Lecture Notes for Math 648

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Fall 2008

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Comments:

These notes are intended to merely augment and not completely reproduce the recorded lectures. I think it's can be a little tough to see what he's writing, so I think this will be helpful. Professor Benedetto gives significant informative exposition during his lectures (which I appreciate). I have to apologize, but I have largely left this out. This is not out of disrespect, but it's hard to do it much justice without pictures (which I'm not drawing here), and my notes on this are poor (I'm just listening). I think this is much better understood through the videos, anyways.

A note on errors: I'd like these to be professional quality, but I'm producing these notes for my own purposes. I have handwriting like a three year old, but I like to keep my lecture notes. I have absolutely no doubt that this document is riddled with typesetting errors. I welcome your help in finding them. Please email thomas.mccullough@gmail.com, and I'll happily fix them. This will help me, and hopefully you too.

I have tried to stick with his notation. I apologize for the places that I've failed, this is not out of disrespect, just simply my bad habits. I have taken liberty on occasion. In particular, theorems quoted for the result with no proof given I have called propositions, and I have pulled some things I call lemmas out of the proofs to streamline. Any other places that I have taken liberty I explicitly mention. I have cited several things in "Real and Complex Analysis" by Walter Rudin [7], which is the only real analysis book that I have. These things can no doubt be found in any decent analysis book, so consult what you've got.

Contents

Lecture on 02 September 2008

I was not in attendance, I apologize.

Lecture on 09 September 2008

HW# is 7-9 from the new homework set and 7-12 from the old homework set.

Proposition 1 (Poisson Summation Formula)**.**

$$T \cdot \sum_{n\in\mathbb{Z}} f(t+nT) = \sum_{n\in\mathbb{Z}} \hat{f}(n/T)e^{2\pi int/T}$$

Proof. Proof omitted here. See [2, pgs. 505-523] for a proof. □

Example 2. *It is the case that* $\exists f \in L^1(\mathbb{R})$*,* f *continuous,* $f(n) = 0,\ \forall n \in \mathbb{Z}$*,* $\hat{f}(n) = 0\ \forall n \neq 0$ *and* $\hat{f}(0) = 1$.

Proof. No proof of this claim will be provided. This example is intended to illustrate that the statement in Proposition 1 doesn't hold universally. The conditions under which it does hold have remained unstated, but there clearly are some. □

For clear notation in what follows, define

$$\mathrm{PW}_\Omega \stackrel{\text{def}}{=} \{f \in \mathrm{L}^2(\mathbb{R}) \mid \mathrm{supp}(\hat{f}) \subset [-\Omega, \Omega]\} \text{ and } \tau_u s(t) \stackrel{\text{def}}{=} s(t-u)$$

where PW stands for Paley-Weiner.

Theorem 3 (Classical Sampling Theorem)**.** *Let* T, Ω *be positive real numbers such that* $0 < 2T\Omega \leq 1$*. Let* $s \in \mathrm{PW}_{1/2T}$ *with* $\hat{s}(\gamma) = 1$ *for* $\gamma \in [-\Omega, \Omega]$*,* $\hat{s}(\gamma) = 0$ *for* $\gamma \notin [-1/2T, 1/2T]$*, and* $\hat{s} \in L^\infty(\hat{R})$*. Then,*

$$\forall f \in \mathrm{PW}_\Omega,\ f = T \cdot \sum_{n\in\mathbb{Z}} f(nT)\tau_{nT} s$$

in the L^2 *norm.*

Proof. Note that by the Plancherel Theorem [7, p. 186], we have:

$$\begin{aligned} \| f - T \cdot \sum_{|n|\leq N} f(nT)\tau_{nT} s \|_{\mathrm{L}^2(\mathbb{R})} &= \| \hat{f}(\gamma) - T \cdot \sum_{|n|\leq N} f(nT) \cdot e^{-2\pi i nT\gamma} \cdot \hat{s}(\gamma) \|_{\mathrm{L}^2(\hat{\mathbb{R}})} \\ &= \| \hat{f}(\gamma) - T \cdot \sum_{|n|\leq N} f(nT) \cdot e^{-2\pi i nT\gamma} \cdot \hat{s}(\gamma) \|_{\mathrm{L}^2(\widehat{[-\frac{1}{2T}, \frac{1}{2T}]})} \end{aligned} \tag{1}$$

Define $G \in \mathrm{L}^2(\mathbb{T}_{1/T})$, where $\mathbb{T}_{1/T} \stackrel{\text{def}}{=} \mathbb{R}/((1/T)\mathbb{R})$, by

$$G(\gamma) = \begin{cases} \hat{f}(\gamma), & |\gamma| < \Omega \\ 0, & \Omega \le \gamma < 1/2T \end{cases}$$

Then, by [7, p. 186], the Fourier series of G is given by

$$\sum_{|n| \le N} \check{G}[n] \cdot e^{-2\pi i n T \gamma}$$

where

$$\check{G} = \int_{\mathbb{T}_{1/T}} G(\gamma) \cdot e^{2\pi i n T \gamma} d\gamma = T \int_{-\Omega}^{\Omega} \hat{f}(\gamma) e^{2\pi i n T \gamma} d\gamma = T f(nT)$$

