# The Balian–Low theorem for the symplectic form on $\mathbb{R}^{2d}$

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**Abstract**

In this paper we extend the Balian–Low theorem, which is a version of the uncertainty principle for Gabor (Weyl–Heisenberg) systems, to functions of several variables.

In particular, we first prove the Balian–Low theorem for arbitrary quadratic forms. Then we generalize further and prove the Balian–Low theorem for differential operators associated with a symplectic basis for the symplectic form on $\mathbb{R}^{2d}$.

## 1 Introduction

For a given function $g \in L^2(\mathbb{R}^d)$ we define the following two unitary operators on $L^2(\mathbb{R}^d)$:

$$M_n(g)(x) = e^{2\pi i n\cdot x} g(x), \quad n \in \mathbb{R}^d,$$

and

$$T_m(g)(x) = g(x-m), \quad m \in \mathbb{R}^d,$$

called *modulation* and *translation* operators, respectively. In 1946 Dennis Gabor [15] proposed to use these operators to define the collections of functions

$$g_{m,n}(x) = e^{2\pi i n\cdot x} g(x-m), \quad m, n \in \mathbb{Z},$$

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to be used in the analysis of information conveyed by communications channels. These systems have been studied extensively in recent years. The edited books by Benedetto and Frazier [7] and by Feichtinger and Strohmer [14], as well as Gröchenig's treatise [16], provide detailed treatments of various issues of the theory. Gabor systems are especially interesting because of their effective role in the time-frequency analysis of a wide variety of signals.

Let us now introduce some terms and notation that will be used throughout this paper. We say that a collection $\{f_k : k = 1, \ldots\} \subset L^2(\mathbb{R}^d)$ of functions is a *frame* for $L^2(\mathbb{R}^d)$, with *frame bounds* $A$ and $B$, if

$$\forall\, f \in L^2(\mathbb{R}^d), \quad A\|f\|_2^2 \leq \sum_k |\langle f, f_k\rangle|^2 \leq B\|f\|_2^2.$$

A frame is *tight* if $A = B$; and a frame is *exact* if it is no longer a frame after removal of any of its elements. For any frame $\{f_k : k = 1, \ldots\}$ there exists a *dual frame* $\{\tilde{f}_k : k = 1, \ldots\}$ such that

$$\forall\, f \in L^2(\mathbb{R}^d), \quad f = \sum_k \langle f, f_k\rangle \tilde{f}_k = \sum_k \langle f, \tilde{f}_k\rangle f_k, \tag{1.1}$$

where the series converge in $L^2(\mathbb{R}^d)$. The choice of coefficients for expressing $f$ in terms of $\{f_k : k = 1, \ldots\}$ or $\{\tilde{f}_k : k = 1, \ldots\}$ is not unique, unless the frame is a *basis. A frame is a basis if and only if it is exact*, e.g., [8].

For a frame $\{f_k : k = 1, \ldots\} \subset L^2(\mathbb{R}^d)$ we define the associated *frame operator* $S$ on $L^2(\mathbb{R}^d)$ by the rule,

$$\forall\, f \in L^2(\mathbb{R}^d), \quad S(f) = \sum_k \langle f, f_k\rangle f_k.$$

$S$ is a bounded and invertible map of $L^2(\mathbb{R}^d)$ onto itself. Given a frame $\{f_k : k = 1, \ldots\}$, our canonical choice of the dual frame $\{\tilde{f}_k : k = 1, \ldots\}$ will be defined by $\tilde{f}_k = S^{-1}(f_k)$. If a frame is exact then $\{f_k : k = 1, \ldots\}$ and $\{\tilde{f}_k : k = 1, \ldots\}$ are *biorthogonal*, that is,

$$\langle f_k, \tilde{f}_l\rangle = \delta_{k,l} \quad k, l = 1, \ldots,$$

where $\delta_{k,l}$ denotes the *Kronecker delta function*, i.e., it is 1 if $k = l$ and 0 otherwise. It is elementary to show that $S^{-1}(g_{m,n}) = (S^{-1}(g))_{m,n}$ for Gabor frames $\{g_{m,n}\}$.

The *Fourier transform* is the unitary transformation $\mathcal{F}$ of $L^2(\mathbb{R}^d)$ onto itself, defined formally by

$$\hat{f}(\xi) = \mathcal{F}(f)(\xi) = \int_{\mathbb{R}^d} f(x) e^{2\pi i x\cdot\xi}\, dx.$$

We write $\mathbb{R}^d$ for arguments of a function $f \in L^2(\mathbb{R}^d)$ and $\widehat{\mathbb{R}}^d$ for arguments of its Fourier transform.

We employ the standard notation in harmonic analysis, e.g., [28].

The following result is a version of the uncertainty principle for Gabor systems for the case $d = 1$. It was first proved independently by Balian [3] and Low [23]. Both proofs contained a gap, which was corrected; and the result was generalized by Coifman, Daubechies, and Semmes from Gabor systems which form orthonormal bases to Gabor systems which form exact frames [12], see also [6], [8]. A different prof of Theorem 1.1 was given by Battle [5]. Battle proved also an analogous result for wavelets [4].

**Theorem 1.1 Balian–Low theorem (BLT).** *Let $g \in L^2(\mathbb{R})$ have the property that $\{g_{m,n} : m, n \in \mathbb{Z}\}$ is a Gabor orthonormal basis for $L^2(\mathbb{R})$. Then*

$$\left(\int_{\mathbb{R}} |g(x)|^2 |x|^2\, dx\right)\left(\int_{\widehat{\mathbb{R}}} |\hat{g}(\xi)|^2 |\xi|^2\, d\xi\right) = \infty. \tag{1.2}$$

**Remark.** Our original goal in this paper was to obtain a generalization of Theorem 1.1 for functions of several variables. In the process, and after having obtained some of our main results, we became aware of the work of Gröchenig, Han, Heil, and Kutyniok [18], in which the authors also extend the Balian–Low theorem to $d$-dimensions. Two of their fundamental results may be compared with our Theorem 2.1 and Theorem 2.5. In fact, Theorem 2.5 is identical with the BLT for non-lattices in [18] and Theorem 2.1 extends the weak BLT for lattices in [18] to more general position and momentum operators. Further, using techniques from the theory of metaplectic representations, the authors in [18] generalize Theorem 2.5 to a Balian–Low type theorem for exact frames on symplectic lattices; for their setting their assertion states that there exists $i \in \{1, \ldots, d\}$ such that (2.8) below holds.

We follow a different path and prove that the choice of coordinates in (2.8) is not canonical, i.e., there is no "preference" for the directional derivatives and for multiplications by the standard basis coordinates. This means that one can work in any representation of $\mathbb{R}^d$, e.g., Theorem 3.6.

