BESOV SPACES FOR THE SCHRÖDINGER OPERATOR WITH BARRIER POTENTIAL

JOHN J. BENEDETTO AND SHIJUN ZHENG

Abstract. Let $H = -\triangle + V$ be a Schrödinger operator on the real line, where $V = \varepsilon^2 \chi_{[-1,1]}$. We define the Besov spaces for H by developing the associated Littlewood-Paley theory. This theory depends on the decay estimates of the spectral operator $\varphi_j(H)$ for the high and low energies. We also prove a Mihlin-Hörmander type multiplier theorem on these spaces, including the L^p boundedness result. Our approach has potential applications to other Schrödinger operators with short-range potentials, as well as in higher dimensions.

1. Introduction

Let $H = -\triangle + V$ be a Schrödinger operator on $\mathbb{R}$, where the potential V is real-valued and belongs to $L^1 \cap L^2$. H is the Hamiltonian in the corresponding time-dependent Schrödinger equation

$$(1) \qquad \begin{aligned} &i\, \partial_t \psi = H\psi, \\ &\psi(0, x) = f(x) \in \mathcal{D}(H), \end{aligned}$$

where the solution is uniquely determined by the initial state: $\psi(t, x) = e^{-itH} f(x),\ t \geq 0$, and where $\mathcal{D}(H) \subseteq L^2$ is the domain of H.

In [15] Jensen and Nakamura introduced Besov spaces associated with H on $\mathbb{R}^d$ and showed that e^{-itH} maps $B_p^{s+2\beta,q}(H)$ into $B_p^{s,q}(H)$ if $s \geq 0$, $1 \leq p, q \leq \infty$, and $\beta > d|\frac{1}{2} - \frac{1}{p}|$, under the condition that $V = V_+ - V_-$ where $V_+ \in K_d^{loc}$ and $V_- \in K_d$, K_d being the Kato class. In this paper we generalize the definition of Besov spaces to $s \in \mathbb{R}$, $0 < p, q < \infty$ and show, in the case of barrier potential, that such a definition is independent of the choice of the dyadic system $\{\Phi, \varphi_j\}$.

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Let $H = \int \lambda dE_\lambda$ be the spectral resolution of H. The spectral operator $\phi(H)$ is defined by functional calculus: $\phi(H) = \int \phi(\lambda) dE_\lambda$.

For $\alpha \in \mathbb{R}$, $0 < p < \infty$, $0 < q \leq \infty$, define the quasi-norm for $f \in L^2$ as

$$(2) \quad \|f\|_{B_p^{\alpha,q}} := \|f\|^{\varphi}_{B_p^{\alpha,q}(H)} = \|\Phi(H)\|_p + \left(\sum_{j=1}^{\infty} (2^{j\alpha} \|\varphi_j(H) f\|_p)^q \right)^{1/q} .$$

The *Besov spaces associated with* H, denoted by $B_p^{\alpha,q} := B_p^{\alpha,q}(H)$, is defined to be the completion of the subspace $L_0^2 := \{f \in L^2 : \|f\|_{B_p^{\alpha,q}} < \infty\}$ of L^2.

Analogous to the Fourier case $H = -\Delta$ and the Hermite case [27, 28, 8, 9], we shall develop the Besov space theory associated with H by considering the Schrödinger operator $H = -\Delta + V$, where $V = \varepsilon^2 \chi_{[-1,1]}$, $\varepsilon > 0$ (called the *barrier potential*) is one of the simplest discontinuous potential models in quantum mechanics.

Peetre's maximal operator plays an important role in the theory of function spaces [27, 28]. In order to establish a Peetre type maximal inequality for H, we need the decay estimates of the kernel of $\varphi_j(H)$ as well as of its derivative. Based on an integral expression of this kernel we obtain the decay estimates by exploiting the analytic behavior of the eigenfunctions $e(x, \xi)$ as ξ approaches ∞ (high energy) and 0 (low energy) in various cases. When the support of Φ contains the origin, we are in the so-called "local energy" case, which usually is harder to deal with for general potentials. We use a "matching" method to put together integrals of the "same type", so that each of the resulting integrals is the Fourier transform of a Schwartz function. This method seems interesting and may have applications to other potentials.

Our first main result (Theorem 3.7) is an equivalence theorem for $B_p^{\alpha,q}(H)$, which states that the Besov space norm can be characterized using Peetre type maximal functions $\varphi_j^*(H)$ in place of $\varphi_j(H)$. This implies that $\|f\|^{\phi}_{B_p^{\alpha,q}}$ and $\|f\|^{\psi}_{B_p^{\alpha,q}}$ are equivalent quasinorms on $B_p^{\alpha,q}(H)$, where $\{\phi_j\}, \{\psi_j\}$ are two given smooth dyadic systems.

Using functional calculus, Jensen and Nakamura [15, 16] obtained smooth multiplier results for certain potentials in the Kato class. For the barrier potential we prove a sharp spectral multiplier theorem on L^p and $B_p^{\alpha,q}(H)$ (Theorem 6.5 and Theorem 6.6). Related results appeared in [13, 6, 9, 7, 23, 20].

The paper is organized as follows. In §2 we give explicit solutions to the eigenfunction equation for H. The proof of Theorem 3.7 is based on decay estimates for the kernel of $\varphi_j(H)$. Detailed proofs of these decay estimates are included in §4 and §5. In §6 we prove a Mihlin-Hörmander type multiplier theorem for H. In §7, we identify these new $B_p^{\alpha,q}(H)$ spaces with the ordinary Besov spaces for a certain range of parameters α, p, q.

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2. Preliminaries

2.1. *Kernel formula for the spectral operator*

Let $e_+(x,\xi)$ and $e_-(x,\xi)$ be two solutions of the equation

$$He(x,\xi) = \xi^2 e(x,\xi) \tag{3}$$

with the asymptotic behavior for $\xi > 0$ and $\xi < 0$, respectively, being

$$e_\pm(x,\xi) \to \begin{cases} T_\pm(\xi)e^{i\xi x} & x \to \pm\infty \\ e^{i\xi x} + R_\pm(\xi)e^{-i\xi x} & x \to \mp\infty. \end{cases} \tag{4}$$

Then the functions $e_\pm(x,\xi)$ are unique for $\xi \in \mathbb{R}$, and equation (3) together with condition (4) is equivalent to the integral equation

$$e(x,\xi) = e^{i\xi x} + (2i|\xi|)^{-1} \int e^{i|\xi||x-y|} V(y) e(y,\xi) dy. \tag{5}$$

These *generalized eigenfunctions* have a physical interpretation in quantum mechanics, where ξ^2 is viewed as a energy parameter; in fact, they represent the transmission and reflection waves when a particle passes through the potential. The coefficients T, R are called the *transmission coefficient* and the *reflection coefficient* (cf., [12, p.4179], also [10]). Under the condition that V is in $L^1 \cap L^2$, the second-named author proved the following two results in [29, 30].

a) The essential spectrum of H is $[0,\infty)$. More precisely, H only has an absolutely continuous spectrum, the singular continuous spectrum being empty; and the discrete spectrum of H is at most countable. Hence, if we denote L^2 by $\mathcal{H}$, then $\mathcal{H} = \mathcal{H}_{ac} \oplus \mathcal{H}_{pp}$.

b) Define the *generalized Fourier transform* $\mathcal{F}$ on L^2 by

$$\mathcal{F}f(\xi) := \text{l.i.m.} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\overline{e}(x,\xi)dx.$$

Then $\mathcal{F}$ is a unitary operator from $\mathcal{H}_{ac}$ onto L^2 and its adjoint is given by

$$\mathcal{F}^* g(x) := \text{l.i.m.} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\xi) e(x,\xi) d\xi.$$

Therefore $\phi(H)|_{\mathcal{H}_{ac}} = \mathcal{F}^*\phi(\xi^2)\mathcal{F}$ for $\phi \in L^\infty$. Suppose H has no point spectrum and $|e(x,\xi)| \le C$ for a.e. $(x,\xi) \in \mathbb{R}^2$, then, if $\phi \in C_c$ (continuous and compactly supported functions), we have

$$\forall\, f \in L^1 \cap L^2, \quad \phi(H)f(x) = \int_{-\infty}^{\infty} \phi(H)(x,y)f(y)dy, \tag{6}$$

where

$$\phi(H)(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi(\xi^2)e(x,\xi)\overline{e}(y,\xi)d\xi. \tag{7}$$

A variant of formula (6) can be found in [12] for short-range potentials defined as a measure. In three dimensions, a similar formula is used by Tao [26] in a scattering problem. Also consult [21] for general references.

Since $V = \varepsilon^2\chi_{[-1,1]}$ is in $L^1 \cap L^2$ and the eigenfunctions of H are uniformly bounded (see §2.3), formula (6) is valid for V. Note that the corresponding point spectrum is empty.

2.2. *Dyadic systems and Besov spaces*

Let $\Phi, \varphi, \Psi, \psi$ be C^∞ functions, satisfying the following conditions:

i) supp Φ, supp $\Psi \subseteq \{|\xi| \le 1\}$; $|\Phi(\xi)|$, $|\Psi(\xi)| \ge c > 0$ if $|\xi| \le \frac{1}{2}$;

ii) supp φ, supp $\psi \subseteq \{\frac{1}{4} \le |\xi| \le 1\}$; and $|\varphi(\xi)|$, $|\psi(\xi)| \ge c > 0$ if $\frac{3}{8} \le |\xi| \le \frac{7}{8}$;

iii) $\forall\, \xi \in \mathbb{R}$, $\Phi(\xi)\Psi(\xi) + \sum_{j=1}^{\infty} \varphi(2^{-j}\xi)\psi(2^{-j}\xi) = 1$.

Such functions exist, e.g., [11], and we shall use the notation $\varphi_j(\xi) = \varphi(2^{-j}\xi)$. The almost orthogonal relation (*iii*) for the dyadic system allows us to write each $f \in L^2$ as

$$f = \Phi(H)\Psi(H)f + \sum_j \varphi_j(H)\psi_j(H)f.$$

If $0 < p < \infty$, $0 < q \le \infty$, $\alpha \in \mathbb{R}$, we define the $B_p^{\alpha,q}(H)$ quasi-norm (2) for $f \in L^2$. Note that when $0 < p < 1$ or $0 < q < 1$, we can always define a metric d on $B_p^{\alpha,q}$, so that the metric space $(B_p^{\alpha,q}, d)$ is topologically isomorphic to the quasi-normed space. In fact, Lemma 3.10.1 in [4] tells that every quasi-normed linear space is metrizable.

The definition of $B_p^{\alpha,q}(H)$ is a natural extension of the classical case where $H = H_0 = -\Delta$, and such a definition leads to the usual Besov space on $\mathbb{R}$: $B_p^{\alpha,q}(H_0) = B_p^{2\alpha,q}(\mathbb{R})$.

2.3. *Generalized eigenfunctions of H*

We now determine eigenfunctions of $H = -\Delta + V$, where $V = \varepsilon^2\chi_{[-1,1]}$, also see, e.g., [10].

First, note that $e(x,\xi)$ must have the following form. If $\xi > \varepsilon$, then

$$e_+(x,\xi) = \begin{cases} A_+ e^{i\xi x} + A'_+ e^{-i\xi x}, & x < -1, \\ B_+ e^{iKx} + B'_+ e^{-iKx}, & |x| \le 1, \\ C_+ e^{i\xi x} + C'_+ e^{-i\xi x}, & x > 1, \end{cases}$$

where $K = \sqrt{\xi^2 - \varepsilon^2}$; and if $0 < \xi < \varepsilon$, then

$$e_+(x,\xi) = \begin{cases} A_+ e^{i\xi x} + A'_+ e^{-i\xi x}, & x < -1, \\ B_+ e^{\rho x} + B'_+ e^{-\rho x}, & |x| \le 1, \\ C_+ e^{i\xi x} + C'_+ e^{-i\xi x} & x > 1, \end{cases}$$

where $\rho = \sqrt{\varepsilon^2 - \xi^2}$.