Then, continuing our calculation

$$\begin{aligned} (1) &= \| \hat{f} - \sum_{|n| \le N} \check{G}(n) \cdot e^{-2\pi i n T \gamma} \cdot \hat{s} \|_{\mathrm{L}^2(\widehat{[-\frac{1}{2T}, \frac{1}{2T}]})} \\ &= \| \hat{f} - G \|_{\mathrm{L}^2(\widehat{[-\frac{1}{2T}, \frac{1}{2T}]})} + \| G - S_N(G) \cdot \hat{s} \|_{\mathrm{L}^2(\widehat{[-\frac{1}{2T}, \frac{1}{2T}]})} \end{aligned}$$

The first term goes to zero, again by [7, p. 186], so we are left with

$$\begin{aligned} \| G - S_N(G) \cdot \hat{s} \|_{\mathrm{L}^2(\widehat{[-\frac{1}{2T}, \frac{1}{2T}]})} &= \| \hat{s}(G - S_N(G)) \|_{\mathrm{L}^2(\widehat{[-\frac{1}{2T}, \frac{1}{2T}]})} \\ &\le \| \hat{s} \|_{\mathrm{L}^\infty(\hat{\mathbb{R}})} \cdot \| G - S_N(G) \|_{\mathrm{L}^2(\widehat{[-\frac{1}{2T}, \frac{1}{2T}]})} \end{aligned}$$

where the last inequality is Hölder's. This last term again goes to 0 by [7, p. 186]. That completes the proof. $\square$

Example 4. *Let $2T\Omega = 1$ and $s_\Omega(t) = d_{2\pi\Omega}(t) = \sin(2\pi\Omega t)/\pi t$. Recall that the inverse Fourier transform of $[-\Omega, \Omega]$ is s_Ω.*

Example 5. *Set $\varphi(t) = s_\Omega(t)/\sqrt{2\Omega}$. Let $V_0 = \overline{\mathrm{span}}\{\tau_{nT}\varphi\}$. This leads directly to an MRA, which will be discussed later. Let $\psi(t) = (1/\sqrt{2\Omega}) \cdot (s_{2\Omega}(t) - s_\Omega(t))$. This is known as the Shannon dyadic wavelet.*

Now, for some applications of PSF (1):

1. $T \sum \delta_{nT} = \sum e^{2\pi i t n/T}$ - engineering notation for dirac delta function (distribution)
2. Classical Sampling Theorem and relations to Locally Compact Abelian Groups
3. Euler-MacLaurin Formula: $T \cdot \sum_0^\infty f(nT) = \int_0^\infty f(t)dt$ + error terms
4. Jacobi Formula: $\vartheta(t) = \sum e^{-\pi n^2 t}$.
   (a) $\forall t > 0, \vartheta(t) = \frac{1}{\sqrt{t}} \cdot \vartheta(\frac{1}{t})$
   (b) Diffusion Equations
   (c) Statistical Mechanics

(d) Automorphic forms & Elliptic functions

(e) Deligne's proof of the Ramanajan conjecture

(f) Selberg trace formula is CST in number theoretic, non-abelian setting.

Given $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$. Define the Fourier transform of $f\colon \mathbb{Z}_n \to \mathbb{C}$ as $F\colon \mathbb{Z}_n \to \mathbb{C}$ where

$$F[n] = \sum_{m\in\mathbb{Z}_n} f(m)e^{-2\pi imn/N}$$

Theorem 6 (Inversion Formula)**.** *Given f and F as stated, then*

$$f(m) = \frac{1}{N}\sum_{n\in\mathbb{Z}_n} F(n)e^{2\pi imn/N}$$

Proof. This shakes out immediately from the fact that

$$\sum_{n=0}^{N} e^{2\pi in/N} = 0.$$

□

Theorem 7 (Discrete Fourier Transform)**.** *Let $\Omega > 0$, $N \in 2\mathbb{N}$, and T s.t. $2\Omega T = 1$. If $f \in \mathrm{PW}_\Omega$, then consider the dilation f_T as a function $f_T\colon \mathbb{Z} \to \mathbb{C}$ (in addition to being a continuous complex valued function on $\mathbb{R}$), by $m \mapsto f_T[m]$. Define $W_N = e^{2\pi i/N}$ for notational purposes. Assume that $f_T \in \ell^1(\mathbb{Z})$ and suppose that $\hat{f}$ is continuous on $[-\Omega, \Omega]$. Then, $\forall n \in (-N/2, N/2)$, we have*

$$\hat{f}\left(\frac{2\Omega n}{N}\right) = \hat{f}\left(\frac{n}{NT}\right) = \sum_{m=0}^{N-1}(f_T)^o[m]\cdot W_N^{mn},\ \text{where}\ (f_T)^o_N = T\sum_{k\in\mathbb{Z}} f((m+kN)\cdot T)$$

Proof. By the CST (3), we have that

$$f = T\sum f(mT)\cdot \tau_{mT}d_{2\pi\Omega} \implies \hat{f} = T\sum f(mT)\cdot e_{-mT}\cdot \mathbb{1}_{[-\Omega,\Omega]}$$

where $e_r(\gamma) = e^{2\pi ir\gamma}$.

