Appendix B

Functional Analysis

This appendix lists results from functional analysis that are used in the book. There are many excellent texts and expositions including [Die81], [GG81], [Hor66], [RSN55], [Rud73], and [Tay58].

B.1 Definition. Compact Set
A set $S \subseteq \mathbf{R}$ is compact if, whenever

$$S \subseteq \bigcup_{\alpha} N_\alpha$$

for a collection of open intervals N_α, there is a finite subcollection $\{N_{\alpha_1}, \ldots, N_{\alpha_n}\}$ for which

$$S \subseteq \bigcup_{j=1}^{n} N_{\alpha_j}.$$

This definition generalizes to topological spaces. In the case of $\mathbf{R}$, compact sets $S \subseteq \mathbf{R}$ are precisely the closed bounded subsets of $\mathbf{R}$, e.g., [Apo57, Chapter 3].

B.2 Definition. Metric Space

a. A *metric space* is a nonempty set M and a function $\rho : M \times M \to [0, \infty)$ satisfying the following properties:

$$\begin{array}{ll} \forall x, y \in M, & \rho(x,y) \geq 0, \\ \forall x, y \in M, & \rho(x,y) = 0 \quad \text{if and only if} \quad x = y, \\ \forall x, y \in M, & \rho(x,y) = \rho(y,x), \\ \forall x, y, z \in M, & \rho(x,z) \leq \rho(x,y) + \rho(y,z). \end{array}$$

The last property is the *triangle inequality*, and the function ρ is a *metric*. An excellent reference for metric spaces is [Gle91].

b. Let $\{x_n : n \in \mathbf{Z}^d\} \subseteq M$ be a sequence. For $N = (N_1, \cdots, N_d) \in \mathbf{N}^d$, let

$$R_N = \{n = (n_1, \cdots, n_d) \in \mathbf{Z}^d : -N_j \leq n_j \leq N_j \ \text{ for } \ j = 1, \cdots, d\}.$$

Also, if $n = (n_1, \cdots, n_d)$, let $|n|^2 = \sum_{j=1}^{d} |n_j|^2$. Thus, if $N \in \mathbf{N}$, we can consider $\{n \in \mathbf{Z}^d : |n| > N\}$ or $\{n \in \mathbf{Z}^d : |n| \leq N\}$.

If $x \in M$, we shall say that $\{x_n\}$ *converges* to x, and write $\lim \rho(x_n, x) = 0$, to mean that

$$\forall \epsilon > 0, \quad \exists N = N(\epsilon) \in \mathbf{N}^d \ \text{ such that } \ \forall n \notin R_N, \quad \rho(x_n, x) < \epsilon.$$

This "rectilinear" notion of convergence is *equivalent* to the "spherical" notion that

$$\forall \epsilon > 0, \ \exists N = N(\epsilon) > 0 \ \text{ such that } \forall n \in \mathbf{Z}^d \ \text{ for which } \ |n| > N, \rho(x_n, x) < \epsilon,$$

see Exercise 5.6.

c. A sequence $\{x_n : n \in \mathbf{Z}^d\} \subseteq M$ is a *Cauchy sequence* if

$$\forall \epsilon > 0, \quad \exists N \ \text{ such that } \ \forall m, n \in \mathbf{Z}^d, \ \text{ for which } \ |m|, |n| > N, \rho(x_m, x_n) < \epsilon.$$

If M is a metric space in which every Cauchy sequence $\{x_n\}$ converges to some element $x \in M$, i.e., $\lim \rho(x_n, x) = 0$, then M is a *complete metric space*.

d. Two metric spaces M_1 and M_2, with metrics ρ_1 and ρ_2, respectively, are *isometric* if there is a bijection $f : M_1 \to M_2$ such that

$$\forall x, y \in M_1, \quad \rho_1(x, y) = \rho_2(f(x), f(y)).$$

In this case, f is an *isometry*. Recall that a function $f : X_1 \to X_2$ is *bijective* if it is *injective* and *surjective*, i.e., if it is one-to-one and onto.

e. Let M be a metric space with metric ρ. A subset $V \subseteq M$ is *closed* if, whenever $\{x_n : n \in \mathbf{Z}^d\} \subseteq V$ and $\lim \rho(x_n, x) = 0$ for some $x \in M$, we can conclude that $x \in V$. The *closure* $\overline{X}$ of a subset $X \subseteq M$ is the set of all elements $x \in M$ for which there is a sequence $\{x_n :\in \mathbf{Z}^d\} \subseteq X$ such that $\lim \rho(x_n, x) = 0$.

M is *separable* if it contains a countable *dense subset* S, i.e., a countable S with the property that

$$\forall y \in M \ \text{ and } \ \forall \epsilon > 0, \ \exists x \in S \ \text{ such that } \ \rho(x, y) < \epsilon.$$

B.3 Definition. Banach Space

a. A vector space $B \neq \{0\}$ over $\mathbf{C}$ is a *normed vector space* if there is a function $\|\ldots\| : B \to [0, \infty)$ such that

$$\begin{array}{ll} \forall x \in B, & \|x\| = 0 \quad \text{if and only if} \quad x = 0, \\ \forall x, y \in B, & \|x + y\| \leq \|x\| + \|y\|, \\ \forall c \in \mathbf{C} \ \text{ and } \ \forall x \in B, & \|cx\| = |c| \, \|x\|. \end{array}$$

The function $\|\ldots\|$ is a *norm.*

b. A normed vector space is a metric space with metric ρ defined by $\rho(x, y) = \|x - y\|$. A complete normed vector space is a *Banach space.* Let $1 \leq p \leq \infty$. $L^p(\mathbf{R})$, with L^p-norm defined in Definition A.10, is a Banach space. This is a restatement of Theorem A.18. If $1 \leq p < \infty$, then $L^p(\mathbf{R})$ is separable.

c. Let B be a normed vector space and let $\{x_n : n = 1, \ldots\} \subseteq B$. $\sum x_n$ *converges* to $x \in B$ if $\lim_{n \to \infty} \|x - \sum_{j=1}^{n} x_j\| = 0$; and $\sum x_n$ is *absolutely convergent* if $\sum \|x_n\| < \infty$. It is straightforward to prove that a normed vector space is a Banach space if and only if every absolutely convergent series is convergent, e.g., [Ben76, page 232].

d. Let B be a Banach space. A subset $V \subseteq B$ is a *linear subspace* of B if V is a vector space over $\mathbf{C}$. If $V \subseteq B$ is a linear subspace, then $\overline{V}$ is a linear subspace.