In Section 2 we prove the generalization of the Balian–Low theorem to $d$-dimensions in the standard coordinate system; this is Theorem 2.1. As a corollary, we prove a Balian–Low theorem for arbitrary non-negative quadratic forms (Corollary 2.3). In Section 3 we state and prove our main result, Theorem 3.6, which asserts a Balian–Low phenomenon (3.6) similar to but more far-reaching than (1.2). The proof depends on our definition of generalized Fourier transforms which, in turn, allows us to reduce a rather general and comprehensive problem to the Balian–Low theorem in the standard coordinates as formulated in Theorem 2.5.

Our approach is both straightforward and natural. This is an essential part of our contribution. It is also based on the quantum mechanical point of view. The second named author presented this work at the Analysis and Probability Related to Solvable Lie Groups conference in Zakopane in June 2002. Afterwards it was brought to our attention by Professor Tim Steger that Theorem 3.6 can also be proved using the theory of metaplectic representations. We gratefully acknowledge Professor Steger for his comments. We also gratefully acknowledge Professor Raymond Johnson for making us aware of the connection between the work of Hörmander and our results.

# 2 Balian–Low theorem in standard coordinates

Let $v, w \in \mathbb{R}^d$ be non-zero vectors. We define the following operators, wherever they make sense in $L^2(\mathbb{R}^d)$:

$$P_v(f)(x) = \left( \sum_{i=1}^{d} v_i x_i \right) f(x)$$

and

$$M_w(f)(x) = \mathcal{F}^{-1} \left( \left( \sum_{i=1}^{d} w_i \xi_i \right) \hat{f}(\xi) \right)(x) = \mathcal{F}^{-1}(P_w(\hat{f}))(x),$$

where $v = (v_1, \ldots, v_d) = \sum v_j u_j$, $u_j = (0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 in the $j$th coordinate, and $v_j \in \mathbb{R}$. These unit vectors $u_j$ define the *standard Euclidean basis* $\{u_j : j = 1, \ldots, d\}$ of $\mathbb{R}^d$. If the vectors $v$ and $w$ in the definitions of $P_v$ and $M_w$ are elements of the standard basis, then we shall use the notation $P_i$ and $M_i$ for the operators induced by the $i$th basis vector $u_i$.

The following result is our first generalization of the Balian–Low theorem. The technique of proof is a well-known method for proving Balian–Low type theorems. $\tilde{g}$ denotes the canonical dual defined in Section 1.

**Theorem 2.1** *Let $\{g_{m,n} : m, n \in \mathbb{Z}^d\}$ be an exact frame for $L^2(\mathbb{R}^d)$. If $v$, $w \in \mathbb{R}^d$ satisfy $v \cdot w \neq 0$, then*

$$\|P_v(g)\|_2 \, \|M_w(g)\|_2 \, \|P_v(\tilde{g})\|_2 \, \|M_w(\tilde{g})\|_2 = \infty. \tag{2.1}$$

**Proof.** We may assume without loss of generality that $|v| = |w| = 1$, where $|\ |$ denotes the Euclidean norm in $\mathbb{R}^d$. We shall proceed with a proof by contradiction; and so we assume that all four functions in (2.1) are elements of $L^2(\mathbb{R}^d)$. Because of the biorthogonality relations for $g$ and $\tilde{g}$ we compute

$$\begin{aligned}
\langle P_v(g), \tilde{g}_{m,n}\rangle &= \langle P_v(g), \tilde{g}_{m,n}\rangle - \left(\sum_{i=1}^{d} v_i m_i\right) \langle g, \tilde{g}_{m,n}\rangle \\
&= \int_{\mathbb{R}^d} \left(\sum_{i=1}^{d} v_i (x_i - m_i)\right) g(x)\overline{\tilde{g}(x-m)} e^{-2\pi i n\cdot x}\, dx \qquad (2.2)\\
&= e^{-2\pi i m\cdot n} \int_{\mathbb{R}^d} \left(\sum_{i=1}^{d} v_i x_i\right) \overline{\tilde{g}(x)} g(x+m) e^{-2\pi i n\cdot x}\, dx \\
&= e^{-2\pi i m\cdot n} \langle g_{-m,-n}, P_v(\tilde{g})\rangle.
\end{aligned}$$

From our assumption that $M_w(g) \in L^2(\mathbb{R}^d)$, it follows that the distributional partial derivative of $g$, $\partial_w g$, belongs to $L^2(\mathbb{R}^d)$. From a standard result about Sobolev spaces, see, e.g., [24], Theorem 1.1, there exists a function $h$ such that $g = h$ *a.e.*, and $h$ is absolutely continuous on almost all straight lines parallel to the vector $w$. Thus the distributional directional derivative of $g$ coincides with the classical directional derivative $D_w(g)$ *a.e.*, and so

$$M_w(g)(x) = \frac{i}{2\pi} D_w(g)(x) \quad a.e.$$

Moreover, our assumptions imply that $D_w(g)$, $D_w(\tilde{g}) \in L^2(\mathbb{R}^d)$. Therefore, using integration by parts, an appropriate change of variables, and the biorthogonality relations between $g$ and $\tilde{g}$, we can compute

$$
\begin{aligned}
\langle g_{m,n}, M_w(\tilde{g})\rangle &= \frac{1}{2\pi i}\int_{\mathbb{R}^d} g(x-m)e^{2\pi i n\cdot x}\overline{D_w(\tilde{g})(x)}\,dx \\
&= \frac{i}{2\pi}\int_{\mathbb{R}^d} D_w\left(g(x-m)e^{2\pi i n\cdot x}\right)\overline{\tilde{g}(x)}\,dx \\
&= \frac{i}{2\pi}\int_{\mathbb{R}^d}\Big(D_w(g)(x-m)e^{2\pi i n\cdot x} \\
&+ (w\cdot n)g(x-m)e^{2\pi i n\cdot x}\Big)\overline{\tilde{g}(x)}\,dx \\
&= \frac{ie^{2\pi i m\cdot n}}{2\pi}\int_{\mathbb{R}^d}\left(D_w(g)(x)+2\pi i(w\cdot n)g(x)\right)e^{2\pi i n\cdot x}\overline{\tilde{g}(x+m)}\,dx \\
&= e^{2\pi i m\cdot n}\left(\langle M_w(g), \tilde{g}_{-m,-n}\rangle + (w\cdot n)\delta_{m,0}\delta_{n,0}\right) \\
&= e^{2\pi i m\cdot n}\langle M_w(g), \tilde{g}_{-m,-n}\rangle.
\end{aligned} \tag{2.3}
$$