If $n \in (-N/2, N/2)$, then

$$\begin{aligned}\hat{f}\left(\frac{2\Omega n}{N}\right) &= T\sum f(mT)\cdot e^{-2\pi imT2\Omega n/N} \qquad (2T\Omega = 1)\\ &= T\sum_m \sum_{p=mN}^{mN+N-1} f(pT)e^{-2\pi ipn/N}\\ &= T\sum_m\sum_{j=0}^{N-1} f((j+mN)T)\cdot e^{-2\pi i(jm/N+mN)}\end{aligned}$$

Rearranging the sum completes the proof. □

Now, for some historical motivation for wavelets.

Definition: Let $g \in \mathrm{L}^2(\mathbb{R})$, and $a, b > 0$. The *Gabor or Weyl-Heisenberg system of Coherent States* is the sequence $\{g_{m,n} \mid (m,n) \in \mathbb{Z} \times \mathbb{Z}\}$. Where

$$g_{m,n}(t) = e^{2\pi itmb} \cdot g(t - na) = e_{mb}(t) \cdot \tau_{na} g(t).$$

Note that $\hat{g}_{m,n}(\gamma) = \tau_{mb}(e_{-na} \cdot \hat{g})(\gamma)$. This arises as a tool in Quantum Mechanics.

Definition: Let $\psi \in \mathrm{L}^2(\mathbb{R})$. The *Dyadic Wavelet or Affine System* for ψ is the sequence $\{\psi_{m,n} \mid (m,n) \in \mathbb{Z} \times \mathbb{Z}\}$, where

$$\psi_{m,n}(t) = 2^{m/2} \cdot \psi(2^m \cdot t - n) \implies \hat{\psi}_{m,n}(\gamma) = 2^{-m/2} \cdot e_{-n} \cdot \hat{\psi}(\gamma \cdot 2^{-m})$$

This arises in conjunction with the so called *Affine Group*, the group of affine transformations of $\mathbb{R}$.

Some wavelets references: for a mathematical treatment see [5], for an applied math treatment see [3], for an engineering treatment see [4].

Wavelets were developed independently in numerous disparate fields, from distinct efforts and without significant cross fertilization until relatively recently.

In mathematics, this work was motivated by work in algebraic bases for function spaces, the study of Fourier tranforms/series, and splines. Significant work was performed by Haar (1909) in his PhD thesis, Franklin (1927), and Stromburg (1970's), Littlewood-Paley theory, and the Calderón formula.

In physics, the work was motivated by the above Gabor systems, in the work of Von Neumann (1920's or 1930's), Heisenberg, and Weyl.

In engineering, the work was motivated by STFT (Short Time Fourier Transform), speech processing (1970's), two aspects of multi-resolution analysis - Quadratic Mirror Filters (1970's) and Image Processing (pyramidal schemes), the radar-ambiguity function (1953), and Walsh functions (primordial wavelet packets).

Proposition 8 (Alberto Calderón)**.** *$\exists \psi$ such that $\forall f \in L^2(\mathbb{R})$*

$$f(t) = \int_{\mathbb{R}} \psi_{1/u} * \psi_{1/u} * f \frac{du}{u}$$

Proof. For a proof, see [1, 2.2.2 (c.)]. It will not be proved here. To see what ψ must be like, let's take the Fourier transform. Note that

$$\hat{f}(\gamma) = \hat{f}(\gamma) \cdot \int_{\mathbb{R}} [\hat{\psi}(u\gamma)]^2 \frac{du}{u} \implies \int_{\mathbb{R}} [\hat{\psi}(u\gamma)]^2 \frac{du}{u} = 1 \text{ (almost everywhere)}$$

The establishes the continuous wavelet transform. □

Proposition 9 (Ingrid Daubechies). *Given $r \geq 1$, then $\exists \psi \in C_c^{(r)}(\mathbb{R})$ such that $\{\psi_{m,n}\}$ is an orthonormal basis for $L^2(\mathbb{R})$.*

Proof. No proof given. This establishes a wavelet basis of arbitrary smoothness for $L^2(\mathbb{R})$. This is an important extension to Haar's work, which established a non-smooth (step function, in fact) wavelet basis for $L^2(\mathbb{R})$. $\square$

Lecture on 16 September 2008

HW# is 34 from the old homework, 'X', 'Y' from the new homework. Look to the instructions.
The next step in the evolution of wavelet theory was *Multi-Resolution Analysis (MRA)*.

Definition: (Intuitive) Set

$$f_M = \sum_{m \le M} \sum_{n \in \mathbb{Z}} \langle f, \psi_{m,n} \rangle \cdot \psi_{m,n}$$

Assume that ψ satisfies $\operatorname{supp}\psi \subseteq [-1/2, 1/2]$. Then, $\operatorname{supp}\psi_{m,n} \subseteq I_{m,n} \overset{\text{def}}{=} [n2^{-m} - 2^{-(m+1)}, n2^{-m} + 2^{-(m+1)}]$. Note that

$$f_{M+1} = f_M + \langle f, \psi_{M+1,n}, \cdot \rangle \psi_{M+1,n}$$

In other words, f_{M+1} *deblurs* f_M by adding details at a finer scale, on intervals of length $2^{-(m+2)}$. This is the essence of *MRA*.