The *span* of a subset $X \subseteq B$, designated by $\mathrm{span}X$, is the set of all finite linear combinations $x \equiv \sum c_n x_n$, where $c_n \in \mathbf{C}$ and $x_n \in X$. (The notion of $\mathrm{span}X$ can be defined in any vector space.) Clearly, $\mathrm{span}X$ is a linear subspace of B, and its closure is designated by $\overline{\mathrm{span}}X$.

B.4 Definition. Hilbert Space

a. A *Hilbert space* $H \neq \{0\}$ is a Banach space for which there is a function $\langle \ldots, \ldots \rangle : H \times H \to \mathbf{C}$ such that

$$\begin{aligned} \forall x, y \in H, \quad \overline{\langle x, y \rangle} &= \langle y, x \rangle, \\ \forall x, y, z \in H, \quad \langle x + y, z \rangle &= \langle x, z \rangle + \langle y, z \rangle, \\ \forall c \in \mathbf{C} \ \text{ and } \ \forall x, y \in H, \quad \langle cx, y \rangle &= c\langle x, y \rangle, \end{aligned}$$

$$\forall x \in H, \quad \|x\| \qquad = \langle x, x\rangle^{1/2},$$

where $\|\ldots\|$ is the Banach space norm. $\langle \ldots, \ldots\rangle$ is an *inner product*.

b. $L^2(\mathbf{R})$ is a Hilbert space with inner product defined by $\langle f, g\rangle = \int f(t)\overline{g(t)}dt$, cf., [Ben76, Example I.2.4, page 234] for a structural converse of the form that every Hilbert space is some type of L^2 space. The integral is well-defined by Hölder's Inequality (Theorem A.15).

c. If H is a Hilbert space (actually, completeness is not required) then the properties of part *a* can be used to give *simple* proofs of the following facts:

$$|\langle x, y\rangle| \leq \|x\| \, \|y\| \tag{1}$$

and

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2). \tag{2}$$

(1) is the *Schwarz Inequality*, which in the case $H = L^2(\mathbf{R})$ is *Hölder's Inequality*. Of course, this does not mean there is a simple proof of Hölder's Inequality since Schwarz' Inequality *assumes* the existence of an inner product, and Hölder's Inequality *shows* the existence of an inner product for $H = L^2(\mathbf{R})$.

(2) is the *parallelogram law*. It can be shown that *a Banach space is a Hilbert space*, i.e., there is an inner product with the required properties, *if and only if the parallelogram law is valid*.

d. (NOTES – FALL 1998) IF INVARIANT SUBSPACE PROBLEM HAS BEEN SOLVED THEN STATE AND REFERENCE IT, AND TIE-IN WITH RESULTS ON $t - i$ SUBSPACE ELSEWHERE IN BOOK.

GET PRECISE STATEMENT OF GOWERS' FIELDS '98 AWARD RELATING B&H SPACES & REFERENCE IT.

B.5 Theorem. Moore-Smith Theorem

Let M be a complete metric space with metric ρ, and let $\{x_{m,n} : m, n \in \mathbf{N}\}$ be given. Assume there are sequences $\{y_m\}, \{z_n\} \subseteq M$ such that

$$\lim_{n\to\infty} \rho(x_{m,n}, y_m) = 0 \text{ uniformly in } m$$

and

$$\forall n \in \mathbf{N}, \ \lim_{m\to\infty} \rho(x_{m,n}, z_n) = 0.$$

Then there is $x \in M$ such that the limits,

$$\lim_{m\to\infty}\lim_{n\to\infty}\rho(x_{m,n},x),\ \lim_{n\to\infty}\lim_{m\to\infty}\rho(x_{m,n},x),\ \text{and}\ \lim_{m,n\to\infty}\rho(x_{m,n},x),$$

all exist and are equal to 0. The last limit signifies that

$$\forall\epsilon>0,\quad \exists N>0,\quad \text{such that}\ \ \forall m,n>N, \rho(x_{m,n},x)<\epsilon.$$

The Moore-Smith Theorem is elementary to prove, e.g., [Ben76, pages 235–237], and should be compared with LDC and its refinements, e.g., [Ben76, Chapter 6].

B.6 Definition. Continuous Linear Functions

a. Let B_1 and B_2 be non-zero normed vector spaces with norms $\|\ldots\|_{B_1}$ and $\|\ldots\|_{B_2}$, respectively. The *norm* of a linear function $L : B_1 \to B_2$ is

$$\|L\| = \sup\{\|Lx\|_{B_2} : x \in B_1 \text{ and } \|x\|_{B_1} \le 1\}. \tag{3}$$

Thus, $\|L\|$ is the smallest constant $C \geq 0$ such that $\|Lx\|_{B_2} \le C\|x\|_{B_1}$ for all $x \in B_1$.

b. Clearly, if $\|L\| < \infty$ then L is continuous. The space of continuous linear functions $L : B_1 \to B_2$ is denoted by $\mathcal{L}(B_1, B_2)$. It is not difficult to show that if B_2 is a Banach space then $\mathcal{L}(B_1, B_2)$ is a Banach space, where $L = c_1L_1 + c_2L_2$ is defined by $Lx = c_1L_1(x) + c_2L_2(x)$, and where $\|L\|$ is defined by (3).

