{ compactly supported} \} \]

G-action: \( g.f(\ast) = f(\ast \cdot g) \)
Yu’s construction and my exhaustion result

\[ G = \text{SL}_2(F), \quad x \in \mathcal{B}(G), \quad r = 0.5, \quad \text{character } \rho_K \]

(Reeder–)Yu

\[ \pi := \text{c-ind}_G^K \rho_K, \quad K = G_{x,0.5} \times \{ \pm 1 \} \]

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Jessica Fintzen

Representations of \( p \)-adic groups
Yu’s construction and my exhaustion result

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e.g. twist of

\[ \begin{pmatrix}
* & * & * & * \\
* & * & * & * \\
* & * & * & * \\
* & * & * & *
\end{pmatrix}
\]

\[ \supseteq
\begin{pmatrix}
* & * & 0 & 0 \\
* & * & 0 & 0 \\
0 & 0 & * & * \\
0 & 0 & * & *
\end{pmatrix}
\]

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\[ \text{GL}_4 = \frac{\begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix}}{\begin{pmatrix} * & * & 0 & 0 \\ * & * & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix}} \supseteq \frac{\begin{pmatrix} * & * & 0 & 0 \\ * & * & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix}}{\begin{pmatrix} * & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix}} \]
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\( K, \rho_K \) such that \( \pi := c\text{-ind}^G_K \rho_K \) is supercuspidal

\( r_1, \phi_1, G_2 \sim \text{“Cent}(\phi_1)\text{”}, \)
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Yu’s construction and my exhaustion result

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$G = G_1 \supsetneq G_2 \supsetneq \ldots \supsetneq G_n \supsetneq G_{n+1}$, $x \in \mathcal{B}(G)$, $\rho$, $r_1 > r_2 > \ldots > r_n > 0$, characters $\phi_1, \phi_2, \ldots, \phi_n$

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Proposition 3 (F., Aug 2019)
There exists a counterexample to the key ingredient of Yu’s proof (Yu’s Prop 14.1 and Thm 14.2, which were based on a misprint).

Theorem 4 (F., Aug 2019)
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There exists a character $\epsilon : K_{Yu} \to \{\pm 1\}$ such that Yu’s Prop 14.1 and Thm 14.2 are satisfied for the twisted representation $\epsilon \rho_{K_{Yu}}$ of $K_{Yu}$. 
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Applications of Theorem 5

- Formula for Harish-Chandra character of these supercuspidal representations (Spice, in progress)
- Candidate for local Langlands correspondence for non-singular representations (Kaletha, Dec 2019)
- Character identities for the LLC for regular supercuspidal representations (in progress)
- Hecke-algebra identities (hope)
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- Candidate for local Langlands correspondence for non-singular representations (Kaletha, Dec 2019)
- Character identities for the LLC for regular supercuspidal representations (in progress)
Theorem 5 (F.–Kaletha–Spice, 2019/2020)

There exists a character $\epsilon : K_{Yu} \to \{\pm 1\}$ such that Yu’s Prop 14.1 and Thm 14.2 are satisfied for the twisted representation $\epsilon \rho_{K_{Yu}}$ of $K_{Yu}$. In particular, $\text{c-ind}_{K_{Yu}}^{G} \epsilon \rho_{K_{Yu}}$ is supercuspidal.

Applications of Theorem 5

- Formula for Harish-Chandra character of these supercuspidal representations (Spice, in progress)
- Candidate for local Langlands correspondence for non-singular representations (Kaletha, Dec 2019)
- Character identities for the LLC for regular supercuspidal representations (in progress)
- Hecke-algebra identities (hope)
The quadratic character $\epsilon$

**Theorem 5’ (F.–Kaletha–Spice, 2019/2020)**

Let $G$ be adjoint, $M$ a twisted Levi subgroup of $G$ that splits over a tamely ramified extension of $F$ (given by a generic element),

There is an explicitly constructed sign character $\epsilon$ of $\mathcal{M}$ with the following property:

For every tame maximal torus $T \in \mathcal{M}$ with $x \in B_p M$, the restriction of $\epsilon$ to $T$ equals a given quadratic character $\epsilon_1 \epsilon_2 \epsilon_3$. 

Construction of $\epsilon$ 

Jessica Fintzen  
Representations of $p$-adic groups
The quadratic character $\epsilon$

**Theorem 5’ (F.–Kaletha–Spice, 2019/2020)**

Let $G$ be adjoint, $M$ a twisted Levi subgroup of $G$ that splits over a tamely ramified extension of $F$ (given by a generic element), $p \neq 2$, $x \in B(M, F) \subset B(G, F)$. 