Because of (2.2), (2.3), and the frame representation property (1.1), we have

$$
\begin{aligned}
\langle P_v(g), M_w(\tilde{g})\rangle &= \sum_{m,n\in\mathbb{Z}^d}\langle P_v(g), \tilde{g}_{m,n}\rangle\langle g_{m,n}, M_w(\tilde{g})\rangle \\
&= \sum_{m,n\in\mathbb{Z}^d}\langle g_{-m,-n}, P_v(\tilde{g})\rangle\langle M_w(g), \tilde{g}_{-m,-n}\rangle \\
&= \sum_{m,n\in\mathbb{Z}^d}\langle M_w(g), \tilde{g}_{m,n}\rangle\langle g_{m,n}, P_v(\tilde{g})\rangle \\
&= \langle M_w(g), P_v(\tilde{g})\rangle.
\end{aligned} \tag{2.4}
$$

It is a fundamental and elementary result in quantum mechanics that

$$
[P_v, M_w] = \frac{1}{2\pi i}(v\cdot w)\ \mathrm{Id}, \tag{2.5}
$$

where the commutator $[P_v, M_w] = P_vM_w - M_wP_v$ and where Id denotes the identity operator, e.g., [25] where (2.5) appears for the position and momentum operators associated with the standard basis vectors; see also the trivial calsulation in [8]. Thus, for functions $g$, $\tilde{g} \in L^2(\mathbb{R}^d)$, such that $P_v(g)$, $P_v(\tilde{g}) \in L^2(\mathbb{R}^d)$ and $M_w(g)$, $M_w(\tilde{g}) \in L^2(\mathbb{R}^d)$, we have

$$
\begin{aligned}
\langle P_v(g), M_w(\tilde{g})\rangle &= \langle M_w(g), P_v(\tilde{g})\rangle + \frac{1}{2\pi i}(v\cdot w)\langle g, \tilde{g}\rangle \\
&= \langle M_w(g), P_v(\tilde{g})\rangle + \frac{1}{2\pi i}(v\cdot w).
\end{aligned}
$$

Since we have assumed that $v \cdot w \neq 0$, we obtain a contradiction with our calculation (2.4). ∎

**Corollary 2.2** *Let $\{g_{m,n} : m, n \in \mathbb{Z}^d\}$ be an exact frame for $L^2(\mathbb{R}^d)$. If $v$, $w \in \mathbb{R}^d$ satisfy $v \cdot w \neq 0$, then*

$$\|P_v(g)\|_2 \, \|M_w(g)\|_2 = \infty.$$

**Proof.** In view of Theorem 2.1, it is enough to show that $P_v(g) \in L^2(\mathbb{R}^d)$ if and only if $P_v(\tilde{g}) \in L^2(\mathbb{R}^d)$, and that $M_w(g) \in L^2(\mathbb{R}^d)$ if and only if $M_w(\tilde{g}) \in L^2(\mathbb{R}^d)$. This, in turn, was proved by Daubechies and Janssen [13] for the position and momentum operators associated with the standard basis vectors, see also [8], Theorem 7.7. The proof for arbitrary operators $P_v$ and $M_w$ is analogous, and it uses the $d$-dimensional Sobolev space argument which we have used in the proof of Theorem 2.1 instead of 1-dimensional considerations . ∎

**Example.** To show that the condition $v \cdot w \neq 0$ is necessary consider $L^2(\mathbb{R}^2)$ with the orthonormal Gabor basis generated by

$$g(x, y) = \chi_{[0,1]}(x) \; \mathcal{F}^{-1}\left(\chi_{[0,1]}\right)(y)$$

and the vectors $v = (1, 0)$ and $w = (0, 1)$. Then

$$\begin{aligned} \|P_v(g)\|_2^2 &= \int_{\mathbb{R}^2} |x g(x, y)|^2 \, dx dy \\ &= \int_{\mathbb{R}} \left| x \chi_{[0,1]}(x) \right|^2 \, dx \int_{\mathbb{R}} \left| \mathcal{F}^{-1}(\chi_{[0,1]})(y) \right|^2 \, dy < \infty \end{aligned}$$

and

$$\begin{aligned} \|M_w(g)\|_2^2 &= \int_{\mathbb{R}^2} |\eta \hat{g}(\xi, \eta)|^2 \, d\xi d\eta \\ &= \int_{\mathbb{R}} \left| \mathcal{F}(\chi_{[0,1]})(\xi) \right|^2 \, d\xi \int_{\mathbb{R}} \left| \eta (\chi_{[0,1]})(\eta) \right|^2 \, d\eta < \infty. \end{aligned}$$

**Corollary 2.3** *Let $\omega(x)$ be any positive quadratic form on $\mathbb{R}^d$ and let $\{g_{m,n} : m,n \in \mathbb{Z}^d\}$ be an exact frame for $L^2(\mathbb{R}^d)$. Then*

$$\left(\int_{\mathbb{R}^d} \omega(x)|g(x)|^2\, dx\right)\left(\int_{\widehat{\mathbb{R}}^d} \omega(\xi)|\hat{g}(\xi)|^2\, d\xi\right) = \infty.$$

**Proof.** Clearly, for any vector $v \neq 0$ we have $v \cdot v \neq 0$. Thus, from Corollary 2.2 it follows that for any $\alpha_k \geq 0$, $k = 1, \ldots, d$, where some $\alpha_k > 0$, either

$$\left(\sum_{k=1}^{d} \alpha_k \int_{\mathbb{R}^d} \left(\sum_{i=1}^{d} v_i^k x_i\right)^2 |g(x)|^2\, dx\right) = \infty$$

or

$$\left(\sum_{k=1}^{d} \alpha_k \int_{\widehat{\mathbb{R}}^d} \left(\sum_{i=1}^{d} v_i^k \xi_i\right)^2 |\hat{g}(\xi)|^2\, d\xi\right) = \infty.$$

The result follows since any quadratic form on $\mathbb{R}^d$ is of the form

$$\omega(x) = \sum_{k=1}^{d} \alpha_k \left(\sum_{i=1}^{d} v_i^k x_i\right)^2,$$

where the $\alpha_k$s are non-negative. ■

We now consider a countable collection $\Lambda$ of vectors in $\mathbb{R}^{2d}$. For any pair $(m,n) \in \Lambda$, $m,n \in \mathbb{R}^d$, we shall associate the *translation-modulation transformation* $T_{m,n}$ defined on $L^2(\mathbb{R}^d)$ as follows:

$$T_{m,n}(g)(x) = e^{2\pi i n \cdot x} g\left(x + m\right).$$

From now on we shall write $g_{m,n} = T_{m,n}(g)$. The study of *nonuniform* Gabor systems, i.e., those Gabor systems which are associated with a set $\Lambda$ which is not a lattice, has increased in recent years because of applications of such systems to problems in signal processing, e.g., [9], [10], [17], [21]. Of course, not all $\Lambda$s generate orthonormal bases or even frames. In order for a Gabor system to have good signal representation properties, $\Lambda$ must satisfy certain density conditions. The most general results so far in this direction were obtained by Ramanathan and Steger [26] and by Christensen, Deng, and Heil [11].