Definition: (Formal) This was probably first formalized by Y. Meyer. The pair $\{V_j\}_{j \in \mathbb{Z}}, \varphi$ is an *MRA* of $\mathrm{L}^2(\mathbb{R})$ if

1. each V_j is a closed subspace (of $\mathrm{L}^2(\mathbb{R})$.
2. $V_j \subseteq V_{j+1}$ for each $j \in \mathbb{Z}$.
3. $\bigcup V_j = \mathrm{L}^2(\mathbb{R})$ and $\bigcap V_j = 0$ (the zero function).
4. $f(t) \in V_j \iff f(2t) \in V_{j+1}$.
5. $f \in V_0 \iff \forall k \in \mathbb{Z}, \tau_k f \in V_0$.
6. $\varphi \in V_0$ and $\{\tau_k \varphi\}_{k \in \mathbb{Z}}$ for an orthonormal basis (ONB) for V_0.

For example, $\varphi = \mathbb{1}_{[0,1)}$ and $V_0 = \overline{\operatorname{span}}\{\tau_k \varphi\}$, this is the Haar system.

Proposition 10. *Given an MRA* $\{V_j\}, \varphi$ *of* $L^2(\mathbb{R})$*, then there exists an* explicitly constructible ψ *s.t.* $\{\psi_{m,n}\}$ *is an ONB for* $L^2(\mathbb{R})$*.*

Outline of Meyer's Algorithm: W_j is defined s.t. $V_j \oplus W_j = V_{j+1}$.

1. $\exists \psi \in W_0$ s.t. $\{\tau_k \psi\}$ is an ONB for W_0 and $\{\psi_{m,n}\}$ form an ONB for $L^2(\mathbb{R})$.
2. $\exists h_0[n]$ s.t.
$$\varphi(t) = \sqrt{2} \cdot \sum h_0[n] \cdot \varphi(2t - n) \implies \sqrt{2} \cdot \hat{\varphi}(2\gamma) = H_0(\gamma) \cdot \hat{\varphi}(\gamma).$$
Where $H_0(\gamma) = \sum h_0[n] \cdot e^{-2\pi i n \gamma}$. Then,
$$\psi(t) = \sqrt{2} \cdot \sum h_1[n] \cdot \varphi(2t - n) \implies \sqrt{2} \cdot \hat{\psi}(2\gamma) = -e^{-2\pi i \gamma} \cdot \overline{H_0}(\gamma + 1/2) \cdot \hat{\varphi}(\gamma)$$
where $h_1[n] = (-1)^n \cdot \overline{h_0}[-n + 1]$. So, $|H_0(\gamma)|^2 + |H_0(\gamma + 1/2)|^2 = 2$.

Proposition 11 (Heisenberg Uncertainty Principle)**.** *If* $f \in L^2(\mathbb{R}),\ t_0 \in \mathbb{R},\ \gamma_0 \in \hat{\mathbb{R}}$*, then*
$$\| f \|^2_{L^2(\mathbb{R})} \leq 4\pi \cdot \| (t - t_0) \cdot f(t) \|_{L^2(\mathbb{R})} \cdot \| (\gamma - \gamma_0) \cdot \hat{f}(\gamma) \|_{L^2(\hat{R})}$$
There are a series of Fourier uncertainty principles spawned by this idea.

This gives rise to two extreme cases:

- Suppose that $f \in L^2_{\text{loc}}(\mathbb{R})$ so that f is $L^2(I)$ for any finite interval I, and $a \in [-3/2, -1/2)$. Suppose that $f(t)$ is asymptotic to $|t|^a$. then, $f \in L^2(\mathbb{R})$ and $\int |t|^2 \cdot |f(t)|^2 dt = \infty$.
- Suppose that $f = \mathbb{1}_{[-T,T]}$ so that $\hat{f} = d_{2\pi T}$ and $\int |\gamma|^2 \cdot |\hat{f}(\gamma)|^2 d\gamma = \infty$.

Example 12. *Suppose that*
$$\begin{aligned} g(t) &= \sqrt{\frac{2s}{\pi}} \cdot e^{-s(t-t_0)^2} \cdot e^{2\pi i t \gamma_0} \implies \\ \sigma^2 t &= 4\pi \| (t - t_0) \cdot g(t) \|^2_{L^2(\mathbb{R})} = \pi / s \\ \sigma^2 s &= 4\pi \| (\gamma - \gamma_0) \cdot \hat{g}(\gamma) \|^2_{L^2(\hat{\mathbb{R}})} = s/\pi. \end{aligned}$$

In this example, we get equality in the uncertainly principle. Apparently, this is an iff situation. If we view the uncertainty principle as a product of variances, then this situation for g seems to maximize the effectiveness of the process. This approach was an important part of Gabor's contribution - "the best utilization of the information area". He wanted to decompose every signal into a collection of gaussians, to minimize the uncertantity. That is, write
$$f(t) = \sum_{m,n \in \mathbb{Z}} c_{m,n} \cdot \sqrt{\frac{2s}{\pi}} \cdot e^{-s(t - n\sigma^2 t)^2} \cdot e^{2\pi i m \sigma^2 \gamma}$$

As it turns out, this doesn't quite work. It was a significant contribution to the field, because it stimulated a lot of ideas.