There is an explicitly constructed sign character $\epsilon_{G}\{M\}$ with the following property:

For every tame maximal torus $T \in M$ with $x \in B(M, F) \subset B(G, F)$, the restriction of $\epsilon_{G}\{M\}$ to $T \cap F_{q}$ equals a given quadratic character $\epsilon_{G}\{M\}$.
**The quadratic character $\epsilon$**

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There is an explicitly constructed sign character $\epsilon_{x}^{G/M} : M_{x} \rightarrow M_{x}/M_{x,0+} \rightarrow \{ \pm 1 \}$ with the following property:
The quadratic character $\epsilon$

**Theorem 5’ (F.–Kaletha–Spice, 2019/2020)**

Let $G$ be adjoint, $M$ a twisted Levi subgroup of $G$ that splits over a tamely ramified extension of $F$ (given by a generic element), $p \neq 2$, $x \in \mathcal{B}(M, F) \subset \mathcal{B}(G, F)$.

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The quadratic character $\epsilon$

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\[
\epsilon_{\#}^{G/M}(\gamma) = \prod_{\alpha \in R(T, G/M)_{\text{asym}}/(\Gamma \times \{\pm 1\}) \atop s \in \text{ord}_{x}(\alpha)} \text{sgn}_{k_{\alpha}}(\alpha(\gamma)) \cdot \prod_{\alpha \in R(T, G/M)_{\text{sym,unram}}/\Gamma \atop s \in \text{ord}_{x}(\alpha)} \text{sgn}_{k_{\alpha}^{-1}}(\alpha(\gamma))
\]

\[
\epsilon_{b,0}^{G/M}(\gamma) = \prod_{\alpha \in R(T, G/M)_{\text{asym}}/(\Gamma \times \{\pm 1\}) \atop \alpha_{0} \in R(Z_{M}, G/M)_{\text{sym,ram}} \atop 2|e(\alpha/\alpha_{0})} \text{sgn}_{k_{\alpha}}(\alpha(\gamma)) \cdot \prod_{\alpha \in R(T, G/M)_{\text{sym,unram}}/\Gamma \atop \alpha_{0} \in R(Z_{M}, G/M)_{\text{sym,ram}} \atop 2|e(\alpha/\alpha_{0})} \text{sgn}_{k_{\alpha}^{-1}}(\alpha(\gamma))
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Let $G$ be adjoint, $M$ a twisted Levi subgroup of $G$ that splits over a tamely ramified extension of $F$ (given by a generic element), $p \neq 2$, $x \in \mathcal{B}(M, F) \subset \mathcal{B}(G, F)$.

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$\epsilon_{G/M}^x : M_x \rightarrow M_x/M_{x,0^+} \rightarrow \{\pm 1\}$

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$(\epsilon_{G/M}^\# \cdot \epsilon_{G/M}^{b,0} \cdot \epsilon_{G/M}^{b,1} \cdot \epsilon_{G/M}^{b,2} \cdot \epsilon_{G/M}^f)$. 

**Construction of $\epsilon$**

$\epsilon_{G/M}^x = \epsilon_1 \cdot \epsilon_2 \cdot \epsilon_3$
The quadratic character $\epsilon$

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**Construction of $\epsilon$**

$$\epsilon_{G/M}^x = \epsilon_1 \cdot \epsilon_2 \cdot \epsilon_3$$

$$
\epsilon_1(g) = \text{sgn}_{F_q} \left( \begin{vmatrix}
\det
\end{vmatrix}_{\alpha \in R(Z_M, G)_{\text{sym,ram}}/\Gamma}
\bigoplus_{t \in (0, \frac{1}{2e_\alpha})} g_{\Gamma, \alpha_0}(F)_x, t / g_{\Gamma, \alpha_0}(F)_x, t^+ \bigoplus
\end{vmatrix}
\right)
$$
Construction of $\epsilon$

$$
\epsilon_{G/M} = \epsilon_1 \cdot \epsilon_2 \cdot \epsilon_3
$$

$$
\epsilon_1(g) = \text{sgn}_{\mathbb{F}_q} \left( \det \left( \bigoplus_{\alpha_0 \in R(Z_M, G)_{\text{sym, ram}} / \Gamma} \bigoplus_{t \in (0, \frac{1}{2e\alpha_0})} g_{\Gamma, \alpha_0}(F)_x, t / g_{\Gamma, \alpha_0}(F)_x, t + \right) \right)
$$
The quadratic character $\epsilon$ continued

