ERRATA to REAL VARIABLE AND INTEGRATION

Corrections for pages 1–16:

• Page 7, Acknowledgment. The beginning of the Acknowledgment should be changed to: "Several colleagues and students have generously helped me: A. Beardon, A. Bruckner, and R Burckel critically read parts of the manuscript; S. Espelie and ...".

• Page 14, line 8. The symbols, "$^{-}x_n$", should be changed to "$\overline{\lim}\, x_n$".

Errata and supplementary notes:

• Page 24, "Proposition 1.17" should be changed to "Proposition 1.7". On line 16, add: "in their interior" (as on line 8). A period should be placed after "a)" on line 21.)

• Page 25, Example 1.5. Refer to Proposition 3.6 at the end of Example 1.5.

• Page 31–32, section 1.4. Add: "Medvedev, F.A.: The work of Henri Lebesgue on the theory of functions (on the occasion of his centenary)." Russian Math. Surveys 30 (1975) 179 – 191".

• Page 40, remark 2.2. Add: "[82, pp. 24 – 26]".

• Page 51, section 2.2. In the discussion of the Banach-Tarski paradox, refer to Dieudonné's biography of von Neumann in the Dictionary of Scientific Biography".

• Page 60. Before Corollary 2.8.1, add: "Theorem 2.8 b should be compared with the characterization of regulated functions in Problem 3.16.".

• Page 66 and 67, Problem 2.10. Omit part ii) of Problem 2.10 b and the reference to part b ii) in part c).

• Page 69, line 1. The Remark should be changed to: "Part b) does not extend to $\mathbb{R}^n$ for $n \geq 3$.".

• Page 76 and 77, Example 3.3. This example should be shanged to: "We say that f: $[a, b] \to \mathbb{R}$ is a *step function* if $f = \sum_1^n a_j \chi_{I_j}$ on $\cup_1^n I_j$, where $I_j = (x_{j-1}, x_j)$ and $x_0 = a, x_n = b$; also g: $[a, b] \to \mathbb{R}$ is a *regulated function* if $\lim_{y\to x^\pm} g(y)$ exists for each $x \in (a, b)$, and $\lim_{t\to a^+} g(y)$ and $\lim_{t\to b^-} g(y)$ exist. For each step function f there is a simple function h such that $f = h$ except for possibly finitely many points. Further, continuous functions are regulated as are functions of bounded variation, e.g., Chapter 4, and each regulated function is bounded m-a.e. A basic fact about regulated functions is the following: *f is regulated if and only if f is the uniform limit of a sequence of step functions*, cf. Theorem 2.8 and Example I.9.".

• Page 79, line -2. In the Remark concerning Denjoy's theory of integration, add the reference, "[80, volume II]".

• Page 87. After the proof of Theorem 3.8 (LDC), add: "Remark. Nets [63, p.65] or filters arise in non-metric convergence, e.g. section I.9. If a net of functions is indexed by a countable set than Theorem 3.8 holds [18, Chapter IV.3.7]. We give an example of a convergent net for which Theorem 3.8 fails. Let N be the set of characteristic functions χ_F, where $F \subset [0,1]$ is a finite set. N is a net since $\chi_{F_1 \cup F_2}$ dominates given functions χ_{F_1} and χ_{F_2}; and N converges to the function $\chi_{[0,1]}$. On the other hand, Theorem 3.8 fails since $\int \chi_F = 0$ and $\int \chi_{[0,1]} = 1$. The deficiency in this example is that $\chi_F = 0$, m-a.e. In this context it is natural to inquire if a genuine failure of Theorem 3.8 for nets is possible; for this question it is well to keep in mind the fact that measurable functions are characterized in terms of *sequences* of measurable functions.".

• Page 92, line -3 and -2. The last sentence of the proof of Proposition 3.3 should be changed to: "(3.17) follows because $R\int_a^b f$ exists and by the discussion in section 3.1.".

• Page 93, line 6. The inequality, "$n \geq 2$", should be changed to $n \geq 4$".

• Page 97. After Proposition 3.7, add: "For perspective we note that Proposition 3.8. If $f \in L^1_m[a,b]$ then there is a sequence $\{\alpha_n > 0 : n = 1, ...\}$ such that if $\beta_n \in (0, \alpha_n), n = 1, ...$, then $\lim_{n\to\infty} f(x + \beta_n) = f(x)$, m-a.e.

This result is proved by choosing $\{\alpha_n : n = 1, ...\}$ for which $\sum \|\tau_{\alpha_n} f - f\|_1 < \infty$, where $\tau_\alpha f(x) = f(x - a\alpha)$. It should be compared with Problem 2.20.".

• Page 98. The first sentence of section 3.5 should be changed to: "The following three results follow from LDC.".

• Page 98, line -1. The symbol, "$\int$", should be changed to "$\int_X$".