The Lippmann-Schwinger equation (5) requires that $e(x,\xi)$ is differentiable in x. By the C^1 condition at ± 1 we can obtain the precise values of the coefficients A, A', B, B', C, C' as follows.

Let

$$\rho = \rho(\xi) = \begin{cases} iK = i\sqrt{\xi^2 - \varepsilon^2}, & |\xi| > \varepsilon, \\ \sqrt{\varepsilon^2 - \xi^2}, & |\xi| \le \varepsilon. \end{cases}$$

Then, for $\xi > 0$,

$$\begin{aligned} C'_+ &= 0, \quad A_+ = 1, \\ C_+ &= \frac{2\rho\xi e^{-2i\xi}}{2\rho\xi \cosh 2\rho + i(\rho^2 - \xi^2)\sinh 2\rho}, \\ A'_+ &= -i\frac{C_+}{2\rho\xi}\varepsilon^2 \sinh 2\rho = -i\frac{\varepsilon^2 \sinh 2\rho e^{-2i\xi}}{2\rho\xi \cosh 2\rho + i(\rho^2 - \xi^2)\sinh 2\rho}, \\ B_+ &= \frac{C_+}{2\rho}(\rho + i\xi)e^{-\rho + i\xi}, \quad \text{and } B'_+ = \frac{C_+}{2\rho}(\rho - i\xi)e^{\rho + i\xi}. \end{aligned}$$

For $\xi < 0$, we obtain similarly, with the same notation $\rho = \rho(\xi)$,

$$e_-(x,\xi) = \begin{cases} A_- e^{i\xi x} + A'_- e^{-i\xi x}, & x < -1, \\ B_- e^{\rho x} + B'_- e^{-\rho x}, & |x| \le 1, \\ C_- e^{i\xi x} + C'_- e^{-i\xi x}, & x > 1, \end{cases}$$

where $C_- = 1$, $A'_- = 0$,

$$\begin{aligned} A_- &= \frac{2\rho\xi e^{2i\xi}}{2\rho\xi \cosh 2\rho - i(\rho^2 - \xi^2)\sinh 2\rho}, \\ C'_- &= i\frac{A_-}{2\rho\xi}\varepsilon^2 \sinh 2\rho = i\frac{\varepsilon^2 \sinh 2\rho e^{2i\xi}}{2\rho\xi \cosh 2\rho - i(\rho^2 - \xi^2)\sinh 2\rho}, \\ B_- &= \frac{A_-}{2\rho}(\rho + i\xi)e^{\rho - i\xi}, \quad \text{and } B'_- = \frac{A_-}{2\rho}(\rho - i\xi)e^{-\rho - i\xi}. \end{aligned}$$

Furthermore, if we define

$$e(x,\xi) = \begin{cases} e_+(x,\xi), & \xi > 0, \\ e_-(x,\xi), & \xi < 0, \end{cases}$$

then

$$e(x,-\xi) = e(-x,\xi), \quad \xi \neq 0, \tag{8}$$

which follows from the following simple relations between the coefficients:

$$A_-(\xi) = \overline{A}_-(-\xi) = \overline{C}_+(\xi) = C_+(-\xi),$$

$$C'_-(\xi) = \overline{C'}_-(-\xi) = \overline{A'}_+(\xi) = A'_+(-\xi),$$

and

$$B_+(-\xi) = B'_-(\xi), \quad B_-(-\xi) = B'_+(\xi).$$

Remark 1. Identity (8) allows us to simplify the estimates in various cases, see §§4–6. Some of the above relations can also be found in [12, Theorem 6.1] for general short-range potentials.

Remark 2. It is easy to observe that A'_+, C_+, hence C'_-, A_-, are real analytic in $\xi \in \mathbb{R}$, while $B_\pm, B'_\pm$ have singularities at $\xi = \pm\varepsilon$. Moreover, for every x, $e(x,\cdot)$ is analytic in $\xi \in \mathbb{R}\setminus\{0\}$. For every ξ, $e(\cdot,\xi)$ is C^∞ in $x \in \mathbb{R}\setminus\{\pm 1\}$, while C^1 (continuously differentiable) at $x = \pm 1$.

3. Peetre type maximal inequality

Let $\Phi, \varphi, \Psi, \psi$ be C^∞ functions, satisfying the conditions given in §2. Recall that if $\phi \in C_c$, the operator $\phi(H)$ has the kernel (7). Note that $e(\cdot,\xi) \in C^1$ ($\xi \neq 0$) implies $\phi(H)(\cdot,\cdot) \in C^1(\mathbb{R}\times\mathbb{R})$.

Lemma 3.1. *Let $K_j(x,y) = \varphi(2^{-j}H)(x,y)$ with supp $\varphi \subseteq [\frac{1}{4},1] \cup [-1,-\frac{1}{4}]$.*
(a) If $j > 4 + 2\log_2 \varepsilon$, then, for each $n \in \{0\}\cup\mathbb{N}$,

$$|K_j(x,y)| \le C_n \sum_{\ell=0}^{2N} 2^{j/2}(1 + 2^{j/2}|x \pm y \pm 2\ell|)^{-n}, \tag{9}$$

where $N := N_n$ is the smallest integer $\ge \max\{1, n/4\}$.
(b) If $-\infty < j \le J := 4 + [2\log_2 \varepsilon]$, then, for each $n \ge 0$,

$$|K_j(x,y)| \le C_n 2^{j/2}(1 + 2^{j/2}|x \pm y|)^{-n}.$$

Notation. In the right hand side of (9) each summand denotes a sum taken over all possible choices of the signs $\pm$. Similar notation applies elsewhere in this paper.

Lemma 3.2. *Let $K(x,y) = \Phi(H)(x,y)$, supp $\Phi \subseteq [-1,1]$. Then, for each $n \geq 0$,*

$$|K(x,y)| \leq C_n(1+|x-y|)^{-n}.$$

We also need decay estimates for the derivative of the kernel.

Lemma 3.3. *Let the notation be as in Lemma 3.1.*
(a) If $j > J$, then, for each $n \geq 0$, there is a constant C_n such that

$$\left|\frac{\partial}{\partial x}K_j(x,y)\right| \leq C_n \sum_{\ell=0}^{2N} 2^j(1+2^{j/2}|x \pm y \pm 2\ell|)^{-n},$$

where N is the the same as in Lemma 3.1(a).
(b) If $-\infty < j \leq J$, then, for each $n \geq 0$, there is a constant C_n such that

$$\left|\frac{\partial}{\partial x}K_j(x,y)\right| \leq C_n 2^j(1+2^{j/2}|x \pm y|)^{-n}.$$

Lemma 3.4. *Let Φ be as in Lemma 3.2. Then, for each $n \geq 0$,*

$$\left|\frac{\partial}{\partial x}K(x,y)\right| \leq C_n(1+|x-y|)^{-n}.$$

Proofs of Lemmas 3.1−3.4 are given in §4 and §5, and they involve oscillatory integral techniques. These lemmas are essential for us to establish a Peetre type maximal inequality (Lemma 3.6).

Given $s > 0$, define the *Peetre maximal functions* for H as follows: if $j > J$, define

$$\varphi_j^* f(x) = \sup_{t\in\mathbb{R}} \frac{|\varphi_j(H)f(t)|}{\min_{\ell,\pm}(1+2^{j/2}|x \pm t \pm 2\ell|)^s},$$

and

$$\varphi_j^{**} f(x) = \sup_{t\in\mathbb{R}} \frac{|(\varphi_j(H)f)'(t)|}{\min_{\ell,\pm}(1+2^{j/2}|x \pm t \pm 2\ell|)^s},$$

where the minimum is taken over $0 \le \ell \le 2N_s$ and N_s is the smallest integer $\ge \max\{1, \frac{[s]+2}{4}\}$; and, if $j \le J$, define

$$\varphi_j^* f(x) = \sup_{t\in\mathbb{R}} \frac{|\varphi_j(H)f(t)|}{\min_\pm (1+2^{j/2}|x \pm t|)^s},$$
$$\Phi^* f(x) = \sup_{t\in\mathbb{R}} \frac{|\Phi(H)f(t)|}{(1+|x-t|)^s},$$
$$\varphi_j^{**} f(x) = \sup_{t\in\mathbb{R}} \frac{|(\varphi_j(H)f)'(t)|}{\min_\pm (1+2^{j/2}|x \pm t|)^s},$$
$$\Phi^{**} f(x) = \sup_{t\in\mathbb{R}} \frac{|(\Phi(H)f)'(t)|}{(1+|x-t|)^s}.$$

We have used the abbreviation $\varphi_j^* f := \varphi_{j,s}^* f$, etc. Notice that

$$\varphi_j^* f(x) \ge |\varphi_j(H)f(x)|. \tag{10}$$

In the following we slightly modify the notation $\varphi_0^* f = \Phi^* f$, etc., in case of no confusion.

Lemma 3.5. *For $s > 0$, there exists a constant $c_s > 0$ such that*

$$\varphi_j^{**} f(x) \le c_s 2^{j/2} \max_{k,\pm} \varphi_j^* f(x \pm 2k),$$

where the maximum is taken over $0 \le k \le 2N_s$ and both $\pm$.

Proof. From the identity

$$\forall f \in L^2, \quad \varphi_j(H)f(x) = \sum_{\nu=-1}^{1} (\varphi\psi)_{j+\nu}(H)\varphi_j(H)f(x),$$

with convention $\varphi_0 = \Phi$ and $\varphi_{-1} = 0$, we derive

$$\frac{d}{dt}(\varphi_j(H)f)(t) = \sum_{\nu=-1}^{1} \int_{\mathbb{R}} \frac{\partial}{\partial t} K_{j+\nu}(t,y)\varphi_j(H)f(y)dy,$$

where $K_j(t,y)$ denotes the kernel of $(\varphi\psi)(2^{-j}H)$.

First, let $j > J$. Apply Lemma 3.3(a) to obtain

$$\frac{|\frac{d}{dt}(\varphi_j(H)f)(t)|}{\min_{k,\pm}(1+2^{j/2}|x \pm t \pm 2k|)^s} \le C_n \sum_{\nu=-1}^{1} \sum_{\ell,\sigma,\mu} \int_{\mathbb{R}}$$

$$\frac{2^{j+\nu}}{(1+2^{\frac{j+\nu}{2}}|t+\sigma y+\mu 2\ell|)^n} \times \frac{|\varphi_j(H)f(y)|}{\min_{k,\pm}(1+2^{j/2}|x \pm t \pm 2k|)^s}\,\mathrm{d}y,$$

where the inner sum is taken over all $0 \le \ell \le 2N$ and $\sigma, \mu \in \{\pm 1\}$.

We shall now prove the following inequality.

(11)
$$\frac{|\varphi_j(H)f(y)|}{\min_{\ell,\pm}(1+2^{j/2}|x\pm t\pm 2\ell|)^s} \le \max_{k,\pm}\varphi_j^* f(x\pm 2k)\min_{\ell,\pm}(1+2^{j/2}|t\pm y\pm 2\ell|)^s.$$

To prove (11), note that for given x, t, there are δ, $\epsilon \in \{\pm 1\}$, and ℓ_0 such that $\min_{\ell,\pm}(1+2^{j/2}|x\pm t\pm 2\ell|) = 1+2^{j/2}|x+\delta t+\epsilon 2\ell_0|$. Then for each σ, μ, and ℓ, the left hand side of (11) is less than or equal to

$$\frac{|\varphi_j(H)f(y)|}{\min_{k,\pm}(1+2^{j/2}|x+\epsilon\cdot 2\ell_0 \pm y\pm 2k|)^s}\cdot\frac{(1+2^{j/2}|x+\epsilon\cdot 2\ell_0+\sigma' y+\mu' 2\ell|)^s}{(1+2^{j/2}|x+\delta t+\epsilon 2\ell_0|)^s}$$
$$\le \varphi_j^* f(x+\epsilon 2\ell_0)(1+2^{j/2}|-\delta t+\sigma' y+\mu' 2\ell|)^s$$
$$\le \max_{k,\pm}\varphi_j^* f(x\pm 2k)(1+2^{j/2}|t+\sigma y+\mu 2\ell|)^s,$$

where we have set $\sigma' = -\delta\sigma$ and $\mu' = -\delta\mu$, and where, for $s>0$, we used the inequality

$$(1+2^{j/2}|x+\epsilon 2\ell_0+\sigma' y+\mu' 2\ell|)^s \le (1+2^{j/2}|x+\delta t+\epsilon 2\ell_0|)^s(1+2^{j/2}|-\delta t+\sigma' y+\mu' 2\ell|)^s.$$

Since σ, μ, ℓ are arbitrary, (11) is proved.