**Example.** One easily constructs examples of uniform orthonormal Gabor bases for $L^2(\mathbb{R}^d)$. The most simple example is $g(x) = \chi_{[0,1]^d}$ with the lattice $\Lambda = \mathbb{Z}^{2d}$. More interestingly there is the work of Liu and Wang [22], where the authors provide examples of nonuniform Gabor bases and frames, i.e., examples where $\Lambda$ is not a lattice.

For $d = 1$ let $\Omega = [0, 1] \cup [3, 4]$ and

$$\Lambda = \{6\mathbb{Z} + \{-1, 0, 1\}\} \times \left\{\frac{1}{2}\mathbb{Z}\right\}.$$

Then $g(x) = (1/\sqrt{2})\chi_\Omega(x)$ forms an orthonormal basis with translations and modulations in $\Lambda$. We would like to stress that although $\Lambda$ is a periodic set it is not a lattice, since in general a sum of two vectors in $\Lambda$ is not an element of $\Lambda$. We note that $\Lambda = -\Lambda$.

[22] also provides an account of differences between nonuniform Gabor bases in 1 and higher dimensions.

To prove Theorem 2.5 we shall need the following lemma, the proof of which is similar to the proof of analogous statements in Theorem 2.1.

**Lemma 2.4** *Let $g \in L^2(\mathbb{R}^d)$, and let $\{g_{m,n} : (m, n) \in \Lambda\}$ be an orthonormal basis for $L^2(\mathbb{R}^d)$. If $P_i(g)$, $M_i(g) \in L^2(\mathbb{R}^d)$, then*

$$\langle g_{m,n}, P_i(g)\rangle = e^{2\pi i m\cdot n}\langle P_i(g), g_{-m,-n}\rangle \tag{2.6}$$

*and*

$$\langle g_{m,n}, M_i(g)\rangle = e^{2\pi i m\cdot n}\langle M_i(g), g_{-m,-n}\rangle. \tag{2.7}$$

**Proof.** Since $\Lambda$ does not posses a lattice structure we cannot use (2.2) and (2.3). Indeed, the fact that a dual to a Gabor frame is also a frame of Gabor type holds only for systems associated with lattices. However, the assumption that $\{g_{m,n} : (m, n) \in \Lambda\}$ is an orthonormal basis for $L^2(\mathbb{R}^d)$ compensates for this lack of structure in $\Lambda$.

$$\begin{aligned}
\langle g_{m,n}, P_i(g)\rangle &= \int_{\mathbb{R}^d} g(x-m)e^{2\pi i n\cdot x}\overline{P_i(g)(x)}\, dx \\
&= e^{2\pi i m\cdot n}\int_{\mathbb{R}^d} g(x)e^{2\pi i n\cdot x}\overline{(x_i + m_i)g(x+m)}\, dx \\
&= e^{2\pi i m\cdot n}\left(\langle P_i(g), g_{-m,-n}\rangle + m_i\langle g, g_{-m,-n}\rangle\right) \\
&= e^{2\pi i m\cdot n}\langle P_i(g), g_{-m,-n}\rangle.
\end{aligned}$$

The last equality above follows from the orthogonality of $\{g_{m,n} : (m,n) \in \Lambda\}$. Similarly, using orthogonality and the integration by parts formula, we calculate

$$\begin{aligned}
\langle g_{m,n}, M_i(g)\rangle &= -\frac{i}{2\pi}\int_{\mathbb{R}^d} g(x-m)e^{2\pi i n\cdot x}\overline{D_i(g)(x)}\,dx \\
&= \frac{i}{2\pi}\int_{\mathbb{R}^d} D_i\left(g(x-m)e^{2\pi i n\cdot x}\right)\overline{g(x)}\,dx \\
&= \frac{ie^{2\pi i m\cdot n}}{2\pi}\int_{\mathbb{R}^d}\left(D_i(g)(x)+2\pi i m_i g(x)\right)e^{2\pi i n\cdot x}\overline{g(x+m)}\,dx \\
&= e^{2\pi i m\cdot n}\left(\langle M_i(g), g_{-m,-n}\rangle + m_i\langle g, g_{-m,-n}\rangle\right) \\
&= e^{2\pi i m\cdot n}\langle M_i(g), g_{-m,-n}\rangle.
\end{aligned}$$

■

**Theorem 2.5** *Let $\Lambda \subset \mathbb{R}^{2d}$ be a countable sequence of vectors with the property that $\Lambda = -\Lambda$. Let $\{T_{m,n} : (m,n) \in \Lambda\}$ be the associated family of translation-modulation transformations, and assume $\{g_{m,n} : (m,n) \in \Lambda\}$ is an orthonormal basis for $L^2(\mathbb{R}^d)$ for some $g \in L^2(\mathbb{R}^d)$. For any $i = 1, \ldots, d$,*

$$\|P_i(g)\|_2\|M_i(g)\|_2 = \infty. \tag{2.8}$$

**Proof.** Because of (2.6), (2.7), the representation property of bases, and the fact that $\Lambda = -\Lambda$, we obtain

$$\begin{aligned}
\langle M_i(g), P_i(g)\rangle &= \sum_{(m,n)\in\Lambda}\langle M_i(g), g_{m,n}\rangle\langle g_{m,n}, P_i(g)\rangle \\
&= \sum_{(m,n)\in\Lambda}\langle g_{-m,-n}, M_i(g)\rangle\langle P_i(g), g_{-m,-n}\rangle \\
&= \langle P_i(g), M_i(g)\rangle.
\end{aligned} \tag{2.9}$$

On the other hand, again using the classical result from [24] used in Theorem 2.1, we note that $M_i(g) \in L^2(\mathbb{R}^d)$ implies that $\partial g/\partial x_i$ exists *a.e.* Thus, integration by parts yields

$$\langle M_i(g), P_i(g)\rangle = \langle P_i(g), M_i(g)\rangle - \frac{1}{2\pi i},$$

which, in turn, leads to a contradiction with the calculation (2.9). ■

# 3 Balian–Low theorem and symplectic forms

The standard *symplectic form* $\Omega$ on $\mathbb{R}^{2d}$ is defined as

$$\Omega((x,y),(\xi,\eta)) = x\cdot\eta - y\cdot\xi,$$

for any $x$, $y$, $\xi$, $\eta \in \mathbb{R}^d$. Note that $\Omega((x,0),(0,\xi)) = x\cdot\xi$. This observation, when compared to Theorem 2.1, suggests a direction which we are going to follow in this section, and which yields our main result, Theorem 3.6.

**Definition. a.** A *symplectic basis* for $\mathbb{R}^{2d}$ with respect to the symplectic form $\Omega$ is a basis $\{a_j, b_j : j = 1,\dots,d\} \subset \mathbb{R}^{2d}$ for $\mathbb{R}^{2d}$ for which

$$\Omega(a_i,a_j) = \Omega(b_i,b_j) = 0$$

and

$$\Omega(a_i,b_j) = \delta_{i,j},$$

for all $i,j = 1,\dots,d$.