• Page 99. Omit line 1.

• Page 99, Theorem 3.16. Theorem 3.16 is false and should be changed to:

"Theorem 3.16. Let $(X, \mathcal{A}, \mu)$ be a measure space and choose an open interval $A \subset \mathbb{R}$. Define the function $f(x,t)$ on $X \times A$ asuming that:

i) $\forall t \in A$, $f(\cdot, t)$ is a μ-measurable function,

ii) $\exists g \in L^1_\mu(X)$ such that $\forall t \in A$, $|f(x,t)| \leq g(x)$, μ-a.e.,

iii) $d/dt f(x,t)$ exists on A, μ-a.e., and

iv) $\exists h \in L^1_\mu(X)$ such that $\forall t \in A$, $|d/dt f(x,t)| \leq h(x)$, μ-a.e.

Then

$$\frac{d}{dt}\int_X f(x,t)d\mu(x) = \int_X \frac{d}{dt} f(x,t)d\mu(x)."$$

• Page 100, equation (3.24). Omit "2π" from the denominator of the RHS.

• Page 102, Problem 3.6 b. After the statement of Steinhaus' result, refer to Problem

1.12; and at the end of problem 3.6 b, add: "Steinhaus' result can be used to characterize absolute continuity as we observe in the remark after Proposition 5.7 at the end of Chapter 5.4.".

- Page 104, Problem 3.10 d. The symbols "$f)(n$", should be replaced with "$f)\hat{}(n$"

- Page 104, Problem 3.11. Problem 3.11 is false and should be changed to:

"3.11. Let $\{f_n : n = 1, ...\}$ be a sequence of non-negative functions defined on $[0, 1]$. Assume that $\sum_n \int_0^1 f_n < \infty$ and prove that $f_n \to 0$, m-a.e. (Hint: $\int \sum f_n = \sum \int f_n < \infty$ and so $\sum f_n(x)$ is finite m-a.e.) This result is an often used technical device, e.g. Proposition 3.8.".

- Page 105, Problem 3.16. Omit the Hint and add: "The characterization of regulated functions in Example 3.3 gives us a means of verifying Proposition 3.7. In fact, *if f is a regulated function then there is a function F such that $F' = f$ except for possibly a countable set.*".

- Page 106, Problem 3.19 c. Add: "Actually, the example can be constructed so that $\lim \|f_n\|_1 = 0$ [18, Chapter IV, p.236].".

- Page 111, Problem 3.26 a. The first sentence of Problem 3.26 a should be changed to: "Given a sequence $n_k : k = 1, ...\} \subset \mathbb{Z}^+$ converging to infinity, we let S_{n_k} denote the set $\{x : \lim_k \sin(n_k x) \text{ exists}\} \subset \mathbb{R}$.".

- Page 112, Problem 3.26. After part e), add:

"f) Suppose that $\sum_1^\infty a_n$ is an absolutely convergent series. By way of considering the converse situation to part e), prove the following: if $\sum_1^\infty a_n/n \cos(nx)$ is not constant on $[0, 2\pi)$, then $\sum_1^\infty a_n \sin(nx)$ cannot converge pointwise m-a.e. to zero.".

- Page 112. After problem 3.27, add:

"3.28 a) prove that $\sum_1^\infty e^{inx}/n$ converges on $[0, 2\pi)$ if and only if $x \neq 0$. (Hint: Use partial summation obtaining a bound on the geometric series $\sum_1^N e^{inx}$.)

b) let $\{a_n : n = 1, ...\}$ be a sequence decreasing to 0 and define $g(x) = \sum_1^\infty a_n \sin(nx)$. By an argument similar to that of part a) the series converges for each x. Prove that $g \in L^1(\mathbb{T})$ if and only if $\sum_1^\infty a_n/n < \infty$, e.g., [35. Volume I, p.115]. Note that $\sum_1^\infty e^{inx}/n \in L_m^2(\mathbb{T})$ by Parseval's equality.".]

- Page 116, line -4. Add: "$\alpha = (1 + \sqrt{5})/2$ is a Pisot number since $|(1 - \sqrt{5})/2| < 1$.".

- Page 117. Omit lines 7 to 13 inclusive.

- Page 119, line 13. The symbols, "$f(b) - f(a)$", should be changed to "$f(a) - f(b)$".

- Page 122, line 14. Before Proposition 4.3, add: "The result is taken from Amer. Math. Monthly 75(1968) 312–313, and a major extension and application is found in J. Functional Analysis 5(1970) 194–203.".

- Page 122. line -1, and Page 123, line 2. The symbol, "e" should be italicized.

- Page 123, line 12. The symbol, "f", should be changed to "h".

- Page 123, line 14. The symbols, "$+h(\frac{x+1}{2})$", should be changed to "$-h(\frac{x+1}{2})$".

