Finite frames and quantum detection

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Outline

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2. A quantum detection problem

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3. Equations of motion

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This is an l^2 theory, but there are relevant analogous l^∞ problems, for example finding Grassmannian frames.

PART 1: Finite frame theory

Frames

Frames $F = \{e_n\}_{n=1}^{N}$ for d-dimensional Hilbert space H, e.g., $H = \mathbb{K}^d$, where $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$.

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- $F \subseteq \mathbb{K}^d$ is A-tight if

$$\forall x \in \mathbb{K}^d,\ A\|x\|^2 = \sum_{n=1}^{N} |\langle x, e_n\rangle|^2$$

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- *Bessel map* – $L : H \longrightarrow \ell^2(\mathbb{Z}_N)$,

$$x \longmapsto \{\langle x, e_n \rangle\}.$$

- *Frame operator* – $S = L^*L : H \longrightarrow H$, in fact,

$$S(x) = \sum_{n=1}^N \langle x, e_n \rangle e_n.$$

Tight frames and applications

Theorem $\{e_n\}_{n=1}^{N} \subseteq \mathbb{K}^d$ is an A-tight frame for $\mathbb{K}^d$ $\Longleftrightarrow$

$$S = L^*L = AI : \mathbb{K}^d \longrightarrow \mathbb{K}^d.$$

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- Robust transmission of data over erasure channels such as the Internet. [Casazza, Goyal, Kelner, Kovačević]
- Multiple antenna code design for wireless communications. [Hochwald, Marzetta, T. Richardson, Sweldens, Urbanke]
- Multiple description coding. [Goyal, Heath, Kovačević, Strohmer, Vetterli]
- Quantum detection.
  - Chandler Davis - mathematics
  - Eldar, Forney, Oppenheim - signal processing
  - Brandt, Kennedy, Helstrom - quantum mechanics quantum detection

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- The geometry of finite tight frames:
  - The vertices of platonic solids are FUN-TFs.
  - Points that constitute FUN-TFs do not have to be equidistributed, e.g., ONBs, Grassmanian frames.
  - FUN-TFs can be characterized as minimizers of a "frame potential function" (with Fickus) analogous to

Coulomb's Law.

Frame force and potential energy

A force

$$F : S^{d-1} \times S^{d-1} \setminus D \longrightarrow \mathbb{R}^d$$

is a *central force* with potential

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if

$$F(a,b) = f(\|a-b\|)(a-b), \qquad P(a,b) = p(\|a-b\|).$$

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Note that

$$\nabla_a P = -F \iff p'(x) = -xf(x).$$

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- Total potential energy for the frame force of $\{x_n\}_{n=1}^N \subset S^{d-1}$

$$TFP(\{x_n\}) = \Sigma_{m=1}^N \Sigma_{n=1}^N \ |\langle x_m, x_n\rangle|^2$$

Local minimizers and frame bounds

Theorem Given d, N, and central force F. $\{x_n\}_{n=1}^N \subset (S^{d-1})^N$ a local minimizer for the total potential energy function $\Rightarrow$

$$\forall m = 1, \ldots, N, \ \exists c_m \in \mathbb{R} \text{ such that } c_m x_m = \sum_{n \neq m} F(x_m, x_n) \in \mathbb{R}^d$$

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- By Theorem and frame operator $S = AI$ characterization of A-tight we were led to definition of frame force.
- $\{x_n\}_{n=1}^N \subset \mathbb{R}^d$ with frame operator S implies

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- This Theorem is a basic input to following characterization.

Characterization of FUN-TFs

For the Hilbert space $H = \mathbb{R}^d$ and N, consider

$$\{x_n\}_1^N \in S^{d-1} \times ... \times S^{d-1}$$

and

$$TFP(\{x_n\}) = \Sigma_{m=1}^N \Sigma_{n=1}^N \, |\langle x_m, x_n \rangle|^2 \, .$$

Theorem Let $N \leq d$. The minimum value of TFP, for the frame force and N variables, is N; and the minimizers are precisely the orthonormal sets of N elements for $\mathbb{R}^d$.

Theorem Let $N \geq d$. The minimum value of TFP, for the frame force and N variables, is N^2/d; and the minimizers are precisely the FUN-TFs of N elements for $\mathbb{R}^d$.

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Problem Find these FUN-TFs analytically, effectively, and computationally.