**b.** If $\{\nu_i : i = 1,\dots,d\} \subset \mathbb{R}^d$ is any orthonormal basis for $\mathbb{R}^d$, then $a_i = (\nu_i,0)$, $b_i = (0,\nu_i)$, $i = 1,\dots,d$, is a symplectic basis for $\mathbb{R}^{2d}$. For a non-trivial example in $\mathbb{R}^4$ take the row vectors of the matrix

$$\begin{pmatrix} 1 & 0 & -\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{6}}{2} & 0 & -\frac{\sqrt{2}}{2} \\ 0 & 1 & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{6}}{2} & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \end{pmatrix}.$$

**c.** Consider the space $\mathbb{R}^{2d}$ with coordinates $(x^1,\dots,x^d,y^1,\dots,y^d)$ and let $\Omega$ be the symplectic form on $\mathbb{R}^{2d}$. A *Lagrangian plane* $\Pi$ in $\mathbb{R}^{2d}$ is a $d$-dimensional subspace with the property that

$$\Omega|_{\Pi\times\Pi} = 0.$$

If $\Pi$ is a Lagrangian plane in $\mathbb{R}^{2d}$ and if $v_1, \ldots, v_d \in \Pi \subset \mathbb{R}^{2d}$ is a basis for $\Pi$ then, in particular, we have

$$\Omega(v_i, v_j) = 0,$$

for all $i, j = 1, \ldots, d$. For classical treatments of these and other related notions see, e.g., [1], [2]. A similar approach is used by Hörmander [20] to define Fourier Integral Operators, a special case of which we consider below. A recent exposition of related results in case of Hermitian symplectic geometry is due to Harmer [19].

We now define the differential operators $\{Q_{v_j},\, j = 1, \ldots, d\}$ associated with a given basis $\{v_j : j = 1, \ldots, d\}$ for a given Lagrangian plane $\Pi$. Each $Q_{v_j}$ is defined by its action on a function $h$ as follows:

$$Q_{v_j}(h)(x) = \frac{i}{2\pi} \nabla_j(h)(x) + f_j(x)h(x), \tag{3.1}$$

where

$$\nabla_j = \sum_{k=1}^{d} v_j^{k+d} \frac{\partial}{\partial x_k}$$

and

$$f_j(x) = \sum_{k=1}^{d} v_j^k x_k.$$

Recall that $v_j^k$ is the $k$th coordinate of the vector $v_j \in \mathbb{R}^{2d}$ and that $x = \sum_{k=1}^{d} x_k u_k \in \mathbb{R}^d$.

The next result serves as the main motivation for our work. It is analogous to a similar observation about commutators of position and momentum operators that was asserted in equation (2.5). Its proof is also a straightforward calculation.

**Proposition 3.1** *Let $\Pi$ be a Lagrangian plane associated with the standard symplectic form $\Omega$ and let $\{v_1, \ldots, v_d\}$ be a basis for $\Pi$. Then*

$$[Q_{v_i}, Q_{v_j}] = \Omega(v_i, v_j)\mathrm{Id},$$

*where the commutator $[A, B] = AB - BA$.*

For the purpose of the next definitions we shall make the following assumption: for given vectors $v_1, \ldots, v_d \in \mathbb{R}^{2d}$, define $B_v(j,k) = v_j^{k+d}$, $j, k = 1, \ldots, d$, to be a $d \times d$ matrix, and assume that it is *non-degenerate*, i.e.,

$$\det B_v \neq 0.$$

As a consequence of Proposition 3.1 we observe that the $Q_{v_j}$s commute with each other since the $v_j$s form a basis for $\Pi$. This commutativity implies, in particular, that $\nabla_k f_j = \nabla_j f_k$, and so we deduce that $f_j(x) = \nabla_j(xF_v x)$, for some quadratic form $F_v$. Thus the common eigenfunction for all the operators $\{Q_{v_j}\}$ has the form:

$$\psi_\xi(x) = \frac{1}{\sqrt{|\det B_v|}} e^{-2\pi i x B_v^{-1}\xi + 2\pi i x F_v x},$$

for any $\xi \in \mathbb{R}^d$. Moreover, let $A_v(j,k) = v_j^k$, $j,k = 1, \ldots, d$. Then, the commutativity of the $Q_{v_j}$s implies that

$$A_v B_v^t - B_v A_v^t = 0, \tag{3.2}$$

where $A^t$ is the adjoint of $A$. It follows from (3.2) that $B_v^{-1}A_v$ is symmetric. It is also easy to see that

$$F_v = \frac{1}{2} B_v^{-1} A_v.$$

Let us now define the following *generalized Fourier transforms* $\mathcal{F}_v$ on the space of tempered distributions on $\mathbb{R}^d$, through their action on the space of Schwartz functions:

$$\mathcal{F}_v(h)(\xi) = \int_{\mathbb{R}^d} h(x)\overline{\psi_\xi(x)} dx = \int_{\mathbb{R}^d} h(x) \frac{1}{\sqrt{|\det B_v|}} e^{2\pi i x B_v^{-1}\xi - \pi i x B_v^{-1} A_v x}\, dx.$$

The operators $\mathcal{F}_v$ are unitary when restricted to $L^2(\mathbb{R}^d)$, since they are combinations of unitary transformations.

We shall now consider two different representations of functions or even distributions associated with two different Lagrangian planes: $\Pi$ with the basis $v_1, \ldots, v_d$, and $\Gamma$ with the basis $w_1, \ldots, w_d$. Assume that $\Pi \cap \Gamma = \{0\}$. Moreover, assume that $B_v$ and $B_w$ are non-degenerate. Define the $d \times d$ matrix $Y_{v,w}(i,j) = \Omega(v_i, w_j)$. Note that $Y_{v,w} = \mathrm{Id}$ if and only if $\{v_1, \ldots, v_d, w_1, \ldots, w_d\}$ forms a symplectic basis for $\mathbb{R}^{2d}$.