If $B = B_1 = B_2$ we denote $\mathcal{L}(B, B)$ by $\mathcal{L}(B)$. If $B = B_1$ and $B_2 = \mathbf{C}$ we denote $\mathcal{L}(B, \mathbf{C})$ by B'. B' is the space of *continuous linear functionals* on B and is called the *dual space* of B.

c. Let B_1 and B_2 be Banach spaces, with duals B_1' and B_2', and let $L \in \mathcal{L}(B_1, B_2)$. The *adjoint* of L is the linear function

$$\begin{aligned} L^* : B_2' &\to B_1' \\ y &\longmapsto L^*y, \end{aligned}$$

where $(L^*y)(x)$ is defined to be $y(Lx)$ for all $x \in B_1$. It is easy to see that $L^* \in \mathcal{L}(B_2', B_1')$.

d. In the case of Hilbert spaces H_1 and H_2, we write

$$\forall x \in H_1 \text{ and } \forall y \in H_2, \quad \langle Lx, y\rangle_{H_2} = \langle x, L^*y\rangle_{H_1} \tag{4}$$

to define the adjoint. In (4), we have used the fact that Hilbert spaces H have the property that $H' = H$, cf., Theorem 14b.

Let $H = H_1 = H_2$ and let $L \in \mathcal{L}(H)$. L is *self-adjoint* or *Hermitian* if $L^* = L$.

It is elementary to check that $L \in \mathcal{L}(H)$ is self-adjoint if and only if $\langle Lx, x\rangle$ is real for all $x \in H$. One direction is trivial: $L = L^*$ implies $\langle Lx, x\rangle = \langle x, Lx\rangle = \overline{\langle Lx, x\rangle}$. Conversely, $\langle Lz, z\rangle$ is always real, then $\langle L(x + cy), (x + cy)\rangle = \langle x + cy, L(x + cy)\rangle$ for all $x, y \in H$ and $c \in \mathbf{C}$; and using the hypothesis again on this equality we can calculate that $\mathrm{Im}\langle cLy, x\rangle = \mathrm{Im}\langle cy, Lx\rangle$, which in turn gives $L = L^*$ by considering $c = 1$ and $c = i$.

[CLEARLY, SELF-ADJOINT IS IMPLIED BY POSITIVE SEMI-DEF – SEE PAGE 205, DEF 3.6.2; I.E., POSITIVE SEMI-DEF OPS ARE HERMITIAN.]

e. If $u \in \mathcal{L}(H)$ is a linear surjective isometry, then u is called a *unitary operator*.

f. Let $U \in \mathcal{L}(H_1, H_2)$. Equivalent to Definition 2c, U is an isometry if $\|Ux\|_{H_2} = \|x\|_{H_1}$ for each $x \in H_1$. It is not difficult to prove that *the following are equivalent*:

i. U is an isometry;

ii. U^*U is the identity mapping I on H_1;

iii. $\forall x, y \in H_1$, $\langle Ux, Uy\rangle_{H_2} = \langle x, y\rangle_{H_1}$.

cf., [Hal58, pages 142–143]. Further, if U is a surjective isometry then UU^* is the identity mapping on H_2.

g. Because of part f, $U \in \mathcal{L}(H)$ *is unitary if and only if* U^{-1} *exists on* H *and* $U^{-1} = U^*$. Thus, unitary operators are characterized by the property that $UU^* = I = U^*U$.

[HAVING THIS IN APPENDIX B ALLOWS US TO PUT MATRIX STUFF IN CH4.]

h. The Fourier transform mapping $\mathcal{F}$ on $L^2(\mathbf{R})$ (Theorem 1.10.2), the DFT mapping $\mathcal{F}_N$ on $L^2(\mathbf{Z}_N)$ (Definition 3.8.1 e),

[NB: Def 3.8.1e WILL MOVE TO CHAPTER 4]

and the Hilbert transform mapping $\mathcal{H}$ on $L^2(\mathbf{R})$ (Theorem 2.5.12) are all unitary operators.

Let M_1 and M_2 be metric spaces with metrics ρ_1 and ρ_2, respectively. A function $f : M_1 \to M_2$ is *uniformly continuous* if

$$\forall \epsilon > 0,\ \exists \delta > 0 \text{ such that } \rho_1(x, y) < \delta \text{ implies } \rho_2(f(x), f(y)) < \epsilon.$$

B.7 Theorem. Uniformly Continuous Extensions

a. Let M_1 be a metric space, let M_2 be a complete metric space, and let V be a subset of M_1. Assume that $f : V \to M_2$ is uniformly continuous. Then f has a unique uniformly continuous extension to $\overline{V} \subseteq M_1$.

b. Let B_1 and B_2 be Banach spaces, and let $L \in \mathcal{L}(B_1, B_2)$. Then L is uniformly continuous.

c. Let B_1 and B_2 be Banach spaces, and let V be a linear subspace of B_1. If $L : V \to B_2$ is a continuous linear function, then L has a unique continuous linear extension to $\overline{V}$.

The following result is also called the *Banach-Steinhaus Theorem.*

B.8 Theorem. Uniform Boundedness Principle

a. Let M be a complete metric space with metric ρ, and let $\mathcal{C}$ be a set of continuous functions $f : M \to \mathbf{C}$. Assume

$$\forall x \in M, \quad \exists C_x > 0 \text{ such that } \forall f \in \mathcal{C}, \quad |f(x)| \leq C_x.$$

Then there is a constant $C > 0$ and a nonempty open ball U, i.e., $U = \{x \in M : \rho(x, x_0) < r\}$ for some $x_0 \in M$ and $r > 0$, such that

$$\forall x \in U \text{ and } \forall f \in \mathcal{C}, \quad |f(x)| \leq C.$$

b. Let B_1 be a Banach space, let B_2 be a normed vector space, and let $\mathcal{C} \subseteq \mathcal{L}(B_1, B_2)$. Then one of the following is true:

i. $\exists C$ such that $\forall L \in \mathcal{C}, \quad \|L\| \leq C$,

or

ii. There is a nonempty set $V \subseteq B_1$ such that $\overline{V} = B_1$ and

$$\forall x \in V, \quad \sup_{L \in \mathcal{C}} \|Lx\| = \infty.$$

(V is also the intersection of a countable family of open sets.)

c. Let B_1 be a Banach space, let B_2 be a normed vector space, and let $\{L_n\} \subseteq \mathcal{L}(B_1, B_2)$ have the property that

$$\forall x \in B_1, \quad \exists Lx \in B_2 \text{ such that } \lim_{n \to \infty} \|L_n x - Lx\| = 0.$$

Then $L \in \mathcal{L}(B_1, B_2)$.