Proposition 13 (Balian - Low Theorem)**.** *Given $a, b > 0$ with $ab = 1$. Let $f \in L^2(\mathbb{R})$, and $f_{m,n}(t) = e^{2\pi imbt} \cdot f(t - na)$. If $\{f_{m,n}\}$ is an ONB for $L^2(\mathbb{R})$, then*

$$\int_{\mathbb{R}} |t \cdot f(t)|^2 dt = \infty \text{ or } \int_{\hat{\mathbb{R}}} |\gamma \cdot \hat{f}(\gamma)|^2 d\gamma = \infty$$

It is clear that Gabor's idea is incompatible with this fact.

Example 14 (Morlet Wavelet)**.** *Let $\psi(t) = e^{-\pi \cdot t^2} \cdot (e^{2\pi i t\gamma_0} - e^{-\pi\gamma_0^2})$ and $\psi_{m,n}(t) = 2^{m/2} \cdot \psi(2^m \cdot t - n)$ be the dyadic system of dialates and translates.*

These are best characterized by a picture, but the real idea is the "same number of cycles for low, medium, and high frequencies". See [6].

Proposition 15 (Gabor Decomposition)**.** *This notation is a bit stiff. Suppose that $T, \Omega > 0$, $2T\Omega \leq 1$, and $g \in \mathrm{PW}\, 1/2T$ with $\hat{g} \in L^\infty(\hat{\mathbb{R}})$, $\hat{g} = 1$ on $[-\Omega, \Omega]$. If $2T\Omega < 1$, then there are other conditions incompletely stated... ($\hat{g}$ continuous, $\hat{g} > 0$ on $[-1/2T, 1/2T]$, maybe more?). Set*

$$G(\gamma) = \sum |\hat{g}(\gamma - mb)|^2 \quad s(t) = (\hat{g}/G)\check{}(t)$$

Then, $\forall f \in L^2(\mathbb{R})$,

$$f = T \cdot \sum \langle \hat{f}, e_{nt}\tau_{mb}\hat{g}\rangle \cdot \tau_{-mT}(e_{mb}s)$$

Then, by theorem 3,

$$\forall f \in \mathrm{PW}_\Omega, f = T \sum f(nT)\tau_{mT}s$$

Theorem 16 (Shannon Wavelet Decomposition)**.** *Let $\Omega > 0$,*

$$\begin{aligned}
\varphi &= (1/\sqrt{2\Omega}) \cdot d_{2\pi\Omega} \\
\psi &= (1/\sqrt{2\Omega}) \cdot (d_{2\pi(2\Omega)} - d_{2\pi\Omega}) \text{ so} \\
\sqrt{2\Omega} \cdot \hat{\psi} &= \mathbb{1}_{[-2\Omega,-\Omega)} + \mathbb{1}_{(\Omega,2\Omega]} \\
\sqrt{2\Omega} \cdot \hat{\psi}_{m,0}(\gamma) &= 2^{-m/2} \cdot \sqrt{2\Omega} \cdot \hat{\psi}\Big(\frac{\gamma}{2^m}\Big)
\end{aligned}$$

Let $f \in L^2(\mathbb{R})$, $\hat{f} = F$ (notational convenience), $\Omega > 0$, φ and ψ as above. Then,

$$\begin{aligned}
f &= \sqrt{2\Omega} \cdot f * \varphi + \sum_{m=0}^{\infty} \sum_{n \in \mathbb{Z}} d_{m,n}\psi_{m,n/(4\Omega)} \implies \\
f &= \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} d_{m,n} \cdot \psi_{m,n/(4\Omega)} \text{ in the } L^2(\mathbb{R}) \text{ norm.}
\end{aligned}$$

$$d_{m,n} = \frac{1}{\sqrt{2\Omega} \cdot 2^{(m/2+1)}} \cdot \int_{-2^{m+1}\Omega}^{2^{m+1}\Omega} F(\gamma) \cdot \Big(\mathbb{1}_{[-2^m\Omega,-2^m\Omega)} + \mathbb{1}_{(2^m\Omega,2^{m+1}\Omega]}\Big)(\gamma) \cdot e^{2\pi i n\gamma/(2^{m+2}\Omega)} d\gamma$$

Proof.

$$F(\gamma) = \sqrt{2\Omega} \cdot \left(F(\gamma) \cdot \hat{\varphi}(\gamma) + \sum_{m=0}^{\infty} F(\gamma) \cdot \hat{\psi}\Big(\frac{\gamma}{2^m}\Big) \right) = \sqrt{2\Omega} \cdot \sum_{m\in\mathbb{Z}} F(\gamma) \cdot \hat{\psi}\Big(\frac{\gamma}{2^m}\Big).$$

Set $F_m(\gamma) = \sqrt{2\Omega} \cdot F(\gamma) \cdot \hat{\psi}(\gamma/2^m)$, and $f_m = \check{F_m}$. Then, $\forall m \in \mathbb{Z}$, $f_m \in \mathrm{PW}_{2^{m+1}\Omega}$ and $\operatorname{supp} F_m \subseteq [-2^{m+1}\Omega, -2^m\Omega) \cup (2^m\Omega, 2^{m+1}\Omega]$.