**Lemma 3.2**

$$\det Y_{v,w} \neq 0.$$

**Proof.** Indeed, if $\Pi \cap \Gamma = \{0\}$, then $\{v_1, \ldots, v_d, w_1, \ldots, w_d\}$ forms a (not necessarily symplectic) basis for $\mathbb{R}^{2d}$. In this basis the matrix of $\Omega$ has the form:

$$\begin{pmatrix} 0 & Y_{v,w} \\ -Y_{v,w} & 0 \end{pmatrix}$$

and so, $(\det Y_{v,w})^2 = \det \Omega \neq 0$. ∎

The matrix $Y_{v,w}$ can be represented, with the use of matrices $A_v$, $A_w$, $B_v$, $B_w$, as:

$$Y_{v,w} = A_v B_w^t - B_v A_w^t.$$

As a consequence, we derive the following formula, which we shall use in the proof of Theorem 3.6:

$$F_v - F_w = \frac{1}{2}\left(B_v^{-1} A_v - B_w^{-1} A_w\right) = \frac{1}{2} B_v^{-1} Y_{v,w} (B_w^{-1})^t. \tag{3.3}$$

**Lemma 3.3** *For any tempered distribution $h$ on $\mathbb{R}^d$, the relationship between its "$v$" and "$w$" generalized Fourier transform representations is*

$$\begin{aligned} \mathcal{F}_w(h)(\eta) &= \frac{1}{\sqrt{|\det Y_{v,w}|}} e^{-\pi i \eta Y_{v,w}^{-1} B_v B_w^{-1} \eta + \pi i \sigma/4} \\ &\times \int_{\mathbb{R}^d} e^{\pi i \xi (Y_{v,w}^{-1})^t \eta + \pi i \eta Y_{v,w}^{-1} \xi - \pi i \xi (B_v^{-1})^t B_w^t Y_{v,w}^{-1} \xi} \mathcal{F}_v(h)(\xi)\, d\xi, \end{aligned}$$

*where $\sigma$ is the difference between the positive and negative squares of the quadratic form $F_v - F_w$.*

**Proof.** The expression in Lemma 3.3 is to be understood in the sense of distributions, and thus it is enough to check its validity on Schwartz functions. Note that the inverse of the generalized Fourier transform $\mathcal{F}_v$ has the form:

$$h(x) = \frac{1}{\sqrt{|\det B_v|}} \int_{\mathbb{R}^d} e^{-2\pi i x B_v^{-1} \xi + 2\pi i x F_v x} \mathcal{F}_v(h)(\xi)\, d\xi.$$

Taking the generalized Fourier transform $\mathcal{F}_w$ of this expression, we obtain

$$\begin{aligned}
\mathcal{F}_w(h)(\eta) &= \frac{1}{\sqrt{|\det B_v B_w|}} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{2\pi i x(B_w^{-1}\eta - B_v^{-1}\xi)+2\pi i x(F_v-F_w)x} \mathcal{F}_v(h)(\xi)\, dx d\xi \\
&= \frac{e^{\pi i\sigma/4}}{\sqrt{|\det Y_{v,w}|}} \int_{\mathbb{R}^d} e^{-\pi i(B_w^{-1}\eta - B_v^{-1}\xi)B_w^t Y_{v,w}^{-1} B_v (B_w^{-1}\eta - B_v^{-1}\xi)} \mathcal{F}_v(h)(\xi)\, d\xi \\
&= \frac{1}{\sqrt{|\det Y_{v,w}|}} e^{\pi i\sigma/4 - \pi i\eta Y_{v,w}^{-1} B_v B_w^{-1}\eta} \\
&\times \int_{\mathbb{R}^d} e^{\pi i\xi(B_v^{-1})^t B_w^t Y_{v,w}^{-1} B_v B_w^{-1}\eta + \pi i\eta Y_{v,w}^{-1}\xi - \pi i\xi(B_v^{-1})^t B_w^t Y_{v,w}^{-1}\xi} \mathcal{F}_v(h)(\xi)\, d\xi.
\end{aligned}$$

In order to finish the proof, it is now enough to observe that

$$B_w^t Y_{v,w}^{-1} B_v = B_v^t (Y_{v,w}^{-1})^t B_w,$$

and that the above representation of $\mathcal{F}_w$ simplifies exactly to the formula in the statement of Lemma 3.3. ■

We shall now introduce two more representations of tempered distributions associated with a collection of vectors $\{v_1, \ldots, v_d, w_1, \ldots, w_d\}$:

$$\tilde{\mathcal{F}}_v(g)(\xi) = e^{-\pi i\xi(B_v^{-1})^t B_w^t Y_{v,w}^{-1}\xi} \mathcal{F}_v(g)(\xi)$$

and

$$\tilde{\mathcal{F}}_w(g)(\eta) = e^{-\pi i\sigma/4} e^{-\pi i\eta Y_{v,w}^{-1} B_v B_w^{-1}\eta} \mathcal{F}_w(g)(\eta).$$

**Remark.** In view of Steger's observation, these modifications of the generalized Fourier transforms may be compared to the metaplectic representations of symplectic transformations which send bases of Lagrangian planes into elements of the standard basis for $\mathbb{R}^{2d}$.

**Proposition 3.4** *If $\{v_1, \ldots, v_d, w_1, \ldots, w_d\}$ forms a symplectic basis in $\mathbb{R}^{2d}$, i.e., $Y_{v,w} = \mathrm{Id}$, then the relation between $\tilde{\mathcal{F}}_v$ and $\tilde{\mathcal{F}}_w$ takes the form of the standard Fourier transform:*

$$\tilde{\mathcal{F}}_w(g)(\eta) = \int_{\mathbb{R}^d} e^{2\pi i\xi\cdot\eta} \tilde{\mathcal{F}}_v(g)(\xi)\, d\xi. \tag{3.4}$$

**Proof.** It follows easily from Lemma 3.3 that

$$\tilde{\mathcal{F}}_w(g)(\eta) = \frac{1}{\sqrt{|\det Y_{v,w}|}} \int_{\mathbb{R}^d} e^{\pi i[\xi (Y_{v,w}^{-1})^t \eta + \eta Y_{v,w}^{-1} \xi]} \tilde{\mathcal{F}}_v(g)(\xi)\, d\xi. \tag{3.5}$$

Since $\{v_1, \ldots, v_d, w_1, \ldots, w_d\}$ is a symplectic basis for $\mathbb{R}^{2d}$, we have $Y_{v,w} = \mathrm{Id}$, and so (3.5) reduces to (3.4). ■

We can view (3.4) as a formal and general expression for the usual Fourier transform of distributions. We also note that $\tilde{\mathcal{F}}_v$ and $\tilde{\mathcal{F}}_w$ are unitary transformations when restricted to $L^2(\mathbb{R}^d)$.

It is evident that the operators $\mathcal{F}_v$ and $\mathcal{F}_w$ composed with operators $Q_{v_j}$ and $Q_{w_j}$, respectively, become multiplications by $j$th coordinates. We use this fact to deduce the following lemma, which we shall use in the proof of our Theorem 3.6.

**Lemma 3.5** *For each $j = 1, \ldots, d$, the operators $Q_{v_j}$ are multiplications by $\xi_j$ in the $\tilde{\mathcal{F}}_v$ representation, and all operators $Q_{w_j}$ are multiplications by $\eta_j$ in the $\tilde{\mathcal{F}}_w$ representation, i.e.,*

$$\begin{aligned} \tilde{\mathcal{F}}_v(Q_{v_j}(g))(\xi) &= \xi_j \tilde{\mathcal{F}}_v(g)(\xi), \\ \tilde{\mathcal{F}}_w(Q_{w_j}(g))(\eta) &= \eta_j \tilde{\mathcal{F}}_w(g)(\eta). \end{aligned}$$

We can now formulate and prove our main result.