B.9 Theorem. Open Mapping Theorem
Let B_1 and B_2 be Banach spaces and let $L \in \mathcal{L}(B_1, B_2)$ be bijective. Then L^{-1} exists and $L^{-1} \in \mathcal{L}(B_2, B_1)$.

B.10 Remark. Baire Category Theorem
The Uniform Boundedness Principle and the Open Mapping Theorem both depend on completeness. In particular, they can be proved using the Baire Category Theorem which asserts that if M is a complete metric space, then the intersection of any countable family of dense open subsets of M is dense in M.

Combining the Open Mapping Theorem with Alaoglu's Theorem [Ben76, pages 246–247], we have the following result.

B.11 Theorem. Injectivity, Surjectivity, and Continuity for L, L^*, L^{-1}, and $(L^*)^{-1}$
Let B_1 and B_2 be normed vector spaces with dual Banach spaces B_1' and B_2' , and let $L \in \mathcal{L}(B_1, B_2)$.

a. $L^*(B_2') = B_1'$ if and only if L^{-1} exists and $L^{-1} \in \mathcal{L}(L(B_1), B_1)$.

b. Let B_1 and B_2 be Banach spaces. $L(B_1) = B_2$ if and only if $(L^*)^{-1}$ exists and $(L^*)^{-1} \in \mathcal{L}(L^*(B_2'), B_2')$. Further, if L^{-1} exists, then it is in $\mathcal{L}(L(B_1), B_1)$.

c. Let B_1 and B_2 be Banach spaces, let L be injective, and assume $\overline{L(B_1)} = B_2$. The following are equivalent:

i. $L(B_1) = B_2$;

ii. There is $C > 0$ such that for all $y \in B_2'$, $\|y\|_{B_2'} \leq C\|L^* y\|_{B_1'}$, i.e., L^* is an open mapping;

iii. $L^*(B_2') = B_1'$.

Let B_1 and B_2 be vector spaces, let $V \subseteq B_1$ be a linear subspace, and let $K : V \to B_2$ be a linear function. Using the Axiom of Choice, it can be shown that there is a linear function $L : B_1 \to B_2$ for which $Lx = Kx$ for all $x \in V$, e.g., [Tay58, pages 40–41]. If B_1 is a normed vector space, $B_2 = \mathbf{C}$, and K is continuous, then the following theorem provides a continuous norm preserving extension $L : B_1 \to \mathbf{C}$ of K, e.g., [Ben76, pages 244–246]. Completeness is not required as it is for Theorems B.8 and B.9.

B.12 Theorem. Hahn-Banach Theorem

Let $V \subseteq B$ be a linear subspace of the normed vector space B. If $K : V \to \mathbf{C}$ is linear and continuous, then there is $L \in B'$ such that $L = K$ on V and $\|L\| = \|K\|$.

B.13 Corollary. Existence of Continuous Linear Functionals
Let B be a normed vector space.

a. If $x, y \in B$ and $x \neq y$, then there is $L \in B'$ such that $Lx \neq Ly$.

b. Let V be a closed linear subspace of B and let $y \in B \backslash V$. There is $L \in B'$ such that $Ly = 1$ and $Lx = 0$ for all $x \in V$.

Proof. qquad **a.** Let V be the linear subspace of B generated by $x - y$, and define a linear function K on V by the rule $K(c(x-y)) \equiv c\|x-y\|$ for $c \in \mathbf{C}$. Note that $\|K\| = 1$ so that K is continuous. Apply Theorem B.12 to obtain an extension $L \in B'$. Then $0 < \|x - y\| = L(x - y) = L(x) - L(y)$. This completes part a.

b. Let $d > 0$ have the property that $\|x - y\| \geq d$ for all $x \in V$, and let $V_y = \{x + cy : x \in V \text{ and } c \in \mathbf{C}\}$. Define the linear function K on V_y by the rule $K(x + cy) = c$. Thus, $K = 0$ on V and $K(y) = 1$. Also, K is continuous on V_y since $|K(x+cy)| = d^{-1}|c|d \leq d^{-1}|c|\|c^{-1}x + y\| = d^{-1}\|x + cy\|$. The result follows from Theorem B.12.□

The Hahn-Banach Theorem allows us to assert that if $\{x_n : n \in \mathbf{Z}^d\} \subseteq B$, a Banach space, then $\overline{\text{span}}\{x_n\} = B$ if and only if, when $Lx_n = 0$ for all n and for a given $L \in B'$, we can conclude that $L = 0$. In fact, the density of the span is a consequence of Corollary 13b; and the other direction follows by choosing $L \in B'$ and $y \in B$ such that $Ly \neq 0$ and $Lx_n = 0$ for all n, and then applying the density hypothesis and continuity of L to obtain the desired contradiction.

The equivalent assertion for Hilbert space H is that $\overline{\text{span}}\{x_n\} = H$ if and only if, when $\langle y, x_n \rangle = 0$ for all n and for a given $y \in H$, we conclude that $y = 0$, cf., Theorem B.23d,e.

B.14 Theorem. L^p-Duality Theorem

a. $L^1(\mathbf{R})' = L^\infty(\mathbf{R})$, where $g : L^1(\mathbf{R}) \to \mathbf{C}$ is well-defined by $g(f) = \int g(t)f(t)\,dt$ for all $f \in L^1(\mathbf{R})$.

b. Let $1 < p < \infty$ and define p' by $\frac{1}{p} + \frac{1}{p'} = 1$. Then $L^p(\mathbf{R})' = L^{p'}(\mathbf{R})$, where $g : L^p(\mathbf{R}) \to \mathbf{C}$ is well-defined by $g(f) = \int g(t)f(t)dt$ for all $f \in L^p(\mathbf{R})$. In particular, the Hilbert space $H = L^2(\mathbf{R})$ has the property that

$L^2(\mathbf{R})' = L^2(\mathbf{R})$.

c. $L^\infty(\mathbf{R})'$ is the space of finitely additive bounded measures on the Borel algebra $\mathcal{B}(\mathbf{R})$, e.g., Chapter IV of N. Dunford and J. T. Schwartz, Linear Operators, Part I, John Wiley and Sons, New York, 1957. ($\mathcal{B}(\mathbf{R})$ was defined in Remark 2.7.4.)