It follows that

$$\frac{|\frac{d}{dt}(\varphi_j(H)f)(t)|}{\min_{k,\pm}(1+2^{j/2}|x\pm t\pm 2k|)^s} \le C_n\sum_{\nu=-1}^{1}\sum_{\ell,\sigma,\mu}\max_{k,\pm}\varphi_j^* f(x\pm 2k)\times$$
$$\int_{\mathbb{R}}\frac{2^{j+\nu}}{(1+2^{\frac{j+\nu}{2}}|t+\sigma y+\mu 2\ell|)^n}\cdot(1+2^{j/2}|t+\sigma y+\mu 2\ell|)^s \mathrm{d}y$$
$$\le C_n \max_{\substack{0\le k\le 2N\\ \pm}}\varphi_j^* f(x\pm 2k)\sum_{\substack{\ell=0\\ \sigma,\mu=\pm 1}}^{2N}\int_{\mathbb{R}}\frac{2^{j+n/2}}{(1+2^{j/2}|t+\sigma y+\mu 2\ell|)^{n-s}}dy$$
$$\le C_{n,s}(2N+1)2^{j/2}\max_{\substack{0\le k\le 2N\\ \pm}}\varphi_j^* f(x\pm 2k),$$

provided $n-s>1$. Thus one may take $n=[s]+2$.

For $j\le J$ similarly we obtain the following inequalities, using Lemma 3.3(b) and Lemma 3.4 in place of Lemma 3.3(a):

$$\varphi_j^{**}f(x)\le C2^{j/2}\varphi_j^* f(x)$$

and

$$\Phi^{**}f(x)\le C\Phi^* f(x).$$

This proves Lemma 3.5. □

We are ready to prove Peetre's maximal inequality for H. Let M be the Hardy-Littlewood maximal operator:

$$Mf(x) = \sup_{x\in I} |I|^{-1} \int_I |f(u)|du,$$

where the supreme is taken over all intervals I containing x.

Lemma 3.6. *Let $0 < r < \infty$. There exists a constant $C > 0$ such that, for any $0 < \epsilon \le 1$,*

$$\varphi_j^* f(x) \le C\epsilon \sum_{\ell=0}^{2N} \varphi_j^* f(x \pm 2\ell) + C\epsilon^{-1/r} \sum_{\ell=0}^{2N} [M(|\varphi_j(H)f|^r)]^{1/r}(\pm x \pm 2\ell).$$

Proof. Let $g \in C^1$. As in [27], the mean value theorem gives, for $z_0 \in \mathbb{R}$ and $\delta > 0$, that

$$|g(z_0)| \le 2\delta \sup_{|z-z_0|\le\delta} |g'(z)| + (2\delta)^{-1/r} \left(\int_{|z-z_0|\le\delta} |g|^r dz \right)^{1/r}.$$

Letting $g(z) = \varphi_j(H)f(\pm x \pm 2\ell - z) \in C^1$, we obtain, with $0 < \delta \le 1, 0 \le \ell \le 2N_s$, that

$$\frac{|\varphi_j(H)f(\pm x \pm 2\ell - z)|}{(1+2^{j/2}|z|)^{1/r}} \le 2\delta \sup_{|u-z|\le\delta} \frac{(1+2^{j/2}|u|)^{1/r} |\frac{d}{dz}(\varphi_j(H)f)(\pm x \pm 2\ell - u)|}{(1+2^{j/2}|z|)^{1/r}(1+2^{j/2}|u|)^{1/r}}$$

$$+ (2\delta)^{-1/r}(1+2^{j/2}|z|)^{-1/r} \left(\int_{|u-z|\le\delta} |\varphi_j(H)f(\pm x \pm 2\ell - u)|^r du \right)^{1/r}$$

$$\le 2\delta(1+2^{j/2}\delta)^{1/r} \sup_{u\in\mathbb{R}} \frac{|\frac{d}{dz}(\varphi_j(H)f)(\pm x \pm 2\ell - u)|}{(1+2^{j/2}|u|)^{1/r}}$$

$$+ (2\delta)^{-1/r}(1+2^{j/2}|z|)^{-1/r} \left(\int_{|u|\le|z|+\delta} |\varphi_j(H)f(\pm x \pm 2\ell - u)|^r du \right)^{1/r}$$

$$\le 2\delta(1+2^{j/2}\delta)^{1/r} \varphi_j^{**} f(x) + \delta^{-1/r} \left(\frac{|z|+\delta}{1+2^{j/2}|z|} \right)^{1/r} [M(|\varphi_j(H)f(\pm x \pm 2\ell)|^r)]^{1/r}$$

$$\le C_r \epsilon \sum_{\ell=0}^{2N_s} \varphi_j^* f(x \pm 2\ell) + (1+\epsilon^{-1})^{1/r} \sum_{\ell=0}^{2N_s} [M(\varphi_j(H)f)^r]^{1/r}(\pm x \pm 2\ell),$$

by taking $\delta = 2^{-j/2}\epsilon$ and using Lemma 3.5. This proves the lemma. □

Remark 1. It is well known that M is bounded on $L^p, 1 < p < \infty$. Lemma 3.6 implies that if $s = 1/r$, then

$$\|\varphi_j^* f\|_p \le c \|\varphi_j(H)f\|_p, \quad 0 < p \le \infty, \tag{12}$$

by taking ϵ small enough and $0 < r < p$ ($s = 1/r > 1/p$).

Remark 2. For $j \le J$, the inequality in Lemma 3.6 takes a simpler form, viz.,

$$\begin{aligned}\varphi_j^* f(x) \le & C_s \epsilon^{-s}[M(|\varphi_j(H)f|^r)]^{1/r}(\pm x),\\ \Phi^* f(x) \le & C_s \epsilon^{-s}[M(|\Phi(H)f|^r)]^{1/r}(x),\end{aligned}$$

cf., the analogue in the Fourier case [27] and Hermite case [8].

A direct consequence of Lemma 3.6 is the Peetre maximal function characterization of the spaces $B_p^{\alpha,q}(H)$.

Theorem 3.7. *Let $\alpha \in \mathbb{R}, 0 < p < \infty$, $0 < q \le \infty$. If $\varphi_j^* f$ and $\Phi^* f$ are defined with $s > 1/p$, we have for $f \in L^2$*

$$\|f\|_{B_p^{\alpha,q}} \approx \|\Phi^* f\|_p + \left(\sum_{j=1}^{\infty} 2^{j\alpha q} \|\varphi_j^* f\|_p^q \right)^{1/q}. \tag{13}$$

Furthermore, $B_p^{\alpha,q}$ is a quasi-Banach space (Banach space if $p \ge 1, q \ge 1$) and it is independent of the choice of $\{\Phi, \varphi_j\}_{j \ge 1}$.

Proof. In view of (10), it is sufficient to show that

$$\|\Phi^* f\|_p + \left(\sum_{j=1}^{\infty} 2^{j\alpha q} \|\varphi_j^* f\|_p^q \right)^{1/q} \le C \|f\|_{B_p^{\alpha,q}},$$

but this follows from (12) immediately.

Next we show that $B_p^{\alpha,q}$ is independent of the generating functions, i.e., given two systems $\{\phi_j, \psi_j\}$ and $\{\tilde\phi_j, \tilde\psi_j\}$, then $\|f\|^{\phi}_{B_p^{\alpha,q}}$ and $\|f\|^{\tilde\phi}_{B_p^{\alpha,q}}$ are equivalent quasi-norms on $B_p^{\alpha,q}$.

Write $\phi_j(H) = \sum_{\nu=-1}^{1} \phi_j(H)(\tilde\phi\tilde\psi)_{j+\nu}(H)$ by the identity that, for all x, $\phi_j(x) = \phi_j(x) \sum_{\nu=-1}^{1} (\tilde\phi\tilde\psi)_{j+\nu}(x)$. We have by Lemma 3.1 that

$$|\phi_j(H)f(x)| \le C \sum_{\nu=-1}^{1} \sum_{\ell,\pm} \int_{\mathbb{R}} \frac{2^{j/2}}{(1+2^{j/2}|x \pm y \pm 2\ell|)^n} |\tilde\phi_{j+\nu}(H)f(y)| dy$$

$$\le C \sum_{\nu=-1}^{1} \tilde\phi^*_{j+\nu} f(x) \sum_{\ell,\pm} \int_{\mathbb{R}} \frac{2^{j/2}}{(1+2^{j/2}|x \pm y \pm 2\ell|)^n} \min_{k,\pm} (1+2^{j/2}|x \pm y \pm 2k|)^s dy$$

$$\le C \sum_{\nu=-1}^{1} \tilde\phi^*_{j+\nu} f(x),$$

provided $n - s > 1$. Thus, for $f \in L^2$,

$$\|f\|^{\phi}_{B_p^{\alpha,q}} = \|\{2^{j\alpha} \|\phi_j(H)f\|_p\}\|_{\ell^q} \le C_\alpha \|\{2^{j\alpha} \|\tilde\phi_j^* f\|_p\}\|_{\ell^q} \approx \|f\|^{\tilde\phi}_{B_p^{\alpha,q}}.$$

This concludes the proof of Theorem 3.7. That $B_p^{\alpha,q}$ are quasi-Banach spaces follows directly from the definition. □

As expected from Lemma 3.6 we can define the homogeneous Besov spaces and obtain a maximal function characterization as well.

Let $\varphi,\ \psi \in C^\infty$ satisfy

i) supp φ, supp $\psi \subset \{\frac{1}{4} \leq |\xi| \leq 1\}$;

$|\varphi(\xi)|,\ |\psi(\xi)| \geq c > 0$ if $\frac{3}{8} \leq |\xi| \leq \frac{7}{8}$;

ii) $\forall\, \xi \neq 0,\ \sum_{j=-\infty}^{\infty} \varphi(2^{-j}\xi)\psi(2^{-j}\xi) = 1$.

Definition. The *homogeneous Besov space* $\dot{B}_p^{\alpha,q} := \dot{B}_p^{\alpha,q}(H)$ associated with H is the completion of the set $\{f \in L^2 : \|f\|_{\dot{B}_p^{\alpha,q}} < \infty\}$ with respect to the norm $\|\cdot\|_{\dot{B}_p^{\alpha,q}}$, where

$$\|f\|_{\dot{B}_p^{\alpha,q}} = \left(\sum_{j=-\infty}^{\infty} (2^{j\alpha}\|\varphi_j(H)f\|_p)^q \right)^{1/q}.$$

Theorem 3.8. *Let $\alpha \in \mathbb{R}, 0 < p, q \leq \infty$. If $\varphi_j^* f$ is defined for $j \in \mathbb{Z}$ with $s > 1/p$, then for $f \in L^2$*

$$\|f\|_{\dot{B}_p^{\alpha,q}} \approx \left(\sum_{j=-\infty}^{\infty} 2^{j\alpha q}\|\varphi_j^* f\|_p^q \right)^{1/q}.$$

Furthermore, $\|\cdot\|_{\dot{B}_p^{\alpha,q}}^{\phi}$ and $\|\cdot\|_{\dot{B}_p^{\alpha,q}}^{\tilde{\phi}}$ are equivalent norms on the quasi-Banach space $\dot{B}_p^{\alpha,q}$ for any given two systems $\{\phi_j\}$ and $\{\tilde{\phi}_j\}$.

The proof is completely implicit in that of Theorem 3.7 and hence omitted.

Moreover, as in the Fourier case and Hermite case [27], [8], the Peetre maximal inequality enables us to define and characterize Triebel-Lizorkin spaces, see [30].