Then, we can consider F_m as a $2^{m+2}\Omega$ periodic function on $\hat{\mathbb{R}}$ with

$$\sqrt{2\Omega} \cdot F(\gamma) \cdot \hat{\psi}\Big(\frac{\gamma}{2^m}\Big) \text{ on } [-2^{m+1}\Omega, 2^{m+1}\Omega]$$

$$S(F_m)(\gamma) = \sum_{m\in\mathbb{Z}} c_{m,n} \cdot e^{-2\pi i m\gamma/(2^{m+2}\Omega)} \text{ (Fourier series)}$$

From the basic facts of Fourier series, we know that

$$\begin{aligned} f_m(t) &= \int_{-2^{m+1}\Omega}^{2^{m+1}\Omega} F_m(\gamma) e^{2\pi i t\gamma} d\gamma \\ &= \sqrt{2\Omega} \cdot \sum_{m\in\mathbb{Z}} c_{m,n} \cdot \underbrace{\int_{-2^{m+1}\Omega}^{2^{m+1}\Omega} \hat{\psi}\Big(\frac{\gamma}{2^m}\Big) \cdot e^{2\pi i (t - m/(2^{m+2}\Omega))\gamma} d\gamma}_{\psi_{m,n/(4\Omega)}} \text{ in } \mathrm{L}^2(\mathbb{R}) \text{ norm.} \end{aligned}$$

But, remember that

$$f = \sum_{m\in\mathbb{Z}} f_m = \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} \sqrt{2\Omega} \cdot c_{m,n} \cdot \psi_{m,n/(4\Omega)}$$

□

Note:

- $\{\psi_{m,n/(4\Omega)}\}$ is not orthogonal in $\mathrm{L}^2(\mathbb{R})$, even though

  $$\left\{ \frac{1}{2^{m/2} \cdot \sqrt{2\Omega}} \cdot e^{-2\pi i (n/(2^{m+2}\Omega))\gamma} \right\}$$

  is an ONB for $\mathrm{L}^2(\mathbb{T}_{2^{m+2}\Omega})$.

- $\{\psi_{m,n/(2\Omega)}\}$ is an orthonormal sequence in $\mathrm{L}^2(\mathbb{R})$. Therefore, if we show

  $$\forall f \in \mathrm{L}^2(\mathbb{R}), \sum_{m,n\in\mathbb{Z}} |\langle f, \psi_{m,n/(2\Omega)}\rangle|^2 = \|f\|^2_{\mathrm{L}^2(\mathbb{R})},$$

  then we can conclude that $\{\psi_{m,n/(2\Omega)}\}$ is an ONB for $\mathrm{L}^2(\mathbb{R})$.

Definition: Let H be a separable Hilbert space, $\{e_n\}_{n\in\mathbb{Z}} \subset H$ is a *frame* for H if $\exists A, B > 0$ s.t.

$$\forall x \in H,\ A\|x\|^2 \leq \sum_{n\in\mathbb{Z}} |\langle x, e_n\rangle|^2 \leq B\|x\|^2$$

If $A = B$, then it is called a *tight frame*.

The Shannon Wavelet Decomposition 16 asserts that $\{\psi_{m,n/(4\Omega)}\}$ is a tight wavelet frame with $A = B$.

Proposition 17. *If $\{e_n\}$ is a frame for H, then*

$$\forall x \in H,\ x = \sum \langle x, S^{-1} e_n \rangle \cdot e_n,\ \textit{where } Sx = \sum \langle x, e_n \rangle \cdot e_n$$

It turns out that Sx defines a topological isomorphism (I think this is directly from open mapping?).

Class on 23 September 2008

HW#11, 12, 20, 21, 39 and new problem Z

Definition: Let $K \subseteq \hat{\mathbb{R}}$ be called τ *congruent* to $[-1/2, 1/2)^d$ iff $\exists\{K_j\}$ a disjoint (except on sets of measure 0) set of Lebesgue measurable subsets of $\hat{\mathbb{R}}^d$, and $\exists\{k_j\} \subseteq \mathbb{Z}^d$ s.t. $\{K_j\}$ is a partition of K and $\{K_j + k_j\}$ is a partition of $[-1/2, 1/2)^d$.

Example 18. *Let $K = [-1, -1/2) \cup [1/2, 1) \subseteq \hat{\mathbb{R}}$.*

1. *K is τ congruent to $[-1/2, 1/2)$. Let $K_- = [-1, -1/2)$ and $K_+ = [1/2, 1)$. Then, $(K_- + 1) \cup (K_+ - 1) = [-1/2, 1/2)$. This example is relevant to the artwork of M.C. Escher.*

2. *$\{2^m K\}_{m\in\mathbb{Z}}$ is a partition of $\hat{\mathbb{R}}$.*

3. *Let $\hat{\psi} = \mathbb{1}_{[-1,-1/2)} + \mathbb{1}_{[1/2,1)} = \mathbb{1}_K$. Note that if $f \in L^2(\mathbb{R})$, and $m, n \in \mathbb{Z}$, then*

$$\begin{aligned}
\int_{\hat{\mathbb{R}}} \hat{\psi}_{m,n} \cdot \hat{f} d\lambda &= \int_{-1}^{-1/2} \hat{(}2^m\lambda) \cdot e^{2\pi i n\lambda} d\lambda + \int_{1/2}^{1} \hat{f}(2^m\lambda) \cdot e^{2\pi i n\lambda} d\lambda \\
&= \int_{0}^{1/2} \hat{f}(2^m(\gamma - 1)) e^{2\pi i n\gamma} d\gamma + \int_{-1/2}^{0} \hat{f}(2^m(\gamma + 1)) \cdot e^{2\pi i n\gamma} d\gamma \\
&= \int_{-1/2}^{1/2} e^{2\pi i n\gamma} \cdot \Big(\hat{f}(2^m(\gamma - 1))\mathbb{1}_{K_-}(\gamma - 1) + \hat{f}(2^m(\gamma + 1))\mathbb{1}_{K_+}(\gamma + 1)\Big) d\gamma
\end{aligned}$$

This example will reappear in later work, and is instrumental throughout this lecture.