Construction of $\epsilon$

$$\epsilon_{x/M}^G = \epsilon_1 \cdot \epsilon_2 \cdot \epsilon_3 : M_x \to M_{x/M_{x,0+}} =: M(\mathbb{F}_q) \to \{\pm 1\}$$

$$\epsilon_1(g) = \text{sgn}_{\mathbb{F}_q} \left( \det \left( \bigoplus_{\alpha_0 \in R(Z_M, G)_{\text{sym,ram}}/\Gamma} \bigoplus_{t \in (0, \frac{1}{2e\alpha_0})} g_{\alpha_0}(F)_x, t/g_{\alpha_0}(F)_x, t+ \right) \right)$$
The quadratic character $\epsilon$ continued

**Construction of $\epsilon$**

$$\epsilon_{x/M} = \epsilon_1 \cdot \epsilon_2 \cdot \epsilon_3 : M_x \to M_x/M_{x,0^+} =: M(\mathbb{F}_q) \to \{\pm 1\}$$

$$\epsilon_1(g) = \text{sgn}_{\mathbb{F}_q} \left( \det \left( g | \bigoplus_{\alpha_0 \in R(Z_M, G)_{\text{sym,ram}}/\Gamma} \bigoplus_{t \in (0, \frac{1}{2e\alpha_0})} g_{\Gamma.\alpha_0}(F)_{x,t}/g_{\Gamma.\alpha_0}(F)_{x,t+} \right) \right)$$

$\epsilon_2$ is constructed via the Galois hypercohomology of the complex $X^*(M) \xrightarrow{2} X^*(M)$ from explicit 1-hypercocycles via $H^1(\Gamma, X^*(M) \to X^*(M)) \to \text{Hom}(M(\mathbb{F}_q), \mathbb{F}_q^\times/(\mathbb{F}_q^\times)^2)$
The quadratic character $\epsilon$ continued

**Construction of $\epsilon$**

$$\epsilon_{G/M} = \epsilon_1 \cdot \epsilon_2 \cdot \epsilon_3 : M_x \to M_x/M_{x,0+} =: M(\mathbb{F}_q) \to \{\pm 1\}$$

$$\epsilon_1(g) = \text{sgn}_{\mathbb{F}_q} \left( \det \left( \bigoplus_{\alpha_0 \in R(Z_M, G)_{\text{sym,ram}}/\Gamma} g_{\Gamma, \alpha_0}(F)(x, t) / g_{\Gamma, \alpha_0}(F)(x, t+) \right) \right)$$

$\epsilon_2$ is constructed via the Galois hypercohomology of the complex $X^*(M) \to X^*(M)$ from explicit 1-hypercocycles via $H^1(\Gamma, X^*(M) \to X^*(M)) \to \text{Hom}(M(\mathbb{F}_q), \mathbb{F}_q^\times / (\mathbb{F}_q^\times)^2)$

$\epsilon_3$ is constructed using the spinor norm:

$$M_x \to O(W, \varphi_W)(\mathbb{F}_q) \xrightarrow{\text{spinor norm}} \mathbb{F}_q^\times / (\mathbb{F}_q^\times)^2 \to \{\pm 1\}$$
The quadratic character $\epsilon$ continued

$$
\epsilon_1(g) = \text{sgn}_{\mathbb{F}_q} \left( \det \left( g \mid \bigoplus_{\alpha_0 \in R(Z_M, G)_{\text{sym,ram}}/\Gamma} \bigoplus_{t \in (0, \frac{1}{2e\alpha_0})} \text{Gr}_{\alpha_0}(F)_x/t \bigoplus \text{Gr}_{\alpha_0}(F)_x/t+ \right) \right)
$$

$\epsilon_2$ is constructed via the Galois hypercohomology of the complex $X^*(M) \rightarrow X^*(M)$ from explicit 1-hypercocycles via $H^1(\Gamma, X^*(M) \rightarrow X^*(M)) \rightarrow \text{Hom}(M(\mathbb{F}_q), \mathbb{F}_q^\times/(\mathbb{F}_q^\times)^2)$

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$$
M_x \rightarrow O(W, \varphi_W)(\mathbb{F}_q) \overset{\text{spinor norm}}{\longrightarrow} \mathbb{F}_q^\times/(\mathbb{F}_q^\times)^2 \rightarrow \{\pm 1\}
$$
The quadratic character $\epsilon$ continued

$$\epsilon_1(g) = \text{sgn}_{\mathbb{F}_q} \left( \det \left( g \mid \bigoplus_{\alpha_0 \in R(Z_M, G)_{\text{sym,ram}}/\Gamma} \bigoplus_{t \in (0, \frac{1}{2e\alpha_0})} g_{\alpha_0}(F)_x, t / g_{\alpha_0}(F)_x, t^+ \right) \right)$$