B.15 Remark. Riesz, Helly, and Hahn-Banach Theorems

a. F. Riesz introduced the L^p spaces, $p \neq 2$, in 1910, and proved the L^p-Duality Theorem (Theorem B.14*b*). Using a complicated argument with Lagrange multipliers, he then proved the following theorem. Let $p > 1$, and let $\{f_n\} \subseteq L^p(\mathbf{R})$ and $\{c_n\} \subseteq \mathbf{C}$ be given sequences; then

$$\exists f \in L^{p'}(\mathbf{R}) \text{ such that } \forall n, \quad c_n = \int f(t) f_n(t) dt \tag{5}$$

if and only if there is $C > 0$ such that for all finite sequences $\{\lambda_n\} \subseteq \mathbf{C}$,

$$|\sum \lambda_n c_n| \leq C \| \sum \lambda_n f_n \|_{L^p(\mathbf{R})}. \tag{6}$$

The proof that (5) implies (6) is clear. If $p = 2$ and $\{f_n\}$ is orthonormal then Riesz' Theorem is equivalent to the Riesz-Fischer Theorem (Remark 3.4.11 *c*).

b. In 1912, the Austrian mathematician E. Helly (1884–1943) (who spent his last years in the USA as an actuary) gave an elementary proof that (6) is a sufficient condition for the validity of (5). In so doing, he proved a result which is *equivalent* to the Hahn–Banach Theorem [Die81].

c. The proof of (5) from (6) is a consequence of the Hahn-Banach Theorem as follows. Assume (6) and let $V = \text{span}\{f_n\} \subseteq L^p(\mathbf{R})$. Define $K : V \to \mathbf{C}$ by the rule that $K(\sum \lambda_n f_n) = \sum \lambda_n c_n$ for any finite sequence $\{\lambda_n\} \subseteq \mathbf{C}$. Clearly, K is linear, and it is continuous on V by (6). Thus, by Theorem B.12, there is $L \in L^{p'}(\mathbf{R})$ for which $L = K$ on V. Hence, $c_n = f(f_n) = \int f(t) f_n(t) dt$ for each n, where we have denoted L by f.

B.16 Remark. Embeddings of Dual Spaces

a. Let $B_1 \subseteq B_2$, where B_1 and B_2 are normed vector spaces; and let $I : B_1 \to B_2$ be the identity mapping with adjoint $I^* : B_2' \to B_1'$. By definition,

$$\forall x \in B_1 \text{ and } \forall y \in B_2', \quad (I^* y)(x) = y(x),$$

i.e., $I^*y = y$ on $B_1 \subseteq B_2$.

b. Assume I, and hence I^*, are continuous. Note that if $y \in B_2'$, then $y|_{B_1}$, the restriction of y to B_1, is an element of B_1'. To see this first note that since $B_1 \subseteq B_2$ and $y \in B_2'$, then $y|_{B_1}$ is linear on B_1. $y|_{B_1}$ is also continuous on B_1 because of the continuity of I. In fact, since y is continuous on B_1 with the induced topology from B_2, then it is continuous on B_1 with its given norm convergence because this latter topology is stronger (finer) than the B_2 criterion. (Continuity of a function for a given topology on its domain implies continuity for any stronger topology on that domain.)

c. To say that B_2' is *embedded* in B_1', in which case we write $B_2' \subseteq B_1'$, we mean that I^* is a continuous injection. This means that whenever $I^*y = 0 \in B_1'$, then $y = 0$, i.e., $y(x) = 0$ for all $x \in B_2$.

d. In the setting of parts *a* and *b*, we further assume $\overline{B}_1 = B_2$. Let $y \in B_2'$ have the property that $I^*y = 0 \in B_1'$. Suppose $x \in B_2$ and $\lim_{n\to\infty} \|x_n - x\|_{B_2} = 0$, where $\{x_n\} \subseteq B_1$. Then $\lim_{n\to\infty} y(x_n) = y(x)$, and $y(x_n) = (I^*y)(x_n) = 0$. Thus, $y(x) = 0$, and so $y \in B_2'$ is the 0-element. Hence, I^* is a continuous injection. I^* is also the identity function, i.e., for all $y \in B_2'$, $I^*y = y$ on a dense linear subspace of B_2.

e. We can summarize parts *a*–*d* by the following Embedding Theorem: let B_1 and B_2 be normed vector spaces; if $B_1 \subseteq B_2$ in the sense that the identity map $I : B_1 \to B_2$ is continuous, and if $\overline{B}_1 = B_2$, then $B_2' \subseteq B_1'$. Several embeddings are listed in Definition 2.4.5 *c* and Example 2.4.6 *b,d,f*. Some of these involve topological structures more general than Banach spaces. This generality does not give rise to any problems since the Embedding Theorem is true quite generally.

B.17 Example. $C_b(\mathbf{R}), M_b(\mathbf{R})$, and Duality

a. Let $C_b(\mathbf{R})$ be the Banach space of continuous bounded functions on $\mathbf{R}$ taken with the L^∞-norm $\|\ldots\|_{L^\infty(\mathbf{R})}$; and let $C_0(\mathbf{R})$ be the closed linear subspace of $C_b(\mathbf{R})$ whose elements f satisfy the condition that $\lim_{|t|\to\infty} f(t) = 0$. Recall from Theorem 2.7.3 (RRT) that $C_0(\mathbf{R})' = M_b(\mathbf{R})$. $C_b(\mathbf{R})$ is a closed linear subspace of $L^\infty(\mathbf{R})$, and $C_b(\mathbf{R})'$ is the space of finitely additive bounded regular measures on the Borel algebra $\mathcal{B}(\mathbf{R})$, cf., Theorem B.14*c* and the reference given there.

b. Since the continuous inclusion map $I : C_0(\mathbf{R}) \to C_b(\mathbf{R})$ is not dense, we can *not* conclude that $C_b(\mathbf{R})' \subseteq M_b(\mathbf{R})$, as is apparent from the characterizations of $C_0(\mathbf{R})'$ and $C_b(\mathbf{R})'$ in part *a*, cf., Remark B.16.

c. The characterizations of $C_0(\mathbf{R})'$ and $C_b(\mathbf{R})'$ in part *a* *do* imply that

$$M_b(\mathbf{R}) \subseteq C_b(\mathbf{R})'. \tag{7}$$

In this regard, if $\mu \in C_0(\mathbf{R})'$, then μ extends to an element $\mu_e \in C_b(\mathbf{R})'$ by the Hahn-Banach Theorem. Of course, there is no apriori guarantee of a unique extension.