PART 2: A quantum detection problem

Positive-operator-valued measures

Let $\mathcal{B}$ be a σ-algebra of sets of X. A *positive operator-valued measure* (POM) is a function $\Pi : \mathcal{B} \to \mathcal{L}(H)$ such that

1. $\forall U \in \mathcal{B},\ \Pi(U)$ is a positive self-adjoint operator,
2. $\Pi(\emptyset) = 0$ (zero operator),
3. $\forall$ disjoint $\{U_i\}_{i=1}^{\infty} \subset \mathcal{B}$ and $x, y \in H$,

$$\left\langle \Pi \left(\bigcup_{i=1}^{\infty} U_i \right) x, y \right\rangle = \sum_{i=1}^{\infty} \langle \Pi(U_i) x, y \rangle,$$

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- A POM Π on $\mathcal{B}$ has the property that given any fixed $x \in H$, $p_x(\cdot) = \langle x, \Pi(\cdot) x \rangle$ is a measure on $\mathcal{B}$. (Probability if $\|x\| = 1$).
- A dynamical quantity Q gives rise to a measurable space $(X, \mathcal{B})$ and POM. When measuring Q, $p_x(U)$ is the probability that the outcome of the measurement is in $U \in \mathcal{B}$.

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- Suppose the state of the electron is given by $x \in H$ with unit norm. Then the probability that the electron is found to be in the region $U \in \mathcal{B}$ is given by

$$p(U) = \langle x, \Pi(U)x \rangle = \int_U |x(t)|^2 \, dt.$$

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- Clear that Π satisfies conditions (1)-(3) for a POM. Since F is Parseval, we have condition (4) ($\Pi(X)x = \sum_{i \in X} \langle x, e_i \rangle e_i = x$). Thus Π defines a POM.

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- Conversely, let $(X, \mathcal{B})$ be a measurable space with corresponding POM Π for a d-dimensional Hilbert space H. If X is countable then there exists a subset $K \subseteq \mathbb{Z}$, a Parseval frame $\{e_i\}_{i \in K}$, and a disjoint partition $\{B_j\}_{j \in X}$ of K such that for all $j \in X$ and $y \in H$,

$$\Pi(j)y = \sum_{i \in B_j} \langle y, e_i \rangle e_i.$$

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- Our goal is to determine what state the system is in by performing a "good" measurement. That is, we want to construct a POM with outcomes $X = \mathbb{Z}_N$ such that if the state of the system is x_i for some $1 \leq i \leq N$, then

$$p_{x_i}(j) = \langle x_i, \Pi(j) x_i \rangle \approx \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

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- Since $\langle x_i, \Pi(i)x_i \rangle$ is the probability of a successful detection of the state x_i, then the probability of a detection error is given by

$$P_e = 1 - \sum_{i=1}^{N} \rho_i \langle x_i, \Pi(i)x_i \rangle.$$

Quantum detection problem

- If we construct our POM using Parseval frames, the error becomes

$$\begin{aligned} P_e &= 1 - \sum_{i=1}^{N} \rho_i \langle x_i, \Pi(i) x_i \rangle \\ &= 1 - \sum_{i=1}^{N} \rho_i \langle x_i, \langle x_i, e_i \rangle e_i \rangle \\ &= 1 - \sum_{i=1}^{N} \rho_i \, |\langle x_i, e_i \rangle|^2 \end{aligned}$$

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$$

- *Quantum detection problem:* Given a unit normed set $\{x_i\}_{i=1}^N \subset H$ and positive weights $\{\rho_i\}_{i=1}^N$ that sum to 1. Construct a Parseval frame $\{e_i\}_{i=1}^N$ that minimizes

$$P_e = 1 - \sum_{i=1}^{N} \rho_i \, |\langle x_i, e_i \rangle|^2$$

over all N-element Parseval frames. ($\{e_i\}_{i=1}^N$ exists by a compactness argument.)

Naimark theorem

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where $\mathcal{P}_H$ is the orthogonal projection $H' \to H$.

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- Minimizing P_e over all N-element Parseval frames for H is equivalent to minimizing P_e over all N-element orthonormal bases for H'.

- Thus we simplify the problem by minimizing P_e over all N-element orthonormal sets in H'.

Quantum detection error as a potential

- Treat the error term as a potential.

$$P = P_e = \sum_{i=1}^{N} \rho_i (1 - |\langle x_i, e_i' \rangle|^2) = \sum_{i=1}^{N} P_i.$$

where we have used the fact that $\sum_{i=1}^{N} \rho_i = 1$ and each

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- For $H' = \mathbb{R}^N$, we have the relation,

$$\|e_i' - x_i\|^2 = 2 - 2\langle x_i, e_i' \rangle$$

  where we have used the fact that $\|e_i'\| = \|x_i\| = 1$. We can rewrite the potential P_i as

$$P_i = \rho_i \left(1 - \left[1 - \frac{1}{2} \|x_i - e_i'\|^2 \right]^2 \right).$$