I took the liberty of introducing this as a lemma. It was presented in lecture as part of a theorem.

Lemma 19. *If $K \subseteq \hat{\mathbb{R}}^d$ is Lebesgue measurable s.t.*

1. *K is τ congruent to $[-1/2, 1/2)^d$.*

2. *$\{2^m K\}$ is a partition (tiling) of $\hat{\mathbb{R}}^d$.*

3. $|K| = 1$.

Let $\hat{\psi} = \mathbb{1}_K$. Then, $\forall f \in L^2(\mathbb{R}^d)$, we have

$$\sum_{\substack{m\in\mathbb{Z}\\ n\in\mathbb{Z}^d}} |\langle f, \psi_{m,n}\rangle|^2 = \| f \|^2_{L^2(\mathbb{R}^d)}$$

Proof. Obviously, our example works for $d = 1$, but it isn't clear that this even works for higher dimensions (it does). No matter for now, suppose we had such a set. Then, note

$$\begin{aligned}
\sum_{\substack{m\in\mathbb{Z}\\ n\in\mathbb{Z}^d}} |\langle f, \psi_{m,n}\rangle|^2 &= \sum_{\substack{m\in\mathbb{Z}\\ n\in\mathbb{Z}^d}} 2^{md} \left| \int_{\hat{R}^d} \hat{f}(\gamma)\cdot 2^{-md}\cdot e^{2\pi i (n2^{-m})\cdot\gamma}\cdot \overline{\hat{\psi}(2^{-m}\gamma)} d\gamma \right|^2 \text{ by Parseval's theorem}\\
&= \sum_{\substack{m\in\mathbb{Z}\\ n\in\mathbb{Z}^d}} 2^{-md} \left| \int_{\hat{R}^d} \hat{f}(2^m\lambda)\cdot e^{2\pi i n\cdot\lambda}\cdot \overline{\hat{\psi}(\lambda)} 2^{md} d\gamma \right|^2\\
&= \sum_{\substack{m\in\mathbb{Z}\\ n\in\mathbb{Z}^d}} 2^{md} \left| \int_K \hat{f}(2^m\lambda)\cdot e^{2\pi i n\cdot\lambda} d\lambda \right|^2\\
&= \sum_{\substack{m\in\mathbb{Z}\\ n\in\mathbb{Z}^d}} 2^{md} \left| \int_{[-1/2,1/2)^d} e^{2\pi i n\cdot\lambda} \underbrace{\left(\sum_j \hat{f}(2^m(\gamma - k_j))\cdot \mathbb{1}_{K-j}(\gamma - k_j)\right)}_{G_m(\gamma)} d\gamma \right|^2\\
&= \sum_{m\in\mathbb{Z}} 2^{md} \left(\int_{[-1/2,1/2)^d} \left| G_m(\gamma)\right|^2 d\gamma \right) \text{ by Parseval's theorem}\\
&= \sum_{m\in\mathbb{Z}} 2^{md} \int_K \left|\hat{f}(2^m\lambda)\right|^2 d\lambda\\
&= \sum_{m\in\mathbb{Z}} \int_{2^m K} \left|\hat{f}(\gamma)\right|^2 d\gamma\\
&= \| \hat{f} \|^2_{\mathrm{L}^2(\hat{\mathbb{R}}^d)} \text{ by Property 2 of } K\\
&= \| f \|^2_{\mathrm{L}^2(\mathbb{R}^d)} \text{ by Plancherel}
\end{aligned}$$

□

Theorem 20. *Let $K = [-1, -1/2) \cup [1/2, 1)$ and $\hat{\psi} = \mathbb{1}_K$. Then, $\{\psi_{m,n}\}$ is an ONB for $L^2(\mathbb{R})$.*

Proof. The set $\{\psi_{m,n}\}$ is orthonormal (asserted last class, and an immediate consequence of the Plancherel theorem). To conclude that it is a basis, it remains to show that

$$\forall f \in \mathrm{L}^2(\mathbb{R}),\ \sum |\langle f, \psi_{m,n}\rangle|^2 = \| f \|^2_{\mathrm{L}^2(\mathbb{R})}$$

This follows immediately from the previous lemma. □

Theorem 21. *Let $K \subseteq \hat{\mathbb{R}}^d$ be a Lebesgue measurable set s.t.*

1. $|K| = 1$.
2. $\{2^m K\}$ *is a partition (tiling) of* $\hat{\mathbb{R}}^d$.
3. K *is* τ *congruent to* $[-1/2, 1/2)^d$.

Set $\psi = \check{\mathbb{1}}_K$. *Then,* $\{\psi_{m,n}\}$ *forms an ONB for* $L^2(\mathbb{R}^d)$.