4. High and low energy estimates

We give proofs of Lemma 3.1 and Lemma 3.3 for the decay estimates of $\varphi_j(H)(x,y)$ and $\frac{\partial}{\partial x}\varphi_j(H)(x,y)$. Recall that $\varphi_j(H) = \int \varphi(2^{-j}\lambda)dE_\lambda = \mathcal{F}^{-1}\varphi_j(\xi^2)\mathcal{F}$ with supp $\varphi_j \subseteq [2^{j-2}, 2^j]$, which means that the spectrum of $\varphi_j(H)$ is bounded away from 0.

When $j > J = 4 + [2\log_2 \varepsilon]$, we treat $K_j(x,y)$, the kernel of the operator $\varphi_j(H)$, as an oscillatory integral as $\xi \to \infty$. When $j \leq J$, we use the asymptotic property (as $\xi \to 0$) of eigenfunctions $e(x,\xi)$ to obtain estimates for the kernel.

Since $e(x,\xi)$ has different expressions as $x>1,\ |x|\le 1$, and $x<-1$, the estimates are divided into nine cases, namely,

$$\begin{aligned}
&1a.\ x>1, y>1;\ 1b.\ x>1, |y|\le 1;\ 1c.\ x>1,\ y<-1;\\
&2a.\ |x|\le 1, y>1;\ 2b.\ |x|\le 1, |y|\le 1;\ 2c.\ |x|\le 1,\ y<-1;\\
&3a.\ |x|<-1, y>1;\ 3b.\ |x|<-1, |y|\le 1;\ 3c.\ x<-1, y<-1.
\end{aligned}$$

By virtue of the relation $e(x,-\xi)=e(-x,\xi)$ and the trivial conjugation relation $\varphi(\lambda^2 H)(x,y)=\overline{\overline{\varphi}(\lambda^2 H)(y,x)}=\varphi(\lambda H)(-x,-y)$, we see, however, that these cases reduce to the following four cases: $1a, 1b, 1c, 2b$.

Let $\lambda=2^{-j/2}$, then $\lambda^{-1}>4\varepsilon$ if and only if $j>J$. In the following we write $\psi(x)=\varphi(x^2)$ and use the notation $\widetilde{O}(\xi^{-m}):=\widetilde{O}_\infty(\xi^{-m})$ to denote a function whose derivatives of arbitrary order ≥ 0 has estimates $O(\xi^{-m})$, as $\xi\to\infty$.

4.1. *High energy estimates $j>J$*

Proof of Lemma 3.1(a). We only show Cases 1a and 2b. Cases 1b and 1c can be shown similarly.

Case $1a$. $x>1, y>1$. Let $I(x,y)=2\pi K_j(x,y)$. Then by (7) and §2.3

$$\begin{aligned}
I(x,y)=&\int_{1/2\lambda}^{1/\lambda}\psi(\lambda\xi)C_+e^{ix\xi}\overline{C_+e^{iy\xi}}d\xi\\
&+\int_{-1/\lambda}^{-1/2\lambda}\psi(\lambda\xi)(e^{ix\xi}+C'_-e^{-ix\xi})\overline{e^{iy\xi}+C'_-e^{-iy\xi}}d\xi:=I^++I^-.
\end{aligned}$$

We now use the notation $\int^+=\int_{1/2\lambda}^{1/\lambda}$ and $\int^-=\int_{-1/\lambda}^{-1/2\lambda}$.

We break the estimate of I^+ into two parts:

$$\begin{aligned}
\int^+&=\int_{1/2\lambda}^{1/\lambda}\psi(\lambda\xi)|C_+|^2e^{i(x-y)\xi}d\xi\\
&=\int_{1/2\lambda}^{1/\lambda}\psi(\lambda\xi)\frac{4K^2\xi^2}{4K^2\xi^2+\varepsilon^4\sin^2 2K}e^{i(x-y)\xi}d\xi\qquad (K=\sqrt{\xi^2-\varepsilon^2})\\
&\le\sum_{p=0}^{N-1}\left|\int_{1/2\lambda}^{1/\lambda}\psi(\lambda\xi)(\frac{\varepsilon^4\sin^2 2K}{4K^2\xi^2})^pe^{i(x-y)\xi}d\xi\right|\\
&+\left|\int_{1/2\lambda}^{1/\lambda}\psi(\lambda\xi)\widetilde{O}(\xi^{-4N})e^{i(x-y)\xi}d\xi\right|:=I_N^++R_N^+,
\end{aligned}$$

where we used

$$\frac{4K^2\xi^2}{4K^2\xi^2+\varepsilon^4\sin^2 2K}=\sum_{p=0}^{\infty}(-1)^p\left(\frac{\varepsilon^4\sin^2 2K}{4K^2\xi^2}\right)^p \tag{14}$$

because $\frac{\varepsilon^4 \sin^2 2K}{4K^2\xi^2} \le \frac{\varepsilon^4}{3\xi^4} \le \frac{1}{3}(\frac{1}{2})^4 < 1$, if $|\xi| \ge 1/2\lambda > 2\varepsilon$ ($K^2 \ge \frac{3}{4}\xi^2$).

If we write $\sin 2K = (2i)^{-1}(e^{i2K} - e^{-i2K})$, the integral in each term of the sum I_N^+ is bounded by a linear combination of the absolute values of the form

$$\int^{+'} = \int^{+} \psi(\lambda\xi) \frac{e^{\pm i2K\ell}}{(4K^2\xi^2)^p} e^{i(x-y)\xi} d\xi$$

$$= \int^{+} \psi(\lambda\xi)(4K^2\xi^2)^{-p} e^{\pm i2(K-\xi)\ell} e^{i(x-y\pm 2\ell)\xi} d\xi,$$

where $0 \le \ell \le 2p$, $0 \le p \le N-1$. The following estimates (as $\xi \sim \lambda^{-1} \to \infty$) will be used.

$$\begin{cases} \frac{d^n}{d\xi^n}(\psi(\lambda\xi)) &= O(\lambda^n), \\ \frac{d^i}{d\xi^i}((K^2\xi^2)^{-p}) &= O(\lambda^{4p+i}), \\ \frac{d^i}{d\xi^i}(e^{\pm i2(K-\xi)\ell}) &= \begin{cases} O(\lambda^{j+1}), & j > 0, \\ O(1), & j = 0. \end{cases} \end{cases}$$

We have

$$\frac{d^n}{d\xi^n}\left[\psi(\lambda\xi)(K^2\xi^2)^{-p} e^{\pm i2\ell(K-\xi)}\right] \le C\lambda^{4p+n}.$$

Integration by parts yields

$$\int^{+'} \le C_{n,\varepsilon} \frac{\lambda^{4p+n-1}}{|x-y\pm 2\ell|^n}.$$

It follows that

$$I_N^+ \le C_{n,\varepsilon} \sum_{p=0}^{N-1} \sum_{\ell=0}^{2p} \frac{\lambda^{4p+n-1}}{|x-y\pm 2\ell|^n}. \tag{15}$$

Also, because of the factor $\widetilde{O}(\xi^{-4N})$ we have

$$R_N^+ \le C \frac{\lambda^{-1}}{|x-y|^n} \lambda^{4N} \le C \frac{\lambda^{n-1}}{|x-y|^n} \quad (4N \ge n) \tag{16}$$

by integration by parts.

Combining (15) and (16) we obtain

$$|I^+| \le I_N^+ + R_N^+ \le C_{N,\varepsilon} \sum_{\ell=0}^{2N-2} \frac{\lambda^{n-1}}{|x-y\pm 2\ell|^n}.$$

For I^-, write

$$
\begin{aligned}
I^- =& \int^- \psi(\lambda\xi) e^{i(x-y)\xi} d\xi + \int^- \psi(\lambda\xi) \overline{C'_-} e^{i(x+y)\xi} d\xi \\
&+ \int^- \psi(\lambda\xi) C'_- e^{-i(x+y)\xi} d\xi + \int^- \psi(\lambda\xi) |C'_-|^2 e^{-i(x-y)\xi} d\xi \\
&:= I_1^- + I_2^- + I_3^- + I_4^-.
\end{aligned}
$$

As in estimating I^+, we have

$$
|I^-| \le C \sum_{\ell=0}^{2N} \frac{\lambda^{n-1}}{|x \pm y \pm 2\ell|^n}.
$$

Hence we obtain that if $x > 1,\ y > 1$, then

$$
2\pi |K_j(x,y)| \le |I^+| + |I^-| \le C \sum_{\ell=0}^{2N} \frac{\lambda^{n-1}}{|x \pm y \pm 2\ell|^n}.
$$

Case 2b. $|x| \le 1, |y| \le 1$. Let the notation be the same as in Case 1a. By symmetry it is enough to deal with $I^+ = \int^+$.

From the expression of B_+, B'_+ in §2.3 we have

$$
\begin{aligned}
I^+ &= \int^+ \psi(\lambda\xi)(B_+ e^{iKx} + B'_+ e^{-iKx}) \overline{B_+ e^{iKy} + B'_+ e^{-iKy}} d\xi \\
&= \int^+ \psi(\lambda\xi) |B_+|^2 e^{iK(x-y)} d\xi + \int^+ \psi(\lambda\xi) B_+ \overline{B'}_+ e^{iK(x+y)} d\xi \\
&+ \int^+ \psi(\lambda\xi) B'_+ \overline{B}_+ e^{-iK(x+y)} d\xi + \int^+ \psi(\lambda\xi) |B'_+|^2 e^{-iK(x-y)} d\xi \\
&:= I_1^+ + I_2^+ + I_3^+ + I_4^+.
\end{aligned}
$$

We estimate these terms separately. For instance,

$$
I_2^+ = \frac{1}{4} \int^+ \psi(\lambda\xi) e^{i(x+y)K} |C_+|^2 e^{-2iK} (1 - \xi^2/K^2) d\xi.
$$

Using the identity

$$
\begin{aligned}
|C_+|^2 =& \frac{4K^2\xi^2}{4K^2\xi^2 + \varepsilon^4 \sin^2 2K} \\
=& \sum_{p=0}^{N-1} (-1)^p \left(\frac{\varepsilon^2 \sin 2K}{2K\xi} \right)^{2p} + \widetilde{O}(\xi^{-4N}),
\end{aligned}
$$

we write

$$4I_2^+ = \sum_{p=0}^{N-1} (-1)^p \int^+ \psi(\lambda\xi) e^{i(x+y-2)K} \left(\frac{\varepsilon^2 \sin 2K}{2K\xi}\right)^{2p} (1-\xi^2/K^2) d\xi$$
$$+ \int^+ \psi(\lambda\xi) e^{i(x+y-2)K} \widetilde{O}(\xi^{-4N})(1-\xi^2/K^2) d\xi := I_{2,N}^+ + R_{2,N}^+.$$

The integral in each term of the sum $I_{2,N}^+$ is bounded by a linear combination of the form

$$\Big| \int^+ \psi(\lambda\xi) e^{i(x+y-2)K} e^{\pm i2K\ell} (2K\xi)^{-2p} (1-\xi^2/K^2) d\xi \Big|.$$

Integration by parts for $I_{2,N}^+$ and $R_{2,N}^+$ gives us

$$|I_2^+| \le C_N \sum_{\ell=0}^{2N-1} \frac{\lambda^{n-1}}{|x+y\pm 2\ell|^n}. \tag{17}$$

The other terms I_1^+, I_3^+, I_4^+ also satisfy the inequality (17) (possibly with $x+y$ replaced by $x-y$); and hence I^+ and I^- satisfy for all $|x| \le 1, |y| \le 1$,

$$|I^\pm| \le C \sum_{\ell=0}^{2N-1} \frac{\lambda^{n-1}}{|x \pm y \pm 2\ell|^n}.$$

This completes the proof of Lemma 3.1(a). □

Proof of Lemma 3.3(a). Note that $\frac{\partial}{\partial x} e(x,\xi) \in L^\infty_{loc}(\mathbb{R}\times\mathbb{R})$ exist for all $\xi \ne 0$. We have

$$\frac{\partial}{\partial x} K_j(x,y) = \frac{1}{2\pi} \int_{\mathbb{R}} \varphi(\lambda^2\xi^2) \frac{\partial}{\partial x} e(x,\xi) \bar{e}(y,\xi) d\xi. \tag{18}$$