A central force corresponds to quantum detection error

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Hence, the force $F_i = -\nabla P_i$ is

$$F_i(x_i, e_i') = f_i(\|x_i - e_i'\|)(x_i - e_i') = -2\rho_i \langle x_i, e_i' \rangle (x_i - e_i'),$$

a multiple of the frame force! The total force is given by

$$F = \sum_{i=1}^{N} F_i.$$

A reformulation of the quantum detection problem

- We reformulate the quantum detection problem in terms of frame force and the Naimark Theorem.
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- The equilibrium position of the points $\{e_i'\}_{i=1}^N$ is the position where all the forces produce no net motion. In this situation, the potential P is minimized.
- For the remainder, let $\{e_i'\}_{i=1}^N$ be an ONB for $\mathbb{R}^N$ that minimizes P. Recall that $\{e_i'\}_{i=1}^N$ exists by compactness. The *quantum detection problem* is to construct or compute $\{e_i'\}_{i=1}^N$.

PART 3: Equations of motion

A parameterization of $O(N)$

- Consider the orthogonal group

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  Hence, for all $\theta \in O(N)$ there is a surjective diffeomorphism b_θ

$$\begin{array}{rcl} & O(N) & \\ & \cup & \\ b_\theta : & \mathcal{U}_\theta & \longrightarrow \mathcal{U} \subset \mathbb{R}^{N(N-1)/2} \end{array}$$

  for relatively compact neighborhoods $\mathcal{U}_\theta \subseteq O(N)$ and $\mathcal{U} \subseteq \mathbb{R}^{N(N-1)/2}$, $\theta \in \mathcal{U}_\theta$.

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- Since any two orthonormal sets are related by an orthogonal transformation, we can smoothly parameterize an orthonormal set $\{e_i\}_{i=1}^N$ with N elements by $N(N-1)/2$ variables, i.e.,

$$\{e_i(q_1, \ldots, q_{N(N-1)/2})\}_{i=1}^N = \{\Theta(q_1, \ldots, q_{N(N-1)/2}) w_i\}_{i=1}^N \subset H'.$$

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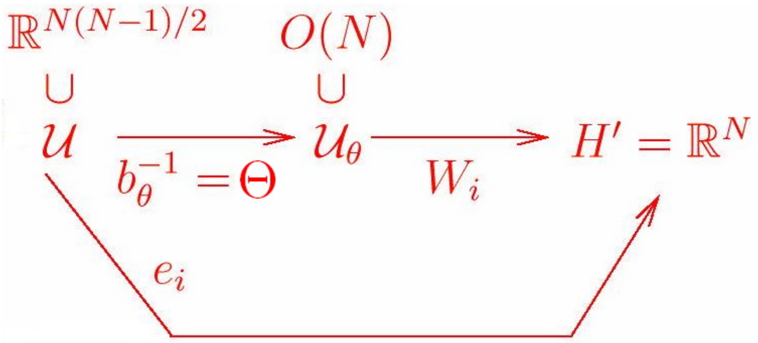

where for all $\Psi \in O(N)$, $W_i(\Psi) = \Psi w_i$.

$$e_i(\vec{q}) = e_i(q_1, \ldots, q_{N(N-1)/2}) = W_i \circ b_\theta^{-1}(\vec{q}) = (b_\theta^{-1}(\vec{q}))w_i \in \mathbb{R}^N.$$

Lagrangian dynamics on $O(N)$

- We now convert the frame force F acting on the orthonormal set $\{e_i\}_{i=1}^N$ into a set of equations governing the motion of the parameterization points $\vec{q}(t) = (q_1(t), \ldots, q_{N(N-1)/2}(t))$, see (1). We define the Lagrangian L and total energy E defined for $\vec{q}(t)$ by:

$$L = T - P_e, \quad E = T + P_e,$$

where

$$T = \frac{1}{2} \sum_{j=1}^{N(N-1)/2} \left(\frac{d}{dt} q_j(t) \right)^2 .$$

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- Using the Euler-Lagrange equations for the potential P_e

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0,$$

we obtain the equations of motion

$$\frac{d^2}{dt^2} q_j(t) = -2 \sum_{i=1}^{N} \rho_i \langle x_i, e_i(\vec{q}(t)) \rangle \left\langle x_i, \frac{\partial e_i}{\partial q_j}(\vec{q}(t)) \right\rangle . \tag{1}$$

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- Choose $\vec{q}\,' \in \mathbb{R}^{N(N-1)/2}$ such that $e_i(\vec{q}\,') = e'_i \in \mathbb{R}^N$ for all $i = 1, \ldots, N$.