**Theorem 3.6** *Let $\Lambda \subset \mathbb{R}^{2d}$ be a countable sequence of points with the property $\Lambda = -\Lambda$. Let $\{T_{m,n} : (m,n) \in \Lambda\}$ be the family of associated translation-modulation transformations $T_{m,n}$. For a function $g \in L^2(\mathbb{R}^d)$, assume that $\{g_{m,n} = T_{m,n}(g) : (m,n) \in \Lambda\}$ forms an orthonormal basis for $L^2(\mathbb{R}^d)$. For any two vectors $v$, $w \in \mathbb{R}^{2d}$ for which the symplectic form is non-vanishing, i.e.,*

$$\Omega(v,w) \neq 0,$$

*we have*

$$\|Q_v(g)\|_2 \|Q_w(g)\|_2 = \infty. \tag{3.6}$$

**Proof.** *i.* Without loss of generality we may assume that $\Omega(v,w) = 1$. There exists a collection of vectors $\{v_2, \ldots, v_d, w_2, \ldots, w_d\} \subset \mathbb{R}^{2d}$ such that if we let $v_1 = v$ and $w_1 = w$, then $\{v_1, \ldots, v_d, w_1, \ldots, w_d\}$ forms a symplectic basis of $\mathbb{R}^{2d}$. (This result is a simple algebraic fact; for its Hermitian version see [19].) With these vectors we associate the corresponding differential operators $Q_{v_1}, \ldots, Q_{v_d}$, $Q_{w_1}, \ldots, Q_{w_d}$, and the induced $d \times d$ matrices $A_v, A_w, B_v, B_w$. For this part of the proof assume that

$$\det B_v \neq 0 \quad \text{and} \quad \det B_w \neq 0.$$

Due to the assumption about the basis $\{v_1, \ldots, v_d, w_1, \ldots, w_d\}$, the matrix

$$\begin{pmatrix} A_v^t & B_v^t \\ A_w^t & B_w^t \end{pmatrix}$$

is symplectic, i.e.,

$$A_v B_v^t - B_v A_v^t = 0, \quad A_w B_w^t - B_w A_w^t = 0, \tag{3.7}$$

and

$$A_v B_w^t - B_v A_w^t = \mathrm{Id}, \quad A_w B_v^t - B_w A_v^t = -\mathrm{Id}. \tag{3.8}$$

Given a vector $(p, q) \in \mathbb{R}^{2d}$, we use translation by $x$ and the symmetry of $B_v^{-1} A_v$ to calculate

$$\begin{aligned}
&\tilde{\mathcal{F}}_v(T_{p,q}(g))(\xi) \\
= {}& \frac{1}{|\sqrt{\det B_v}|} e^{-\pi i \xi (B_v^{-1})^t B_w^t \xi} \int_{\mathbb{R}^d} h(x) g(x+p) e^{2\pi i x (B_v^{-1}\xi + q) - \pi i x B_v^{-1} A_v x} \, dx \\
= {}& c_{p,q} e^{-\pi i \xi (B_v^{-1})^t B_w^t \xi - 2\pi i p B_v^{-1} \xi} \mathcal{F}_v(g)(\xi + B_v q + A_v p),
\end{aligned}$$

where $c_{p,q}$ is a complex constant of absolute value equal to 1. Recall that for a symplectic basis, $Y_{v,w} = \mathrm{Id}$. Because of this and the symmetry of $B_w B_v^{-1}$, which, in turn, follows from (3.3), we obtain

$$\tilde{\mathcal{F}}_v(T_{p,q}(g))(\xi) = c_{p,q} e^{2\pi i (-(B_v^{-1})^t p + B_w q + B_w B_v^{-1} A_v p) \cdot \xi} \tilde{\mathcal{F}}_v(g)(\xi + B_v q + A_v p).$$

Therefore we can write

$$\tilde{\mathcal{F}}_v(T_{p,q}(g))(\xi) = c_{p,q} T_{(p',q')} \left( \tilde{\mathcal{F}}_v(g) \right)(\xi),$$

where

$$\begin{pmatrix} p' \\ q' \end{pmatrix} = \begin{pmatrix} A_v p + B_v q \\ -(B_v^{-1})^t p + B_w q + B_w B_v^{-1} A_v p \end{pmatrix}. \tag{3.9}$$

(3.3) yields $A_w = -(B_v^{-1})^t + B_w(B_v^{-1}A_v)^t$. Thus, using the symmetry of $B_v^{-1}A_v$, we can write (3.9) in a more familar form

$$\begin{pmatrix} p' \\ q' \end{pmatrix} = \begin{pmatrix} A_v & B_v \\ A_w & B_w \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}.$$

Overall, we obtain that in the $\tilde{\mathcal{F}}_v$ representation, a Gabor system remains a Gabor system, but associated with a new set $\Lambda'$:

$$\tilde{\mathcal{F}}_v(T_{m,n}(g)) = c_{m',n'}(\tilde{\mathcal{F}}_v(g))_{m',n'},$$

where the primes indicate the elements of the new sequence. Since we know that $\tilde{\mathcal{F}}_v$ is unitary on $L^2(\mathbb{R}^d)$, if $\{g_{m,n} : (m,n) \in \Lambda\}$ is an orthonormal basis for $L^2(\mathbb{R}^d)$ then so is $\{c_{m',n'}(\tilde{\mathcal{F}}_v(g))_{m',n'} : (m',n') \in \Lambda'\}$, where $\Lambda' = -\Lambda'$. Thus, using Theorem 2.5 and invoking Proposition 3.4, we obtain that

$$\|\xi_1 \tilde{\mathcal{F}}_v(g)(\xi)\|_2 \|\eta_1 \tilde{\mathcal{F}}_w(g)(\eta)\|_2 = \infty. \tag{3.10}$$

Moreover, because of Lemma 3.5, we know that $Q_{v_1}$ becomes multiplication by $v_1$ in the $\tilde{\mathcal{F}}_v$ representation, and similarly $Q_{w_1}$ becomes multiplication by $w_1$ in the $\tilde{\mathcal{F}}_w$ representation, both in the sense of distributions. Thus (3.10) is equivalent to (3.6), since the generalized Fourier transforms are unitary on $L^2(\mathbb{R}^d)$ and because of Lemma 3.5.

*ii.* First, let us observe that, since $\Omega(v,w) = 1$, we cannot have both $(v^{1+d}, \ldots, v^{2d}) = 0$ and $(w^{1+d}, \ldots, w^{2d}) = 0$. Therefore, without loss of generality, we assume that $(v^{1+d}, \ldots, v^{2d}) \neq 0$.