Let $\delta(x) = x\psi(x)$, where $\psi(x) = \varphi(x^2)$. Further, let $j > J$ and $\lambda = 2^{-j/2}$ so that $\operatorname{supp} \delta(\lambda\,\cdot) \subseteq \{1/2\lambda \le |\xi| \le 1/\lambda\}$, with $2\varepsilon < 1/2\lambda$.
Case 1. $x > 1,\ y \in \mathbb{R}$. By (18)

$$\frac{\partial}{\partial x} K_j(x,y) = \int i\xi\psi(\lambda\xi)(Ce^{ix\xi} - C'e^{-ix\xi})\bar{e}(y,\xi) d\xi$$
$$= i\lambda^{-1} \int \delta(\lambda\xi)(Ce^{ix\xi} - C'e^{-ix\xi})\bar{e}(y,\xi) d\xi,$$

where $\delta(x)$ satisfies the same conditions as $\psi(x)$: (i) $\delta \in C^\infty$ (ii) $\operatorname{supp} \delta \subseteq \{\frac{1}{2} \le |\xi| \le 1\}$ (except for ψ being even, which is unimportant). Thus we obtain, similar to the case for $K_j(x,y)$,

$$\Big|\frac{\partial}{\partial x} \int \Big| \le C_N \sum_{\ell=0}^{2N} \frac{\lambda^{-2}}{(1+\lambda^{-1}|x\pm y\pm 2\ell|)^n}.$$

Case 2. $|x| \le 1,\ y \in \mathbb{R}$. Writing

$$B(\xi) = \begin{cases} B_+, & \xi > 0, \\ B_-, & \xi < 0, \end{cases}$$

and the corresponding $B'(\xi)$ (see §2.3), we have

$$\begin{aligned} \frac{\partial}{\partial x}\int &= \int iK\psi(\lambda\xi)(Be^{iKx} - B'e^{-iKx})\bar{e}(y,\xi)d\xi \\ &= i\lambda^{-1}\int \delta(\lambda\xi)(Be^{iKx} - B'e^{-iKx})K/\xi\ \overline{e}(y,\xi)d\xi. \end{aligned}$$

Thus we obtain, similar to the case for $K_j(x,y)$,

$$|\frac{\partial}{\partial x}\int| \le C_N \sum_{\ell=0}^{2N} \frac{\lambda^{-2}}{(1+\lambda^{-1}|x \pm y \pm 2\ell|)^n}.$$

Case 3. $x < -1,\ y \in \mathbb{R}$. The relation $\varphi(\lambda H)(x,y) = \varphi(\lambda H)(-x,-y)$ implies

$$\frac{\partial}{\partial x}[\varphi(\lambda H)(x,y)] = -[\frac{\partial}{\partial x}\phi(\lambda H)](-x,-y)].$$

Therefore, if $x < -1$, then, by Case 1,

$$\begin{aligned} &\big|\frac{\partial}{\partial x}\varphi(\lambda H)(x,y)\big| = \big|\frac{\partial}{\partial x}\big[\varphi(\lambda H)(x,y)\big](-x,-y)\big| \\ \le& C\sum_{\ell=0}^{2N} \frac{\lambda^{-2}}{(1+\lambda^{-1}|-x \mp y \pm 2\ell|)^n}. \end{aligned}$$

This concludes the proof of Lemma 3.3(a). □

4.2. *Low energy estimates* $j \le J$

Proof of Lemma 3.1(b). As in the high energy case, we only need to check the four cases 1a, 1b, 1c, and 2b. Outlines will be given for 1a, 2b only.
Case 1a. $x > 1,\ y > 1$ ($1/\lambda \le 4\varepsilon$; $\lambda = 2^{-j/2} \to \infty$ as $j \to -\infty$).

$$2\pi K_j(x,y) = \int_{\mathbb{R}} \psi(\lambda\xi)e(x,\xi)\bar{e}(y,\xi)d\xi = \int^+ + \int^- .$$

We obtain by integration by parts that

$$|\int^+| \le C\frac{\lambda^{n-1}}{|x-y|^n},$$

where we used (as $\xi \sim \lambda^{-1} \to 0$)

$$\begin{cases} \frac{d^n}{d\xi^n}(\psi(\lambda\xi)) & = O(\lambda^n), \\ \frac{d^i}{d\xi^i}(|C_+|^2) & = \begin{cases} O(\lambda^{-2}), & i = 0, \\ O(\lambda^{-1}), & i = 1, \\ O(1), & i > 1. \end{cases} \end{cases}$$

We also obtain

$$|\int^{-}| \leq C_n \frac{\lambda^{n-1}}{|x-y|^n},$$

using the facts that

$$\frac{d^i}{d\xi^i}(|C'_-|^2) = \frac{d^i}{d\xi^i}\Big(\frac{\varepsilon^4}{(2\rho/\sinh 2\rho)^2\xi^2 + \varepsilon^4}\Big) = O(1), \quad \xi \to 0^-$$

and

$$\frac{d^i}{d\xi^i} C'_- = O(1), \quad \xi \to 0^-,$$

where we note that $C'_- = C'_-(\xi)$ is C^∞.

Case 2b. $|x| \leq 1,\ |y| \leq 1$. Let $I^+(x,y) := \int^+$, $I^-(x,y) := \int^-$. Then

$$|I^+(x,y)| = |\int^+ \psi(\lambda\xi)(B_+e^{\rho x} + B'_+e^{-\rho x})\overline{B_+e^{\rho y} + B'_+e^{-\rho y}}d\xi|$$

$$=|\int^+ |C_+|^2(\cosh\rho(1-x) - i\xi/\rho\sinh\rho(1-x))(\cosh\rho(1-y) + i\xi/\rho\sinh\rho(1-y))d\xi|$$

$$\leq C\lambda^{-1} \leq 3^n C \frac{\lambda^{-1}}{1+\lambda^{-1}|x\pm y|)^n},$$

where we note that $\cosh\rho(1-x) - i\xi/\rho\sinh\rho(1-x)$ and $\cosh\rho(1-y) + i\xi/\rho\sinh\rho(1-y)$ are uniformly bounded in $|x| \leq 1$ and $|y| \leq 1$.

The term $I^-(x,y)$ satisfies the same inequality since $I^-(x,y) = I^+(-x,-y)$. □

Proof of Lemma 3.3(b). The same argument in proving Lemma 3.1(b) is valid for the proof of Lemma 3.3(b). □

5. Local energy estimates

Let $\Phi \in C^\infty$ have support contained in $\{|\xi| \leq 1\}$. Then the spectrum of $\Phi(H)$ includes the *local energy*, a neighborhood of 0. We use the term "local energy" to distinguish from the low energy case, where the support of φ_j ($j \leq J$) keeps away from 0. Since $0 \in \operatorname{supp}\Phi$ and $e(x,\xi)$ is *discontinuous* at the origin $\xi = 0$, we need to treat the corresponding kernel more carefully. The proof is more delicate and requires a "matching" method.

Proof of Lemma 3.2. As in §4, the estimates rely on the four cases, 1a, 1b, 1c, and 2b. We use $\hat{f}$ and $\check{f}$ to denote the ordinary Fourier transform and its inverse, respectively.

Case 1a. $x > 1, y > 1$. Let $K(x,y) = \Phi(H)(x,y)$, $\Psi(x) = \Phi(x^2)$.

$$\begin{aligned}2\pi K(x,y) =& \int_0^1 \Psi(\xi) C_+ e^{ix\xi} \overline{C_+ e^{iy\xi}} d\xi + \int_{-1}^0 \Psi(\xi)(e^{ix\xi} + C'_- e^{-ix\xi}) \overline{e^{iy\xi} + C'_- e^{-iy\xi}} d\xi \\ =& I^+ + I^-.\end{aligned}$$

We write

$$\begin{aligned}I^- =& \int_{-1}^0 \Psi(\xi) e^{i(x-y)\xi} d\xi + \int_{-1}^0 \Psi(\xi) C'_- e^{-i(x+y)\xi} d\xi \\ &+ \int_{-1}^0 \Psi(\xi) \overline{C'_-} e^{i(x+y)\xi} d\xi + \int_{-1}^0 \Psi(\xi) |C'_-|^2 e^{-i(x-y)\xi} d\xi \\ :=& I_1^- + I_2^- + I_3^- + I_4^-.\end{aligned}$$

The relations $C'_-(-\xi) = A'_+(\xi) = \overline{C'_-}(\xi)$ and $|C_+|^2 + |A'_+|^2 = 1$ imply that

$$\begin{aligned}I^+ + I_1^- + I_4^- =& \int_0^1 \Psi(\xi) |C_+|^2 e^{i(x-y)\xi} d\xi \\ &+ \int_{-1}^0 \Psi(\xi) e^{i(x-y)\xi} d\xi + \int_0^1 \Psi(\xi) |C'_-|^2 e^{i(x-y)\xi} d\xi \\ =& \int_{-1}^1 \Psi(\xi) e^{i(x-y)\xi} d\xi = \sqrt{2\pi}\Psi^\vee(x-y).\end{aligned}$$

Also, the relation $C'_-(-\xi) = \overline{C'_-}(\xi)$ gives

$$\begin{aligned}I_2^- + I_3^- =& \int_{-1}^0 \Psi(\xi) C'_- e^{-i(x+y)\xi} d\xi + \int_0^1 \Psi(\xi) C'_-(\xi) e^{-i(x+y)\xi} d\xi \\ =& \sqrt{2\pi}(\Psi(\xi)C'_-(\xi))^\wedge(x+y).\end{aligned}$$

Since $\Psi \in C_c^\infty$ and $C'_- \in C^\infty$, we have, for $x > 1$ and $y > 1$, that

$$\begin{aligned}2\pi|K(x,y)| \leq& |I^+ + I_1^- + I_4^-| + |I_2^- + I_3^-| \\ \leq& \frac{C_n}{(1+|x-y|)^n} + \frac{C_n}{(1+|x+y|)^n} \leq 2\frac{C_n}{(1+|x-y|)^n}\end{aligned}$$

by the rapid decay for the Fourier transform of C_c^∞ functions, where

$$C'_-(\xi) = i\frac{\varepsilon^2 e^{2i\xi} \sinh 2\rho/2\rho}{\xi \cosh 2\rho - i(\rho^2 - \xi^2)\sinh 2\rho/2\rho} \in C^\infty.$$

Case 1b. $x > 1, |y| \leq 1$.