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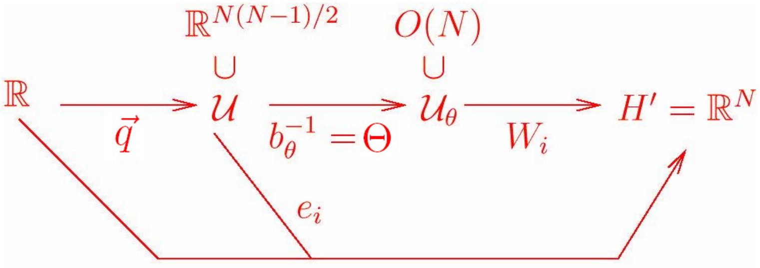

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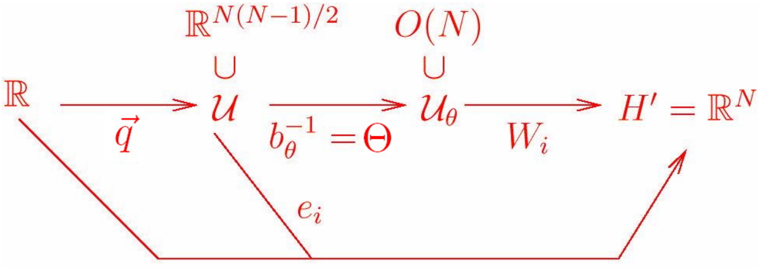

Theorem Constant function $\tilde{q} : \mathbb{R} \to \mathbb{R}^{N(N-1)/2}$ is a minimum energy solution of (1).

Results

It can be shown that

- **Theorem** Denote by $\vec{q}(t) = (q_1(t), \ldots, q_{N(N-1)/2}(t))$ a solution of the equations of motion that minimizes the energy E and denote by $\mathcal{P}_H$ the orthogonal projection from H' into H. Then $\vec{q}(t)$ is a constant solution and the set of vectors

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- **Theorem** A minimum energy solution, a minimizer of P_e, satisfies the expression

$$\sum_{i=1}^N \rho_i \langle x_i, e_i \rangle \left\langle x_i, \frac{\partial e_i}{\partial q_j} \right\rangle = 0.$$

Numerical problems

- The use of Lagrangia provides a point of view for computing the TF minimizers of P_e. (Some independent, direct calculations are possible (Kebo), but not feasible for large values of d and N.)
- The minimum energy solution theorem opens the possibility of using numerical methods to find the optimal orthonormal set. For example, a type of Newton's method could be used to find the zeros of the function

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- With the parameterization of $SO(N)$, the error P_e is a smooth function of the variables $(q_1, \ldots, q_{N(N-1)/2})$, that is,

$$P_e(q_1, \ldots, q_{N(N-1)/2}) = 1 - \sum_{i=1}^{N} \rho_i \left| \langle x_i, e_i(q_1, \ldots, q_{N(N-1)/2}) \rangle \right|^2 .$$

  A conjugate gradient method can be used to find the minimum values of P_e.

Another error criterion

- *Problem* Given a unit normed set $\{x_i\}_{i=1}^N \subset H$, where H is d-dimensional, and positive weights $\{\rho_i\}_{i=1}^N \subset \mathbb{R}$ that sum to 1. Construct the Parseval frame $\{e_i\}_{i=1}^N$ that minimizes

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Theorem Let $\{x_i\}_{i=1}^N$ be a frame for H with frame operator S. $\{S^{-1/2}x_i\}_{i=1}^N$ is the unique Parseval frame such that

$$\sum_{i=1}^{N} \|x_i - S^{-1/2}x_i\|^2 = \inf\left\{ \sum_{i=1}^{N} \|x_i - e_i\|^2 : \{e_i\}_{i=1}^N \text{ Parseval frame for } H \right\}$$

and, with S having eigenvalues $\{\lambda_j\}_{j=1}^d$,

$$\sum_{i=1}^{N} \|x_i - S^{-1/2}x_i\|^2 = \sum_{j=1}^{d} (\lambda_j - 2\sqrt{\lambda_j} + 1).$$

Geometrically uniform frames

$\mathcal{Q} = \{U_i \in \mathcal{L}(H) : 1 \leq i \leq N\}$-finite Abelian group of unitary linear operators.

A set of vectors $\{x_i \in H : 1 \leq i \leq N\}$ is *geometrically uniform* if there exists $x \in H$ such that

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Minimizers of the least-squares error are also minimizers of the quantum detection error when the given set is a geometrically uniform frame. (Bölcskei, Edlar, Forney):

Theorem Let H be a Hilbert space, let $\{x_i\}_{i=1}^N \subset H$ be a frame for H, and let S be its frame operator. If $\{x_i\}_{i=1}^N$ is geometrically uniform then,

1. $\{S^{-1/2}x_i\}_{i=1}^N$ minimizes the detection error P_e when the weights are all equal,
2. $\{S^{-1/2}x_i\}_{i=1}^N$ is a geometrically uniform set under the same abelian group $\mathcal{Q}$.

Thats all folks!