Proof. As in the previous, this follows from Plancherel and the lemma. □

Definition: Let $I_{m,n} = \{x \in \mathbb{R} \mid 2^{-m} n \leq x < 2^{-m}(n+1)\}$. This is called a *dyadic interval*. Note that $I_{m+1,2n}$ is the lower half of $I_{m,n}$ and $I_{m+1,2n+1}$ is the upper half.

I have taken liberties with this result, to pick the low hanging fruit for the fact that the Haar wavelets form an orthogonal system.

Lemma 22. *Let $I_{m,n}$ and $I_{p,q}$ be dyadic intervals and assume WLOG that $m \leq p$. Then, exactly one of the following holds:*

1. $I_{m,n} = I_{p,q} \iff m = p$ *and* $n = q$.
2. $I_{m,n} \cap I_{p,q} = \emptyset$.
3. $m < p$ *and* $I_{p,q} \subseteq I_{m+1,2n} \subset I_{m,n}$.
4. $m < p$ *and* $I_{p,q} \subseteq I_{m+1,2n+1} \subset I_{m,n}$.

Proof. Suppose not, for a contradiction. The first two cases are trivial, so it suffices to assume that $m < p$. Then, it must be the case that

$$\frac{q}{2^p} < \frac{n}{2^m} = \frac{2n}{2^{m+1}} < \frac{q+1}{2^p} \text{ or}$$
$$\frac{q}{2^p} < \frac{2n+1}{2^{m+1}} < \frac{q+1}{2^p} \text{ or}$$
$$\frac{q}{2^p} < \frac{n+1}{2^m} = \frac{2n+2}{2^{m+1}} < \frac{q+1}{2^p}$$

These all reduce to the case of $M \leq p$, and

$$\frac{q}{2^p} < \frac{N}{2^M} < \frac{q+1}{2^p}$$

But, note that $p - M \geq 0$, so $N \cdot 2^{p-M}$ is an integer, and

$$q < N \cdot 2^{p-M} < q + 1$$

Then, we have found an integer strictly between q and $q+1$, so we have reached the desired contradiction. Note that the actual conclusion of this is that $I_{p,q}$ is in one half of $I_{m,n}$. □

Theorem 23. *Let $\psi = \mathbb{1}_{[0,1/2} - \mathbb{1}_{[1/2,1)}$ be the Haar function. Then,*

1. *$\forall m, n \in \mathbb{Z}$, $\operatorname{supp} \psi_{m,n} = I_{m,n}$.*
2. *$\{\psi_{m,n}\}_{m,n\in\mathbb{Z}}$ is orthonormal.*

Proof. Part (1.) Note that $x \in \operatorname{supp} \psi_{m,n} \iff 2^m x - n \in [0,1) \iff 2^m x \in [n, n+1) \iff x \in [2^{-m}n, 2^{-m}(n+1)) = I_{m,n}$.

Part (2.) Consider $\langle \psi_{m,n}, \psi_{p,q} \rangle$. From the previous lemma, there are only four cases.
Case (1.) $m = p$ and $n = q$, and (by definition of ψ) we have $\langle \psi_{m,n}, \psi_{p,q} \rangle = \| \psi_{m,n} \|_{\mathrm{L}^2(\mathbb{R})} = 1$.
Case (2.) Obviously, $\langle \psi_{m,n}, \psi_{p,q} \rangle = 0$, because $\operatorname{supp} \psi_{m,n} \cap \operatorname{supp} \psi_{p,q} = \emptyset$.
Cases (3.) and (4.) From the lemma, we can conclude that $\psi_{m,n}$ is constant (± 1) on $\operatorname{supp} \psi_{p,q}$. □

Theorem 24. *Let ψ be the Haar function. Then, $\{\psi_{m,n}\}$ is an ONB for $L^2(\mathbb{R})$.*

Proof. That it is an orthonormal system was just shown. Now, it suffices to show that

$$\forall f \in \mathrm{L}^2(\mathbb{R}),\ f = \sum \langle f, \psi_{m,n} \rangle \psi_{m,n} \text{ in } \mathrm{L}^2(\mathbb{R}) \text{ norm.}$$

This will be done as follows: Given $\varepsilon > 0$ and $g \in \mathrm{L}^2(\mathbb{R})$.
First, show that $\exists N \in \mathbb{N}$, $\exists g_N \in \mathrm{L}^2(\mathbb{R})$ s.t. $\operatorname{supp} g_N \subseteq [-2^N, 2^N]$ and

$$\| g - g_N \|_{\mathrm{L}^2(\mathbb{R})} < \varepsilon/3.$$

Second, show that $\exists M \in \mathbb{N}$ and $\exists f \in \mathrm{L}^2(\mathbb{R})$ s.t. $\operatorname{supp} f \subseteq [-2^N, 2^N]$, f is constant on intervals $I_{m,n}$, and

$$\| g_N - f \|_{\mathrm{L}^2(\mathbb{R})} < \varepsilon/3.$$

Lastly (and most difficult), $\exists \{c_{m,n} \in \mathbb{C} \mid m, n \in F_1 \times F_2\ F_j \subseteq \mathbb{Z} \text{ finite}\}$ s.t.

$$\| f - \sum c_{m,n} \psi_{m,n} \|_{\mathrm{L}^2(\mathbb{R})} < \varepsilon/3.$$

This outline will be completed in two weeks. □

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