Using $e_+(y,\xi) = C_+ e^{i\xi}[\cosh\rho(1-y) - i\xi/\rho \sinh\rho(1-y)]$ and $A_- = 2\rho\xi e^{2i\xi}/(2\rho\xi\cosh 2\rho - i(\rho^2-\xi^2)\sinh 2\rho)$, we have

$$2\pi K(x,y) = \int_0^1 \Psi(\xi)e^{i(x-1)\xi}|C_+|^2(\Re + i\Im)[\cosh\rho(1-y) + i\xi/\rho\sinh\rho(1-y)]d\xi$$
$$+\int_{-1}^0 \Psi(\xi)e^{i(x-1)\xi}(\Re+i\Im)\frac{2\rho\xi[\cosh\rho(1+y) - i\xi/\rho\sinh\rho(1+y)]}{2\rho\xi\cosh 2\rho + i(\rho^2-\xi^2)\sinh 2\rho}$$
$$+\int_{-1}^0 \Psi(\xi)e^{-i(x-1)\xi}(\Re+i\Im)\frac{i\varepsilon^2\sinh 2\rho\cdot 2\rho\xi}{4\rho^2\xi^2+\varepsilon^4\sinh^2 2\rho}[\cosh\rho(1+y) - i\xi/\rho\sinh\rho(1+y)]d\xi$$
$$:= \text{“}\mathcal{R}e\text{ ”} + i\text{“}\mathcal{I}m\text{”},$$

where we break each of the above three integrals into two parts; then let “$\mathcal{R}e$” be the sum of the three integrals involving $\Re$ only, and let “$\mathcal{I}m$” be the sum of the three integrals involving $\Im$ only. We have

$$\begin{aligned}
\text{“}\mathcal{R}e\text{”} &= \int_0^1 \Psi(\xi)|C_+|^2 e^{i(x-1)\xi}\cosh\rho(1-y)d\xi \\
&\quad+\int_{-1}^0 \Psi(\xi)e^{i(x-1)\xi}\Re\Big[\frac{2\rho\xi(\cosh\rho(1+y) - i\xi/\rho\sinh\rho(1+y))}{2\rho\xi\cosh 2\rho + i(\rho^2-\xi^2)\sinh 2\rho}\Big]d\xi \\
&\quad+\int_{-1}^0 \Psi(\xi)e^{-i(x-1)\xi}\frac{2\varepsilon^2\xi^2\sinh 2\rho\sinh\rho(1+y)}{4\rho^2\xi^2+\varepsilon^4\sinh^2 2\rho}d\xi \\
&= \int_0^1 \Psi(\xi)e^{i(x-1)\xi}\frac{4\rho^2\xi^2\cosh\rho(1-y) + 2\varepsilon^2\xi^2\sinh 2\rho\sinh\rho(1+y)}{4\rho^2\xi^2+\varepsilon^4\sinh^2 2\rho} +
\end{aligned}$$
$$\int_{-1}^0 \Psi(\xi)e^{i(x-1)\xi}\frac{4\rho^2\xi^2\cosh 2\rho\cosh\rho(1+y) - 2\xi^2(\rho^2-\xi^2)\sinh 2\rho\sinh\rho(1+y)}{4\rho^2\xi^2+\varepsilon^4\sinh^2 2\rho}d\xi.$$

Noting that $\rho^2-\xi^2 = 2\rho^2-\varepsilon^2$ and $\cosh 2\rho\cosh\rho(1+y) - \sinh 2\rho\sinh\rho(1+y) = \cosh\rho(1-y)$, we obtain

(19)
$$\text{“}\mathcal{R}e\text{”} = \sqrt{2\pi}\Big[\Psi(\xi)e^{-i\xi}\frac{(4\rho^2+2\varepsilon^2)\xi^2\cosh\rho(1-y) + 2\varepsilon^2\xi^2\sinh 2\rho\sinh\rho(1+y)}{4\rho^2\xi^2+\varepsilon^4\sinh^2 2\rho}\Big]^{\vee}(x).$$

For the "imaginary part",

$$\text{“}\mathcal{I}m\text{”} = \int_0^1 \Psi(\xi)|C_+|^2 e^{i(x-1)\xi}\xi/\rho \sinh\rho(1-y)d\xi$$
$$+\int_{-1}^0 \Psi(\xi)e^{i(x-1)\xi}\Im\{\frac{2\rho\xi[\cosh\rho(1+y)-i\xi/\rho\sinh\rho(1+y)]}{2\rho\xi\cosh 2\rho + i(\rho^2-\xi^2)\sinh 2\rho}\}d\xi$$
$$+\int_{-1}^0 \Psi(\xi)e^{-i(x-1)\xi}\frac{\varepsilon^2\sinh 2\rho 2\rho\xi}{4\rho^2\xi^2+\varepsilon^4\sinh^2 2\rho}\cosh\rho(1+y)d\xi$$
$$=\int_0^1 \Psi(\xi)e^{i(x-1)\xi}\frac{2\rho\xi}{4\rho^2\xi^2+\varepsilon^4\sinh^2 2\rho}[2\xi^2\sinh\rho(1-y)-\varepsilon^2\sinh 2\rho\cosh\rho(1+y)]d\xi +$$
$$\int_{-1}^0 \Psi(\xi)e^{i(x-1)\xi}\frac{-2\rho\xi}{4\rho^2\xi^2+\varepsilon^4\sinh^2 2\rho}\cdot$$
$$\cdot[2\xi^2\cosh 2\rho\sinh\rho(1+y)+(\rho^2-\xi^2)\sinh 2\rho\cosh\rho(1+y)]d\xi.$$

Noting that $\rho^2-\xi^2=\varepsilon^2-2\xi^2$ and $\sinh 2\rho\cosh\rho(1+y)-\cosh 2\rho\sinh\rho(1+y)=\sinh\rho(1-y)$, we obtain

(20)

$$\text{“}\mathcal{I}m\text{”} = \sqrt{2\pi}\big[\Psi(\xi)e^{-i\xi}\frac{2\rho\xi}{4\rho^2\xi^2+\varepsilon^4\sinh^2 2\rho}(2\xi^2\sinh\rho(1-y)-\varepsilon^2\sinh 2\rho\cosh\rho(1+y))\big]^\vee(x).$$

Since the functions in the square brackets of (19) and (20) are C_c^∞, it follows that for all $x>1$, $|y|\le 1$,

$$|K(x,y)|\le\frac{C_n}{(1+|x|)^n}\le\frac{2^nC_n}{(1+|x-y|)^n}.$$

Case 1c. $x>1, y<-1$. The proof is similar to that of Case 1a and hence omitted.

Case 2b. $|x|\le 1, |y|\le 1$. Since $|e(x,\xi)|\le C_\varepsilon$, for all x,ξ, the result is straightforward:

$$|K(x,y)|\le C'_\varepsilon\sim\frac{C_n}{(1+|x-y|)^n}$$

This concludes the proof of Lemma 3.2. □

Proof of Lemma 3.4. With the convention $\int^+=\int_0^1$, $\int^-=\int_{-1}^0$, we have

$$2\pi\frac{\partial}{\partial x}K(x,y)=\frac{\partial}{\partial x}\int_{-1}^1\Psi(\xi)e(x,\xi)\bar{e}(y,\xi)d\xi$$
$$:=\frac{\partial}{\partial x}\int^+ +\frac{\partial}{\partial x}\int^-.$$

The function $\xi \mapsto \frac{\partial}{\partial x} e(x,\xi)$ is discontinuous at $\xi = 0$. As suggested by the treatment of $K(x,y)$ we want to "match" different parts of the above integrals properly so that $\frac{\partial}{\partial x} K(x,y)$ can be written as a linear combination of the Fourier transform of C_c^∞ functions.

We only need to check five cases $1a, 1b, 1c, 2a, 2b$. Estimates for the other cases follow readily from the relation $\frac{\partial}{\partial x} K(x,y) = \frac{\partial}{\partial x}[K(-x,-y)] = -(\frac{\partial}{\partial x} K)(-x,-y)$. We outline the proofs for 1a, 1b and 2b only, since 1c and 2a can be dealt with similarly.

Case 1a. $x > 1, y > 1$. Let $\Delta(\xi) = i\xi\Psi(\xi) \in C_c^\infty$.

$$\frac{\partial}{\partial x}\int^+ = \int^+ \Psi(\xi) i\xi |C_+|^2 e^{i(x-y)\xi} d\xi = \int^+ \Delta(\xi)|C_+|^2 e^{i(x-y)\xi} d\xi.$$

$$\begin{aligned}
\frac{\partial}{\partial x}\int^- &= \int^- i\xi\Psi(\xi)(e^{ix\xi} - C'_- e^{-ix\xi})\overline{e^{iy\xi} + C'_- e^{-iy\xi}} d\xi \\
&= \int^- \Delta(\xi) e^{i(x-y)\xi} d\xi - \int^- \Delta(\xi)|C'_-|^2 e^{-i(x-y)\xi} d\xi \\
&\quad + \int^- \Delta(\xi)\overline{C'_-} e^{i(x+y)\xi} d\xi - \int^- \Delta(\xi) C'_- e^{-i(x+y)\xi} d\xi \\
&= \int^- \Delta(\xi) e^{i(x-y)\xi} d\xi + \int^+ \Delta(\xi)|C'_-|^2 e^{i(x-y)\xi} d\xi \\
&\quad - \int^+ \Delta(\xi) C'_- e^{-i(x+y)\xi} d\xi - \int^- \Delta(\xi) C'_- e^{-i(x+y)\xi} d\xi,
\end{aligned}$$

where we note that $\Delta(\xi)$ is odd and $C'_-(\xi) = \overline{C'_-}(-\xi)$. We have, by the relation $|C_+|^2 + |C'_-|^2 = 1$, that

$$\begin{aligned}
\frac{\partial}{\partial x}\int^+ + \frac{\partial}{\partial x}\int^- &= \int \Delta(\xi) e^{i(x-y)\xi} d\xi - \int \Delta(\xi) C'_- e^{-i(x+y)\xi} d\xi \\
&= \sqrt{2\pi}[\Delta(\xi)]^\vee (x-y) - \sqrt{2\pi}[\Delta(\xi) C'_-]^\wedge (x+y).
\end{aligned}$$

Since $\Delta \in C_c^\infty, C'_-(\xi) \in C^\infty$, the inequality in Lemma 3.4 holds for all $x > 1, y > 1$.

Case 1b. $x > 1, |y| \le 1$. Let the notation be as in Case 1a. Then

$$\frac{\partial}{\partial x}\int^+ = \int^+ \Delta(\xi)|C_+|^2 e^{i(x-1)\xi}(\Re + i\Im)[\cosh\rho(1-y) + i\xi/\rho \sinh\rho(1-y)] d\xi$$

$$\frac{\partial}{\partial x}\int^- = \int^- \Delta(\xi) e^{i(x-1)\xi}(\Re + i\Im)\Big[\frac{2\rho\xi(\cosh\rho(1+y) - i\xi/\rho\ \sinh\rho(1+y))}{2\rho\xi\cosh\rho + i(\rho^2 - \xi^2)\sinh 2\rho}\Big] d\xi$$

$$-\int^- \Delta(\xi) e^{-i(x-1)\xi}(\Re + i\Im)\Big[\frac{i\varepsilon^2 \sinh 2\rho \cdot 2\rho\xi}{4\rho^2\xi^2 + \varepsilon^4 \sinh^2 2\rho}(\cosh\rho(1+y) - i\xi/\rho \sinh\rho(1+y))\Big] d\xi.$$

As in the case for $K(x,y)$, we split each integral into two parts and let "$\mathcal{R}e$" and "$\mathcal{I}m$" denote the sum of integrals involving only those consisting of "real parts" and "imaginary parts", respectively. As a result,

$$2\pi\frac{\partial}{\partial x}K(x,y) = \text{“}\mathcal{R}e\text{”} + i\,\text{“}\mathcal{I}m\text{”},$$

where we find, by noting that Δ is odd, that "$\mathcal{R}e$" and "$\mathcal{I}m$" have the same expressions as in (19) and (20), respectively, except that Ψ should be replaced by Δ. Case 1b is so verified by this observation.

Finally, the decay estimate for Case 2b ($|x|, |y| \leq 1$) follows easily from the fact that $e(y,\xi) \in L^\infty(\mathbb{R}\times\mathbb{R})$ and $\frac{\partial}{\partial x}e(x,\xi) \in L^\infty_{loc}(\mathbb{R}\times\mathbb{R})$. In fact, we have

$$\begin{aligned}|2\pi K(x,y)| &= \Big|\int_{|\xi|\leq 1}\Psi(\xi)\frac{\partial}{\partial x}e(x,\xi)\bar{e}(y,\xi)d\xi\Big| \\ &\leq C_\varepsilon \sim \frac{C_n}{(1+|x-y|)^n},\end{aligned}$$

where, by §2.3,

$$\frac{\partial}{\partial x}e_+(x,\xi) = C_+e^{i\xi}[-\rho\sinh\rho(1-x) + i\xi\cosh\rho(1-x)],$$

$$\frac{\partial}{\partial x}e_-(x,\xi) = A_-e^{-i\xi}[\rho\sinh\rho(1+x) + i\xi\cosh\rho(1+x)].$$

This completes the proof of Lemma 3.4. □

6. Spectral multipliers for L^p and $B_p^{\alpha,q}(H)$

Throughout this section we assume that $m : \mathbb{R} \to \mathbb{C}$ is bounded and $|m'(\xi)| \leq C|\xi|^{-1}$ for $\xi \neq 0$. We prove that under this same differentiability condition on m as in the Fourier case, $m(H)$ has a bounded extension on L^p from $L^p \cap L^2$. To do so we need to show that the kernel $m(H)(x,y)$ satisfies a Hörmander type condition:

$$\int_{z>2|y-\bar{y}|}|m(H)(x,y) - m(H)(x,\overline{y})|dx \leq A, \tag{21}$$

where $z = \min_\pm(|x \pm \overline{y}|)$ (compare the Fourier case [25]).