On the other hand, and without invoking the characterization of $C_b(\mathbf{R})'$ stated in part *a*, we can see the validity of (7) in the following way.

If $f \in C_b(\mathbf{R})$ we can choose $\{f_n\} \subseteq C_0(\mathbf{R})$ for which $\lim_{n\to\infty} f_n = f$ pointwise on $\mathbf{R}$ and $\sup_n \|f_n\|_{L^\infty(\mathbf{R})} = \|f\|_{L^\infty(\mathbf{R})} < \infty$. Then we apply a form of LDC for $L^1_\mu(\mathbf{R})$ which allows us to assert that $f \in L^1_{|\mu|}(\mathbf{R})$ and $\lim_{n\to\infty} \|f_n - f\|_{L^1_{|\mu|}(\mathbf{R})} = 0$, e.g., [Ben76, Chapter 3]. The integral $\mu(f)$ is well-defined, i.e., it is independent of the sequence $\{f_n\} \subseteq C_0(\mathbf{R})$. Further, $\mu : C_b(\mathbf{R}) \to \mathbf{C}$ is linear. To prove the continuity of μ, let $f \in C_b(\mathbf{R})$, let $\epsilon > 0$, and choose $\{f_n\}$ as above. Then

$$\exists N > 0 \text{ such that } \forall n \geq N, \quad |\mu(f - f_n)| < \epsilon;$$

and so, for such n,

$$|\mu(f)| \leq \epsilon + |\mu(f_n)| \leq \epsilon + \|\mu\| \|f\|_{L^\infty(\mathbf{R})}.$$

This is true for all $\epsilon > 0$, and so $\mu \in C_b(\mathbf{R})'$. We designate μ so defined on $C_b(\mathbf{R})$ by μ^*.

The inclusion (7) is accomplished by the mapping $\mu \mapsto \mu^*$. The fact that many extensions μ_e of μ exist does not contradict (7). In fact, $\nu_e \equiv \mu_e - \mu^* \in C_b(\mathbf{R})'$ vanishes on $C_0(\mathbf{R})$; and if ν_e is not identically 0 on $C_b(\mathbf{R})$, then μ_e is not countably additive on $\mathcal{B}(\mathbf{R})$ and so it does not correspond to an element of $M_b(\mathbf{R})$.

Because of our presentation in Chapter 5, we shall devote the remainder of Appendix B to the special case of Hilbert spaces (Definition B.4).

B.18 Definition. Orthonormal Set and Orthonormal Basis (ONB)

a. Let H be a Hilbert space. Elements $x, y \in H$ are *orthogonal* if $\langle x, y\rangle = 0$; and this property is denoted by $x \perp y$. An element $x \in H$ is *orthogonal to the set* $S \subseteq H$, denoted by $x \perp S$, if $\langle x, y\rangle = 0$ for all $y \in S$. A set $S \subseteq H$ is an *orthogonal set* if $x \perp y$ for all $x, y \in S$ for which $x \neq y$. A set $S \subseteq H$ is an *orthonormal set* if it is orthogonal and if $\|x\| = 1$ for each $x \in S$.

b. A countable orthonormal set $S = \{x_n : n \in \mathbf{Z}^d\}$ is an *orthonormal basis* (ONB) for H if

$$\forall x \in H, \quad \exists \{c_n : n \in \mathbf{Z}^d\} \subseteq \mathbf{C} \; \text{ such that} x = \sum a_n x_n \; \text{ in } \; H,$$

cf., Definition 3.4.10 *a* for the special case $H = L^2(\mathbf{T}_{2\Omega})$.

B.19 Proposition.
Let $S = \{x_\alpha\}$ be an orthonormal set in a separable Hilbert space H. Then S is a countable set.
Proof. By separability, let $D = \{y_n : n \in \mathbf{Z}^d\}$ be a countable dense subset of H. Since S is orthonormal, we can assert that

$$\forall \alpha, \beta \; \text{ for which } \; \alpha \neq \beta, \quad \|x_\alpha - x_\beta\| = \sqrt{2}. \tag{8}$$

Using the density, we have

$$\forall \alpha, \; \exists n = n(\alpha) \in \mathbf{Z}^d, \text{ such that } \|x_\alpha - y_{n(\alpha)}\| < \frac{\sqrt{2}}{2}. \tag{9}$$

We are forced into choosing a different $n(\alpha)$ for each α, for otherwise, if $y_{n(\alpha)}$ corresponds to both x_α and x_β in the sense of (9), then

$$\|x_\alpha - x_\beta\| \leq \|x_\alpha - y_{n(\alpha)}\| + \|y_{n(\alpha)} - x_\beta\| < \sqrt{2},$$

and this contradicts (8). Thus, (9) gives rise to an injective mapping $S \to D$, and, hence, S is countable. □

B.20 Example. Hilbert Spaces and ONBs

a. By Definition 3.4.10 and Theorem 3.4.12, $H = L^2(\mathbf{T}_{2\Omega})$ is a Hilbert space with inner product defined by $\langle F, G\rangle = \int_{\mathbf{T}_{2\Omega}} F(\gamma)\overline{G(\gamma)}d\gamma$. The sequence $\{e^{-\pi i n\gamma/\Omega} : n \in \mathbf{Z}\}$ is an ONB for $L^2(\mathbf{T}_{2\Omega})$.