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**Spinor norm**

1 $\to \mu_2 \to \text{Pin}(W, \varphi_W) \to O(W, \varphi_W) \to 1$ leads to

1 $\to \mu_2(\mathbb{F}_q) \to \text{Pin}(W, \varphi_W)(\mathbb{F}_q) \to O(W, \varphi_W)(\mathbb{F}_q) \to H^1(\text{Gal}(\overline{\mathbb{F}_q}, \mathbb{F}_q), \mu_2) \to \cdots$
The quadratic character $\epsilon$ continued

$$\epsilon_1(g) = \text{sgn}_{\mathbb{F}_q} \left( \det \left( g \bigg| \bigoplus_{\alpha_0 \in R(Z_M, G)_{\text{sym,ram}}} \bigoplus_{t \in (0, \frac{1}{2 e \alpha_0})} g_{\Gamma \cdot \alpha_0}(F)_x, t / g_{\Gamma \cdot \alpha_0}(F)_x, t^+ \right) \right)$$

$\epsilon_2$ is constructed via the Galois hypercohomology of the complex $X^*(M) \to X^*(M)$ from explicit 1-hypercocycles via $H^1(\Gamma, X^*(M) \to X^*(M)) \to \text{Hom}(M(\mathbb{F}_q), \mathbb{F}_q^\times / (\mathbb{F}_q^\times)^2)$

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$\epsilon_2$ is constructed via the Galois hypercohomology of the complex $X^*(M) \to X^*(M)$ from explicit 1-hypercocycles via $H^1(\Gamma, X^*(M) \to X^*(M)) \to \text{Hom}(M(F_q), F_q^\times/(F_q^\times)^2)$

$\epsilon_3$ is constructed using the spinor norm:

$M_x \to O(W, \varphi_W)(F_q) \xrightarrow{\text{spinor norm}} F_q^\times/(F_q^\times)^2 \to \{\pm1\}$

**Spinor norm**

1. $\mu_2 \to \text{Pin}(W, \varphi_W) \to O(W, \varphi_W) \to 1$ leads to
2. $1 \to \mu_2(F_q) \to \text{Pin}(W, \varphi_W)(F_q) \to O(W, \varphi_W)(F_q) \xrightarrow{\text{spinor norm}} H^1(\text{Gal}(\overline{F}_q, F_q), \mu_2) = F_q^\times/(F_q^\times)^2 \to \ldots$
The quadratic character $\epsilon$ continued

$$\epsilon_1(g) = \text{sgn}_{\mathbb{F}_q} \left( \det \left( g \bigg| \bigoplus_{\alpha_0 \in R(Z_M,G)_{\text{sym,ram}}/\Gamma} \bigoplus_{t \in (0, \frac{1}{2e\alpha_0})} g\Gamma.\alpha_0(F)x,t / g\Gamma.\alpha_0(F)_x,t+ \right) \right)$$

$\epsilon_2$ is constructed via the Galois hypercohomology of the complex $X^*(M) \rightarrow X^*(M)$ from explicit 1-hypercocycles via $H^1(\Gamma, X^*(M) \rightarrow X^*(M)) \rightarrow \text{Hom}(M(\mathbb{F}_q), \mathbb{F}_q^\times/\mathbb{F}_q^\times 2)$

$\epsilon_3$ is constructed using the spinor norm:

$$M_x \rightarrow O(W, \varphi_W)(\mathbb{F}_q) \xrightarrow{\text{spinor norm}} \mathbb{F}_q^\times/\mathbb{F}_q^\times 2 \rightarrow \{ \pm 1 \}$$

$W = \bigoplus_{\alpha_0 \in R(Z_M,G)_{\text{sym,ram}}/\Gamma} g\Gamma.\alpha_0(F)x,0 / g\Gamma.\alpha_0(F)_x,0+$

### Spinor norm

$$1 \rightarrow \mu_2 \rightarrow \text{Pin}(W, \varphi_W) \rightarrow O(W, \varphi_W) \rightarrow 1$$

leads to

$$1 \rightarrow \mu_2(\mathbb{F}_q) \rightarrow \text{Pin}(W, \varphi_W)(\mathbb{F}_q) \rightarrow O(W, \varphi_W)(\mathbb{F}_q) \xrightarrow{\text{spinor norm}} H^1(\text{Gal}(\overline{\mathbb{F}_q}, \mathbb{F}_q), \mu_2) = \mathbb{F}_q^\times/\mathbb{F}_q^\times 2 \rightarrow \ldots$$
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