We begin with two technical lemmas that will be proved at the end of this section. Let $\{\delta_j\}_{-\infty}^{\infty}$ be a smooth dyadic resolution of unit and let $m_j(x) = m\delta_j(x)$. Denote by K_j the kernel of $m_j(H)$.

Lemma 6.1. *Let $z = \min|x \pm y|$ and $\lambda = 2^{-j/2}$. Then there exists a constant C independent of y so that*

$$\|K_j(\cdot, y)\|_2 \le C\lambda^{-1/2}, \tag{22}$$

$$\|zK_j(\cdot, y)\|_2 \le C\lambda^{1/2}, \tag{23}$$

$$\int_{|z|>t} |K_j(x, y)|dx \le Ct^{-1/2}\lambda^{1/2}. \tag{24}$$

Lemma 6.2. *Let z, λ be as above. Then there exists a constant C, independent of y so that*

$$\|\frac{\partial}{\partial y}K_j(\cdot, y)\|_2 \le C\lambda^{-3/2}, \tag{25}$$

$$\|z\frac{\partial}{\partial y}K_j(\cdot, y)\|_2 \le C\lambda^{-1/2}, \tag{26}$$

$$\int_{|z|>t}\Big|\frac{\partial}{\partial y}K_j(x, y)\Big|dx \le Ct^{-1/2}\lambda^{-1/2}. \tag{27}$$

To show (21) an immediate question arises: what is the kernel expression for $m(H)$? Since m may not necessarily have compact support, the answer is not so immediate. For $f \in L^2$, $m(H)f = \sum_{-\infty}^{\infty} m_j(H)f$ in L^2. This suggests that $m(H)(x, y)$ may have the (pointwise) expression $\sum_{-\infty}^{\infty} K_j(x, y)$. Our next lemma shows that this is true in an appropriate sense.

Lemma 6.3. *Let m be bounded and let $|m'(\xi)| \le C|\xi|^{-1}$ for $\xi \in \mathbb{R} \setminus \{0\}$. Then, for $f \in L^2_c = \{f \in L^2 : supp\ f$ is compact$\}$, $m(H)f$ has the expression*

$$m(H)f(x) = \int K(x, y)f(y)dy$$

for a.e. $x \notin \pm supp\ f$, where $K(x, y) = \sum_{-\infty}^{\infty} K_j(x, y)$.

Proof. Since $\sum_{-\infty}^{\infty} m_j(H)f$ converges to $m(H)f$ in L^2, it suffices to show the series $\sum m_j(H)f(x)$ converges pointwise for each $x \notin \pm \text{supp} f$.

Let $t > 0$ be the distance from x to the set $(\text{supp } f) \cup (-\text{supp } f)$. Then $\text{supp } f \subset \{y : \min(|y + x|, |y - x|) \ge t\}$. By Lemma 6.1 we have, for $x \notin \pm\text{supp } f$ and $J \in \mathbb{Z}$, that

$$\sum_{-\infty}^{J} \Big|\int K_j(x, y)f(y)dy\Big| \le \|f\|_2 \sum_{-\infty}^{J} \|K_j(x, \cdot)\|_2$$

$$\le C\|f\|_2 \sum_{-\infty}^{J} 2^{j/4} \le C_J\|f\|_2;$$

and, writing $\min|y \pm x| = \min(|y+x|, |y-x|)$, we have

$$\sum_{J+1}^{\infty}\Big|\int K_j(x,y)f(y)dy\Big| = \sum_{J+1}^{\infty}\Big|\int_{\min|y\pm x|>t} K_j(x,y)f(y)dy\Big|$$
$$\leq \|f\|_2 \sum_{J+1}^{\infty}\Big(\int_{\min|y\pm x|>t}|K_j(x,y)|^2dy\Big)^{1/2}$$
$$= C\|f\|_2 t^{-1}\sum_{J+1}^{\infty}2^{-j/4} \leq C_J\|f\|_2 t^{-1}.$$

This shows that $\sum m_j(H)f(x)$ converges for all $x \notin \pm\mathrm{supp} f$.

□

We are ready to verify the Hörmander condition for $m(H)$.

Lemma 6.4. *Let $z = \min|x \pm \overline{y}|$, $t = |y - \overline{y}|$ and $\lambda = 2^{-j/2}$. Then*

$$\int_{|z|>2t}|K_j(x,y) - K_j(x,\overline{y})|\mathrm{d}x \leq C\begin{cases} t^{1/2}\lambda^{-1/2} & if \quad t\lambda^{-1} \leq 1 \\ t^{-1/2}\lambda^{1/2} & if \quad t\lambda^{-1} > 1. \end{cases} \tag{28}$$

Moreover,

$$\int_{|z|>2t}|K(x,y) - K(x,\overline{y})|\mathrm{d}x \leq A, \tag{29}$$

where $K(x,y)$ denotes the kernel of $m(H)$ as given in Lemma 6.3.

Proof. Let $y \in \overline{y} + I, I = [-t,t]$. If $t\lambda^{-1} \leq 1$, then, by Lemma 6.2,

$$\int_{\{|x-\bar{y}|>2t\}\cap\{|x+\bar{y}|>2t\}}|K_j(x,y) - K_j(x,\bar{y})|dx$$
$$= \int_{z>2t}\Big|\int_{\bar{y}}^{y}\frac{\partial}{\partial\xi}K_j(x,\xi)d\xi\Big|dx$$
$$\leq \int_{\bar{y}}^{y}d\xi\int_{z>2t}\Big|\frac{\partial}{\partial\xi}K_j(x,\xi)\Big|dx$$
$$\leq t\sup_{\xi\in\bar{y}+I}\int_{\{|x-\bar{y}|>2t\}\cap\{|x+\bar{y}|>2t\}}\Big|\frac{\partial}{\partial\xi}K_j(x,\xi)\Big|dx$$
$$\leq C_\varepsilon t^{1/2}\lambda^{-1/2},$$

for all $\bar{y}$.

If $t\lambda^{-1} > 1$, then, by Lemma 6.1,

$$\begin{aligned}
&\int_{|z|>2t} |K_j(x,y) - K_j(x,\bar{y})|dx \\
\leq &\int_{|z|>2t} |K_j(x,y)|dx + \int_{|z|>2t} |K_j(x,\bar{y})|dx \\
\leq &\int_{\min|x\pm y|>t} |K_j(x,y)|dx + \int_{\min|x\pm\bar{y}|>2t} |K_j(x,\bar{y})|dx \\
\leq &Ct^{-1/2}\lambda^{1/2}.
\end{aligned}$$

This proves (28).

The inequality (29) follows easily from Lemma 6.3 and (28). □

Theorem 6.5. *Suppose $m \in L^\infty$ satisfies $|m'(\xi)| \leq C|\xi|^{-1}$. Then $m(H)$ is bounded on $L^p, 1 < p < \infty$, and of weak type $(1,1)$.*

As a consequence of Theorem 6.5, we shall show that $m(H)$, initially defined for $f \in L^2$, has a bounded linear extension to the Banach spaces $B_p^{\alpha,q}(H)$, $1 < p < \infty$.

Theorem 6.6. *Suppose $m \in L^\infty$ is as above. Then $m(H)$ extends to a bounded linear operator on $B_p^{\alpha,q}(H)$ for $1 < p < \infty$, $1 \leq q \leq \infty$, $\alpha \in \mathbb{R}$.*

Proof of Theorem 6.5. Applying the Calderón-Zygmund decomposition and using Lemma 6.4, we can obtain the *weak* $(1,1)$ result for $m(H)$. Then the L^p result, $1 < p < \infty$, follows by means of Marcinkiewicz interpolation and duality. For completeness, we prove the *weak* type $(1,1)$ estimate. By density, it is enough to assume $f \in L^1 \cap L^2$.

Given $f \in L^1, s > 0$. According to the Calderón-Zygmund lemma there is a decomposition $f = g + b$ with $b = \sum b_k$ and a countable collection of disjoint open intervals I_k such that the following properties hold.

i) $|g(x)| \leq Cs$, a.e.

ii) Each b_k is supported in $I_k, \int b_k dx = 0$, and

$$s \leq \frac{1}{|I_k|}\int_{I_k} |b_k|dx \leq 2s.$$

iii) Let $D_s = \cup_k I_k = \cup_k(\bar{y}_k - t_k, \bar{y}_k + t_k)$, where $2t_k = |I_k| > 0$ and $\bar{y}_k$ is the center of I_k. Then

$$|D_s| \leq Cs^{-1}\|f\|_1.$$

iv) $g \in L^1 \cap L^2$, $g(x) = f(x)$ if $x \notin D_s$, and

$$\|g\|_2^2 \leq Cs\|f\|_1, \ \|b\|_1 \leq 2\|f\|_1. \tag{30}$$

Now let $f \in L^1 \cap L^2$. Then $b = \sum b_k$ converges both a.e. and in $L^1 \cap L^2$, by the definition of b_k and properties (ii) and (iii), where

$$b_k(x) = \begin{cases} f(x) - \frac{1}{|I_k|}\int_{I_k} f dy, & x \in I_k, \\ 0, & x \notin I_k. \end{cases}$$

It follows from Lemma 6.4 and properties (ii) and (iv) that

$$\begin{aligned} \int_{\mathbb{R}\setminus D_s^*} |m(H)b(x)|dx \leq & \sum_k \int_{\mathbb{R}\setminus D_s^*} |m(H)b_k(x)|\mathrm{d}x \\ \leq & \sum_k \int_{I_k} |b_k(y)|dy \int_{\mathbb{R}\setminus I_k^*} |K(x,y) - K(x,\overline{y}_k)|\mathrm{d}x \\ \leq & A\sum_k \int |b_k(y)|dy \leq 2A\|f\|_1, \end{aligned} \tag{31}$$

where $D_s^* = \cup_k I_k^*$ and $I_k^* = (\overline{y}_k - 2t_k, \overline{y}_k + 2t_k) \cup (-\overline{y}_k - 2t_k, -\overline{y}_k + 2t_k)$. Since $|D_s^*| \leq 4|D_s|$, we obtain the weak $(1,1)$ estimate from (30) and (31). □

Proof of Theorem 6.6. For $g \in L^2 \cap B_p^{\alpha,q}(H)$,

$$\begin{aligned} \|m(H)g\|_{B_p^{\alpha,q}} =& \|\Phi(H)m(H)g\|_p + \left\{ \sum_{j=1}^{\infty} (2^{j\alpha}\|\varphi_j(H)m(H)g\|_p)^q \right\}^{1/q} \\ =& \|\{2^{j\alpha}\varphi_j(H)m(H)g\}\|_{\ell^q(L^p)}. \end{aligned}$$

Using $\varphi_j(H) = \sum_{\nu=-1}^{1}(\varphi\psi)_{j+\nu}(H)\varphi_j(H)$, with the convention that $\phi_0 = \Phi, \phi_{-1} = 0$, we have

$$\|\{2^{j\alpha}\varphi_j(H)m(H)g\}\|_{\ell^q(L^p)} \leq C_{p,q} \sum_{\nu=-1}^{1} \left\{ \sum_{j=0}^{\infty} 2^{j\alpha q}\|m_{j+\nu}(H)\varphi_j(H)g\|_p^q \right\}^{1/q},$$

where $m_j = m(\varphi\psi)_j$. Therefore it is sufficient to show that the $m_j(H)$ are uniformly bounded on $L^p, 1 < p < \infty$. However, according to Theorem 6.5, this is true because each $m_j = m\psi_j$ verifies the condition

$$|m_j^{(k)}(\xi)| \leq C|\xi|^{-k},$$

$k = 0, 1$, with C independent of j. □

Proof of Lemma 6.1. Assuming $\|zK_j(\cdot,y)\|_2 \le C\lambda^{1/2}$, the Schwarz inequality gives

$$\begin{aligned}
&\int_{|z|>t} |K_j(x,y)|dx = \\
&\int_{\{|x-y|>t\}\cap\{|x+y|>t\}} (\min_{\pm}|x\pm y|)^{-1}\big|(\min_{\pm}|x\pm y|)K_j(x,y)\big|dx \\
\le& \Big(\int_{\{|x-y>t\}\cap\{|x+y|>t\}} (\min|x\pm y|)^{-2}dx\Big)^{1/2}\|zK_j(\cdot,y)\|_2 \le Ct^{-\frac12}\lambda^{\frac12}.
\end{aligned}$$

Next we need to show $\|zK_j(\cdot,y)\|_2 \le C\lambda^{1/2}$. Clearly,

$$\begin{aligned}
&\|zK_j(\cdot,y)\|_2 \le \|zK_j(x,y)\chi_{\{x>1\}}\|_2 \\
+&\|zK_j(x,y)\chi_{\{|x|\le 1\}}\|_2 + \|zK_j(x,y)\chi_{\{x<-1\}}\|_2.
\end{aligned}$$

Each of these three terms is in fact $\le C_\varepsilon\lambda^{1/2}$. We shall prove the estimate for the first term only since the other two terms can be proved similarly. The discussion is divided into three cases: $y>1$, $|y|\le 1$, and $y<-1$. Again here, we indicate the proof for the case $y>1$ only.