b. $\ell^2(\mathbf{Z}^d)$ is defined to be the vector space of all sequences $f : \mathbf{Z}^d \to \mathbf{C}$ with the property that

$$\|f\|_{\ell^2(\mathbf{Z}^d)} = \left(\sum |f[n]|^2\right)^{1/2} < \infty.$$

With this norm, $H = \ell^2(\mathbf{Z}^d)$ is a Hilbert space, and its inner product is given by
$$\forall f, g \in \ell^2(\mathbf{Z}^d), \quad \langle f, g\rangle = \sum f[n]\overline{g[n]}.$$
Let $u_n \in \ell^2(\mathbf{Z}^d)$ be defined by $u_n[m] = \delta(m,n)$, for $m, n \in \mathbf{Z}^d$, where
$$\delta(m,n) = \begin{cases} 1 & \text{if} \quad m = n, \\ \\ 0 & \text{if} \quad m \neq n, \end{cases}$$
cf., Definition 3.4.10 *a* where $\delta(m,n)$ is defined for $m, n \in \mathbf{Z}$. It is easy to check that the sequence $\{u_n\}$ is an ONB for $\ell^2(\mathbf{Z}^d)$.

c. As we observed in Definition 4*b*, $H = L^2(\mathbf{R})$ is a Hilbert space with inner product defined by $\langle f, g\rangle = \int f(t)\overline{g(t)}\,dt$. We noted in Remark 2.4.11 that the Hermite functions are an ONB for $L^2(\mathbf{R})$. The concept of a multiresolution analysis in wavelet theory leads to the construction of many other ONBs for $L^2(\mathbf{R})$, see Chapter 5. [GIVE MORE PRECISE REFERENCE LATER.]

d. If $H = PW_\Omega$, then H is a closed linear subspace of $L^2(\mathbf{R})$, and hence H is a Hilbert space with inner product induced from $L^2(\mathbf{R})$. The sequence $\{(\frac{1}{\sqrt{2\Omega}})\tau_{n/(2\Omega)}d_{2\pi\Omega} : n \in \mathbf{Z}\}$ is an ONB for $PW_\varnothing$, cf., the Classical Sampling Theorem (Theorem 4.10.10). [FORMLERLY THEOREM 3.10.10 IN HAA 1st.] In light of our mention of multiresolution in part *c*, we note that $PW_\varnothing$ can be considered as part of a multiresolution analysis of $L^2(\mathbf{R})$ for the so-called Shannon wavelet system, see Chapter 5. [GIVE MORE PRECISE REFERENCE LATER.]

The following is an immediate useful consequence of the Schwarz Inequality (1).

B.21 Proposition. Continuity of the Inner Product
Let H be a Hilbert space. The inner product is continuous on $H \times H$, i.e., if $\{x_n : n \in \mathbf{Z}^d\} \subseteq H$ converges to $x \in H$ and $\{y_n : n \in \mathbf{Z}^d\} \subseteq H$ converges to $y \in H$, then
$$\lim\langle x_n, y_n\rangle = \langle x, y\rangle.$$

B.22 Theorem. Consequences of Orthonormality
Let H be a Hilbert space and let $\{x_n : n \in \mathbf{Z}^d\}$ be an orthonormal sequence.

a. Bessel Inequality. The mapping

$$\begin{array}{rcl} L : H & \longrightarrow & \ell^2(\mathbf{Z}^d) \\ y & \longmapsto & \{\langle y, x_n \rangle\} \end{array} \tag{10}$$

is well-defined, linear, and continuous; in fact

$$\forall y \in H, \quad \sum |\langle y, x_n \rangle|^2 \leq \|y\|^2,$$

cf., (3.4.22).

b. For each $y \in H$, $\sum \langle y, x_n \rangle x_n$ converges in H.

c. $\sum c_n x_n$ converges in H if and only if $c = \{c_n : n \in \mathbf{Z}^d\} \in \ell^2(\mathbf{Z}^d)$.

d. If $y = \sum c_n x_n$ converges in H, then each $c_n = \langle y, x_n \rangle$.

Proof. i. Let $y \in H$, let $F \subseteq \mathbf{Z}^d$ be finite, and suppose $\{c_n : n \in F\} \subseteq \mathbf{C}$. Using orthonormality, two direct calculations yield

$$\| \sum_{n \in F} c_n x_n \|^2 = \sum_{n \in F} |c_n|^2 \tag{11}$$

and

$$0 \leq \|y - \sum_{n \in F} \langle y, x_n \rangle x_n \|^2 = \|y\|^2 - \sum_{n \in F} |\langle y, x_n \rangle|^2. \tag{12}$$

ii. Bessel's Inequality (part a) is immediate from (12). In particular $\{\langle y, x_n \rangle\} \in \ell^2(\mathbf{Z}^d)$. Since H is complete, to prove part b we need only show that

$$\{s_N\} = \Big\{ \sum_{n \in R_N} \langle y, x_n \rangle x_n \Big\}, \quad N = (N_1, \ldots, N_d) \text{ and } N_j > 0,$$

is a Cauchy sequence in H. This is a consequence of (11) and the fact that $\{\langle y, x_n \rangle\} \in \ell^2(\mathbf{Z}^d)$, which, in turn, was a consequence of part a. Part c also follows from (11).

iii. To prove part d, we use the orthonormality and the continuity of inner products (Proposition B.21) to compute

$$\langle y, x_n \rangle = \lim \Big\langle \sum_{m \in R_N} c_m x_m, x_n \Big\rangle = c_n. \quad \square$$

The following result is also elementary to verify. One efficient route is to prove the implications a implies b implies c implies d implies e implies a. cf., Theorem 3.4.12.

B.23 Theorem. Parseval Formula and ONB
Let H be a Hilbert space and let $\{x_n : n \in \mathbf{Z}^d\}$ be an orthonormal sequence. The following are equivalent.

a. ON $\{x_n\}$ is an ONB for H.

b. Parseval formula.

$$\forall x, y \in H, \quad \langle x, y\rangle = \sum \langle x, x_n\rangle \overline{\langle y, x_n\rangle}.$$

c. The mapping L of (10) (in Theorem B.22a) is a linear surjective isometry, and, in fact,

$$\forall y \in H, \quad \|y\| = \left(\sum |\langle y, x_n\rangle|^2\right)^{1/2}.$$

d. $\overline{\mathrm{span}}\{x_n\} = H$.

e. If $\langle y, x_n\rangle = 0$ for each $n \in \mathbf{Z}^d$ then $y = 0$.