Recall that according to (3.1) we write $Q_v = \frac{i}{2\pi}\nabla_v + f_v$. Thus we may always find a (non-unique) non-degenerate linear transformation of $\mathbb{R}^d$ such that the operator $\frac{i}{2\pi}\nabla_v$ becomes $\frac{i}{2\pi}\frac{\partial}{\partial \tilde{x}^1}$, in the new coordinates. The operator $Q_v$ can be then written as

$$Q_v = \frac{i}{2\pi}\frac{\partial}{\partial \tilde{x}^1} + \tilde{a}_v^1 \tilde{x}^1 + \ldots + \tilde{a}_v^d \tilde{x}^d.$$

We also note that

$$\tilde{a}_v^1 \tilde{x}^1 + \ldots + \tilde{a}_v^d \tilde{x}^d = \frac{\partial}{\partial \tilde{x}^1}\left(\tilde{a}_v^1 \frac{(\tilde{x}^1)^2}{2} + \tilde{a}_v^2 \tilde{x}^1 \tilde{x}^2 + \ldots + \tilde{a}_v^d \tilde{x}^1 \tilde{x}^d\right) = \frac{\partial}{\partial \tilde{x}^1} q(\tilde{x}).$$

We define a unitary transformation $U$ of $L^2(\mathbb{R}^d)$ to be

$$U(g)(\tilde{x}) = e^{-2\pi i q(\tilde{x})} g(\tilde{x}).$$

It is easy to verify that the operator $Q_v$ takes the form $\frac{i}{2\pi}\frac{\partial}{\partial \tilde{x}^1}$ in this new representation, i.e.,

$$U(Q_v(g))(\tilde{x}) = \frac{i}{2\pi}\frac{\partial}{\partial \tilde{x}^1}U(g)(\tilde{x}).$$

Also, the operator $U \circ Q_w \circ U^{-1}$ may be written in an analogous form:

$$\tilde{b}_w^1 \frac{i}{2\pi}\frac{\partial}{\partial \tilde{x}^1} + \ldots + \tilde{b}_w^d \frac{i}{2\pi}\frac{\partial}{\partial \tilde{x}^d} + \tilde{a}_w^1 \tilde{x}^1 + \ldots + \tilde{a}_w^d \tilde{x}^d.$$

We shall consider three different possibilities for the differential part of the operator $U \circ Q_w \circ U^{-1}$.

*ii.a.* In case $\tilde{b}_w(\tilde{b}_w^1, \ldots, \tilde{b}_w^d) = 0$, $U \circ Q_w \circ U^{-1}$ has the form

$$\tilde{a}_w^1 \tilde{x}^1 + \ldots + \tilde{a}_w^d \tilde{x}^d.$$

Since $\Omega(v,w) = 1$, we have $\tilde{a}_w^1 = 1$. We make the following non-degenerate linear transformation in $\mathbb{R}^d$:

$$z^1 = \tilde{a}_w^1 \tilde{x}^1 + \ldots + \tilde{a}_w^d \tilde{x}^d, \quad z^2 = \tilde{x}^2, \quad \ldots, \quad z^d = \tilde{x}^d. \tag{3.11}$$

Thus we obtain

$$U \circ Q_v \circ U^{-1} = \frac{i}{2\pi}\frac{\partial}{\partial z^1}, \quad U \circ Q_w \circ U^{-1} = z^1,$$

and the problem reduces to the standard Balian-Low theorem, Theorem 2.1.

*ii.b.* If $\tilde{b}_w = \alpha b_v = (\alpha, 0, \ldots, 0)$ and $\alpha \neq 0$ then, since we again have $\tilde{a}_w^1 = 1$, by making the same transformation (3.11) as in part *ii.a*, we obtain:

$$U \circ Q_v \circ U^{-1} = \frac{i}{2\pi}\frac{\partial}{\partial z^1}, \quad U \circ Q_w \circ U^{-1} = \alpha \frac{i}{2\pi}\frac{\partial}{\partial z^1} + z^1.$$

It is again easy to see that our result follows from the standard Balian-Low theorem.

*ii.c.* Finally we consider the case $\tilde{b}_w \neq \alpha b_v$ for all $\alpha$. We can make a linear transformation in $\mathbb{R}^d$ such that $U \circ Q_v \circ U^{-1}$ remains the differentiation with respect to the first coordinate $z^1$, and the differential part of $U \circ Q_w \circ U^{-1}$ becomes the differentiation with respect to the second coordinate $z^2$, i.e.,

$$U \circ Q_v \circ U^{-1} = \frac{i}{2\pi}\frac{\partial}{\partial z^1}, \quad U \circ Q_w \circ U^{-1} = \frac{i}{2\pi}\frac{\partial}{\partial z^2} + z^1 + c_w^2 z^2 + \ldots + c_w^d z^d.$$

We now define the following two families of vectors in $\mathbb{R}^{2d}$:

$$\begin{aligned} v_1 &= (0, \dots, 0; 1, 0, \dots, 0) \\ v_2 &= (0, c_w^2, c_w^3, \dots, c_w^d; 0, 1, 0, \dots, 0) \\ v_3 &= (0, 0, c_w^3, 0, \dots, 0; 0, 0, 1, 0, \dots, 0) \\ &\dots \\ v_d &= (0, \dots, 0, c_w^d; 0, \dots, 0, 1) \end{aligned}$$

and

$$\begin{aligned} w_1 &= (1, \dots, c_w^2, c_w^3, \dots, c_w^d; 0, 1, 0, \dots, 0) \\ w_2 &= (0, 1, 0, \dots, 0; 1, 0, \dots, 0) \\ w_3 &= (0, c_w^3, 1, 0, \dots, 0; 0, 0, 1, 0, \dots, 0) \\ &\dots \\ w_d &= (0, c_w^3, 0, \dots, 0, 1; 0, \dots, 0, 1). \end{aligned}$$

It is not difficult to verify that $\{v_1, \dots, v_d; w_1, \dots, w_d\}$ forms a symplectic basis in $\mathbb{R}^{2d}$ and that the matrices $B_v$ and $B_w$ are both non-degenerate. Thus we have reduced this situation to the case described in part $i$. ■

**Remark.** We used the notion of a symplectic matrix in the proof of Theorem 3.6. A matrix $M$ is *symplectic* if it preserves the symplectic form $\Omega$, i.e., $\Omega(Mv, Mw) = \Omega(v, w)$, for all $v, w \in \mathbb{R}^{2d}$. The collection of all such matrices forms a group, the so-called *symplectic group*, which plays a significant role in the study of Hamiltonian systems. In fact, the symplectic matrices generate invertible transformations which take a Hamiltonian system into another such system of differential equations, see, e.g., [1], [2], [27].

Following [18] we say that a lattice $\Lambda \subset \mathbb{R}^{2d}$ is *symplectic* if

$$\Lambda = rM(\mathbb{Z}^{2d})$$

for some $r \in \mathbb{R} \setminus \{0\}$ and $M$ a symplectic matrix.

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