Let $y>1$, $x>1$ and consider first the high frequency case $j>J=4+[2\log_2\varepsilon]$. Recall that $j>J$ if and only if $\lambda^{-1}>4\varepsilon$. Then

$$\begin{aligned}
&2\pi\min_{\pm}|x\pm y|K_j(x,y) \\
=&z\int_{\xi>2\varepsilon}^{+} m_j(\xi^2)|C_+|^2e^{i(x-y)\xi}d\xi + z\int_{\xi<-2\varepsilon}^{-} m_j(\xi^2)(e^{ix\xi}+C'_-e^{-ix\xi})\overline{e^{iy\xi}+C'_-e^{-iy\xi}} \\
:=&I^+(x,y)+I^-(x,y).
\end{aligned}$$

Integrating by parts, we obtain

$$\begin{aligned}
&|I^+(x,y)| \le |x-y|\cdot\Big|\int^{+} m_j(\xi^2)|C_+|^2e^{i(x-y)\xi}\mathrm{d}\xi\Big| \\
=&\Big|\int^{+}\frac{d}{d\xi}(m_j(\xi^2)|C_+|^2)e^{i(x-y)\xi}\mathrm{d}\xi\Big| \\
=&\sqrt{2\pi}\left|\left[\frac{d}{d\xi}(m_j(\xi^2)|C_+|^2\chi_{\{\xi>0\}})\right]^{\vee}(x-y)\right|.
\end{aligned}$$

By the Plancherel formula,

$$\|I^+(x,y)\chi_{\{x>1\}}\|_2 \le \sqrt{2\pi}\|\frac{d}{d\xi}(m_j(\xi^2)|C_+|^2\chi_{\{\xi>0\}})\|_2 \le C_\varepsilon\lambda^{1/2}, \tag{32}$$

where we used the following facts when $1/2\lambda \le |\xi| \le 1/\lambda$:

$$\begin{cases} m_j(\xi^2) & = O(1), \\ \frac{d}{d\xi}(m_j(\xi^2)) & = O(1/\xi), \\ |C_+|^2 & = O(1), \\ \frac{d}{d\xi}\big(|C_+|^2\big) & = O(1/\xi^4). \end{cases}$$

Similarly, one can show that

$$\|I^-(x,y)\chi_{\{x>1\}}\|_2 \le C_\varepsilon \lambda^{1/2}. \tag{33}$$

Combing (32), (33), we obtain

$$\|z\, K_j(x,y)\chi_{\{x>1\}}\|_2 \le C_\varepsilon \lambda^{1/2}. \tag{34}$$

Estimation for the low energy case $j \le J$ can be obtained by following the same line of reasoning (with a suitable modification when necessary) for the high energy case, except that we use certain asymptotic properties near the origin instead of ∞ (cf., §4).

We are left with the first inequality (22) concerning the "size" of the kernel. The proof of (22) is similar to but easier than that of (23) and is omitted. This completes the proof of Lemma 6.1. □

Outline of the proof of Lemma 6.2. Lemma 6.2 can be proved in the same fashion as Lemma 6.1. Assuming (26) for the moment, we can apply the Schwarz inequality to obtain (27) for all y. Inequalities (25) and (26) measure the L^2 -norm of $\frac{\partial}{\partial y}K_j(\cdot,y)$ and $z\frac{\partial}{\partial y}K_j(\cdot,y)$, which are derivative analogues of (22) and (23) in Lemma 6.1.

We now indicate some steps for proving (26). (25) is easier to deal with.

Consider first the high energy case $j > J$. To prove (26) we break the function $x \mapsto z\frac{\partial}{\partial y}K_j(x,y)$ into three parts: its restrictions to the sets $\{x>1\}, \{|x|\le 1\}$, and $\{x<-1\}$. As before we are able to show that the L^2-norm of these restrictions (in x) is bounded by $C\lambda^{-1/2}$.

For instance, in the case $y > 1$, $x > 1$, the identities

$$\begin{cases} \frac{\partial}{\partial y}e_+(y,\xi) & = i\xi e_+(y,\xi) \\ \frac{\partial}{\partial y}e_-(y,\xi) & = i\xi(e^{iy\xi} - C'_- e^{-iy\xi}) \end{cases}$$

tell us that the integral expression of $z\frac{\partial}{\partial y}K_j(x,y)$ differs from that of $zK_j(x,y)$ only by a factor $i\xi$ (up to a $\pm$ sign), for which reason we use the estimate $\frac{d}{d\xi}[\xi m_j(\xi^2)] = O(1),\ \xi \to \infty$ in place of the estimate $\frac{d}{d\xi}[m_j(\xi^2)] = O(1/\xi)$.

The remaining two parts are also straightforward.

The corresponding inequality is valid for the low energy case, based on some simple asymptotic estimates as $\xi \to 0$. □

7. Identification of $B_p^{\alpha,q}(H), 0 < \alpha < 1$

Generalized Besov space methods have been considered in [14, 17, 19, 20] in the study of perturbations of Schrödinger operators. In applications to PDE problems it is of interest to identify these spaces.

The spaces $B_p^{\alpha,q}(H)$ we have defined using (2) and the system $\{\Phi, \varphi_j\}$ are essentially of the same type as those defined in [15] for $p, q \geq 1$ and $\alpha \geq 0$. In [15], sufficient conditions are given on V so that $B_p^{\alpha,q}(H)$ can be identified with ordinary Besov spaces. The proof is based on a real interpolation result, where the interpolation spaces are defined by means of the semigroup method. The following result is a variant of Theorem 5.1 in [15].

Let $\mathcal{K} := \{V : V = V_+ - V_- \text{ so that } V_+ \in K_d^{loc},\ V_- \in K_d\}$, where K_d denote the Kato class (see §1, [15] or [24]). Let W_p^s be the ordinary Sobolev space of order s on $\mathbb{R}^d$.

Theorem 7.1. *Suppose $V \in \mathcal{K}$ and $\mathcal{D}(H^m) = W_p^{2m}$ for some $m \in \mathbb{N}$ and $1 \leq p < \infty$. Then for $1 \leq q \leq \infty$ and $0 < \alpha < m$, $B_p^{\alpha,q}(H) = B_p^{2\alpha,q}(\mathbb{R}^d)$ (with equivalent norms).*

Theorem 7.1 can be proved directly by following the proof of Theorem 5.1 in [15] with obvious modifications. Indeed, noting that $\mathcal{D}(H^m) = W_p^{2m}$, the proof is implicit in the commutative diagram

$$\begin{array}{ccc} B_p^{\alpha,q}(H) & \xrightarrow{=} & (L^p, \mathcal{D}(H^m))_{\theta,q}, \\ \uparrow & & \uparrow \\ B_p^{2\alpha,q}(\mathbb{R}^d) & \xrightarrow{=} & (L^p, W_p^{2m})_{\theta,q} \end{array}$$

with $\theta = \frac{\alpha}{m}$.

Remark 1. Note that the Besov norm was defined in [15] using the 4-adic system, while we have used the dyadic system in this paper. We also note that in the condition $B(p, m)$ of [15], W_p^m should be W_p^{2m}.

Remark 2. The condition on the domain of H^m is equivalent to Assumption $B(p, m)$ in [15], which assumes that for some $M > 0$, $(H+M)^{-m}$ is a bounded map from $L^p(\mathbb{R}^d)$ to $W_p^{2m}(\mathbb{R}^d)$ with a bounded inverse.

It is essential to verify the domain condition on H^m or the assumption $B(p, m)$. In his communication to the second author, A. Jensen explained that, using the Kato-Rellich theorem it is easy to show that if V is bounded relative to Δ on $L^p(\mathbb{R}^d)$ with relative bound less than one, then the condition $B(p,m)$ is satisfied for $m=1$. For $m>1$, the condition $B(p,m)$ is valid for all $m\geq 1$ and all p if V is C^∞ with all bounded derivatives.

In the following let V be the barrier potential defined in §1. Obviously $V << -\Delta$ with relative bound zero, satisfying the conditions in Theorem 7.1. Thus $B_p^{\alpha,q}(H) = B_p^{2\alpha,q}(\mathbb{R})$ for $1\leq p<\infty, 1\leq q\leq\infty$, $0<\alpha<1$. This, combined with Theorem 6.6 implies the following multiplier result on ordinary Besov spaces.

Proposition 7.2. *Suppose $m\in L^\infty$ is as in Theorem 6.6. Then $m(H)$ is bounded on $B_p^{\alpha,q}(\mathbb{R})$ for $1<p<\infty$, $1\leq q\leq\infty$, $0<\alpha<2$.*

Another interesting result follows from the discussion above for barrier potential and Theorem 4.6 and Remark 4.7 in [15].

Proposition 7.3. *Suppose $1\leq p<\infty$, $1\leq q\leq\infty$, and $0<\alpha<2-2\beta$ with $\beta=|\frac{1}{2}-\frac{1}{p}|$. Then e^{-itH} maps $B_p^{\alpha+2\beta,q}(\mathbb{R})$ continuously to $B_p^{\alpha,q}(\mathbb{R})$. Moreover, e^{-itH} maps $B_p^{2\beta,q}(\mathbb{R})$ continuously to L^p. In both cases the operator norm is less than or equal to $C\langle t\rangle^\beta$, where $\langle t\rangle=(1+|t|^2)^{1/2}$.*

We conclude with the following conjecture, for the barrier potential, concerning the identification of $B_p^{\alpha,q}(H)$. For $m=2$, we have reason to doubt the verification of the domain condition for H^m that is assumed in Theorem 7.1.

Conjecture. $B_p^{\alpha,q}(H)\neq B_p^{\alpha,q}(H_0), \quad \alpha=2.$

To see the rationale for the conjecture we compare H^2 and H_0^2. Write $H^2 = H_0^2 + H_0V + VH_0 + V^2$. The only term that could cause a problem is H_0V, which formally involves Dirac delta distributions and their first derivatives. On the other hand, Theorem 3.2.2 in [1] tells us that the domain of the operator $H_0 + c_1\delta + c_2\delta'$ consists of functions $u\in W_2^2(\mathbb{R}\setminus\{0\})$, with u satisfying certain boundary condition at the origin. Thus, if $\mathcal{D}(H^2)=\mathcal{D}(H_0^2)$ we would have that the domain of H_0V is W_p^4, $p=2$, which is not the case by the above-mentioned theorem in [1].

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(John J. Benedetto) Department of Mathematics, University of Maryland, College Park, Maryland 20742
E-mail address: jjb@math.umd.edu
URL: http://www.math.umd.edu/~jjb

(Shijun Zheng) Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803

and

Department of Mathematics, University of Maryland, College Park, Maryland 20742
E-mail address: szheng@math.lsu.edu
URL: http://www.math.lsu.edu/~szheng