Because of Example B.20a, the coefficients $\langle y, x_n\rangle$ for an ONB $\{x_n\} \subseteq H$ are called the *Fourier coefficients* of $y \in H$.

B.24 Theorem. Hilbert Space Fourier Series
Let H be a Hilbert space and let $\{x_n : n \in \mathbf{Z}^d\}$ be an ONB for H. Then

$$\forall y \in H, \quad y = \sum \langle y, x_n\rangle x_n \quad H.$$

Proof. If $y \in H$, then $\sum \langle y, x_n\rangle x_n = x$ in H for some $x \in H$ by Theorem B.22b. Hence,

$$\forall n \in \mathbf{Z}^d, \quad \langle x, x_n\rangle = \langle y, x_n\rangle$$

by Theorem B.22d. The result follows by Theorem B.23, using either the equivalence of parts a and c or of parts a and e. $\square$

B.25 Remark. Separability and Bases

a. By the definition of an ONB, *if H contains an* ONB*, then H is separable*. The converse is also true: *if H is a separable Hilbert space, then H contains an* ON The proof of the converse has four elementary steps. First, if $S = \{x_n : n \in \mathbf{Z}^d\}$ is a countable dense subset of H, then $\overline{\mathrm{span}}\{x_n\} = H$. Next, we choose a linearly independent subset $\{y_n\}$ of $\{x_n\}$, which also has the property that $\overline{\mathrm{span}}\{y_n\} = H$. This can be accomplished both constructively and iteratively by throwing-out those x_n which are linear combinations of finite sets $\{x_j : j \in F \text{ and } j \neq n\}$. Third, the Gram-Schmidt orthogonalization procedure, e.g., [GG81, pages 21–22], constructs $\{u_n\}$ in terms of $\{y_n\}$ with the properties that $\{u_n\}$ is orthonormal and $\overline{\text{span}}\{u_n\} = H$. Finally, we invoke Theorem B.23 to complete the proof.

Because of this remark and the focus of harmonic analysis to decompose signals in terms of sequences of prescribed harmonics, the Hilbert spaces considered in this book are all separable.

b. The notion of a basis for a Banach space B was defined in Definition 5.1.2. The situation in part a leads to the question of whether or not every separable Banach space contains a basis. Using Walsh functions, Enflo proved that there are separable Banach spaces having no basis [Enf73]. Walsh functions are defined in Definition 5...
[INSERT PRECISE REFERENCE ABOUT WHERE WALSH FUNCTIONS ARE DEFINED].

B.26 Definition. Direct Sum and Orthogonal Complement

a. Let B be a Banach space, and let $X, Y \subseteq B$ be linear subspaces of B for which $X \cap Y = \{0\}$ and $X + Y \equiv \{x + y : x \in X, y \in Y\} = B$. We denote this situation by

$$B = X \oplus Y,$$

and B is the *direct sum* of X and Y.

b. Suppose $B = X \oplus Y$. Let $z \in B$ and suppose $z = x_1 + y_1 = x_2 + y_2$, where $x_j \in X$ and $y_j \in Y$. Then $x_1 - x_2 = y_2 - y_1$. Thus, $x_1 - x_2$, $y_2 - y_1 \in X \cap Y$ and so $x_1 = x_2$ and $y_1 = y_2$. Therefore, *if $B = X \oplus Y$, then each $z \in B$ has a unique representation $z = x + y$ for some $x \in X$ and $y \in Y$.*

c. Let H be a Hilbert space, and let $X \subseteq H$ be a subset of H. The *orthogonal complement* $X^\perp$ of X is the set $\{y \in H : \forall x \in X, x \perp y\}$.

d. Let X be a closed linear subspace of H. It is not difficult to prove that *for each $z \in H$ there are unique elements $x \in X$ and $y \in X^\perp$ such that $z = x + y$.* The proof requires the following two results:

i. For each $z \in H$ there is a unique element $x \in X$ such that

$$\|z - x\| = \inf\{\|z - w\| : w \in X\};$$

ii. If $z \in H$ and $x \in X$, then $\langle z - y, w\rangle = 0$ for all $w \in X$ if and only if

$$\|z - x\| = \inf\{\|z - u\| : u \in X\}.$$

e. From part d, we see that *if* $X \neq \{0\}$ *is a closed linear subspace of* H*, then* $X^\perp$ *is a closed linear subspace of* H,

$$H = X \oplus X^\perp,$$

and $(X^\perp)^\perp = X$. We refer to $X \oplus X^\perp$ as an *orthogonal complement direct sum*, see Exercise 5.27.

Finally, we rewrite Theorem B.11 for the case of Hilbert spaces H_1 and H_2. We shall also make use of the orthogonal complement (Definition B.26) and of the range and kernel of an operator $L \in \mathcal{L}(H_1, H_2)$. The *range* of L, also called the *image* of L, is defined as $\mathcal{R}(L) = \{Lx : x \in H_1\} \subseteq H_2$; and the *kernel* of L, also called the *null space* of L, is defined as the closed linear subspace $\ker L = \{x \in H_1 : Lx = 0 \in H_2\}$.

B.27 Theorem.
Let H_1 and H_2 be Hilbert spaces, and let $L \in \mathcal{L}(H_1, H_2)$.

a. $L^*(H_2) = H_1$ if and only if L^{-1} exists and $L^{-1} \in \mathcal{L}(L(H_1), H_1)$.

b. $L(H_1) = H_2$ if and only if $(L^*)^{-1}$ exists and $(L^*)^{-1} \in \mathcal{L}(L^*(H_2), H_2)$. Further, if L^{-1} exists, then it is in $\mathcal{L}(L(H_1), H_1)$.

c. If L^{-1} exists and $L^{-1} \in \mathcal{L}(H_2, H_1)$, then $(L^*)^{-1} \in \mathcal{L}(H_1, H_2)$ and

$$(L^*)^{-1} = (L^{-1})^*.$$

d. $\ker L = (\mathcal{R}L^*)^\perp$, $\ker L^* = \mathcal{R}L)^\perp$, $\overline{\mathcal{R}L} = (\ker L^*)^\perp$, and $\overline{\mathcal{R}L^*} = (\ker L)^\perp$.

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