- Page 124, line 11. Before equation (4.14), add: "As a convention we set $f(x+h) = 0$ if f is defined on $[a, b]$, $x \in [a, b]$, and $x + h \notin [a, b]$.".

- Page 124, line -9. The phrase, "of basic", should be changed to "of the basic".

- Page 124, Theorem 4.4. The hypothesis, "$f \in L^{\infty}_{m}[a, b]$" should be changed to "$f \in L^{1}_{m}[a, b]$".

- Page 125, line 7. The symbols, "$(K + \|f\|_{\infty})|h|$", should be changed to "$K|h|$".

- Page 126, line -3. The sum on the RHS of the equation should be changed to a difference.

- Page 127, line -9. The parentheses should be closed between "discontinuous" and "part".

- Page 127. Proposition 4.6 from page 130 should be inserted before Theorem 4.5, and line -5 on page 127 should be changed to: "From Proposition 4.5, Proposition 4.6, and the above remark we conclude with".

- Page 128, Example 4.3. The symbols, "$\{\xi_k : k = 1\}$" should be changed to "$\{\xi_k : k = 1, ...\}$".

- Page 130. As noted above, line 6 to 9 inclusive should be moved to page 127.

- Page 130, line 10. Omit: "We also omit the details of".

- Page 142, line 6. The sentence, "If $x \in L$ then x is in some L_n", should be changed to "If $x \in L$ then x is in $L_{n(\epsilon,r)}$".

- Page 144, line 14. The closed interval, "$[a, c]$", should be changed to "(a, c)".

- Page 145, equation (4.41). Each "n" should be italicized.

- Page 145, Example 4.8. The penultimate sentence should be changed to: "Theorem 4.13, which follows from FTC, is used to prove (4.41).".

- Page 146, line 2. The word, "leads", should be changed to "lead".

- Page 148, line 8. The symbols "FG" should be italicized.

• Page 150, line -4. In the definition of $F(x)$ the interval, "$[0,1]$", should be changed to "$(0,1]$".

• Page 155, Problem 4.15. The hint should be changed to: "(Hint. Use Corollary 4.10.1 or the Fourier uniqueness theorem.)"; and then add: "Note that if $f \in L^1_m[0,1]$ and

$$\|\tau_{-h}f - f\|_1 = o(|h|^2), \ \ |h| \to 0,$$

then $f = 0$, m-a.e.".

• Page 156, Problem 4.23. The following hint should be given. The problem is conveniently divided into two parts. In the first part, assume that $|f_n'(x)| \leq M$ for each n and x; and use LDC and FTC. In the second part use the change of variables formula from Problem 4.28.

• Page 157, Problem 4.26. Add: "Although it is not apparent, there are continuous, non-constant, locally recurrent functions.".

• Page 160, Problem 4.35 c. At the end of the problem, add: "In particular, if $f = 0$ on the cantor set $\mathcal{C} \subset [0,1]$ then f is identically 0 on $\mathbb{R}$. This latter statement follows directly from Problem 1.12 and the fact that $y = nx$, $x \in [0,1]$, and $f(y) = nf(x)$ for a given $y \in \mathbb{R}$.".

• Page 161, Problem 4.35. After point e), add:

"f) Let f satisfy the functional equation and assume $f = 0$ on some non-measurable set. Does $f = 0$ on $\mathbb{R}$? (Hint: Use the material about Hamel bases in Problem 2.10 d), e), f).)".

• Page 162. After Problem 4.42, add:

"4.43. Let $f \in L^1_m[0,2\pi]$. Prove that $\sum |\hat{f}(n)n|^2 < \infty$ if and only if f is absolutely continuous on $[0,2\pi]$, $f' \in L^2_m[0,2\pi]$, and $f(0) = f(2\pi)$.

4.44. In Proposition 4.9 we used LDC to prove that $F(x) = \int_a^x f$ is continuous if $f \in L^1_m[a,b]$. Now prove the following continuity property: let $f : [a,b] \to \mathbb{R}^+$ be Lebesgue measurable and pick any $y \in [0, \int_a^b f]$; then there is $A \in \mathcal{M}$ for which $y = \int_A f$.

(Hint: a) Assume $\int_a^b f < \infty$. Then F is continuous so that by the intermediate value theorem there is t for which $F(t) = y$; hence, set $A = [a,t]$.

b) Assume $\int_a^b f = \infty$. $A_n = \{x \in [a,b] : f(x) < n\} \in \mathcal{M}$, $A_n \subset A_{n+1}$, $\cup A_n = [a,b]$, and $\lim \int_{A_n} f = \infty$. If $y = \infty$ let $A = [a,b]$. If $y < \infty$ let $\int_{A_n} f > y$ for some n and apply part a) to $f_n = \chi_{A_n} f \in L^1_m(A_n)$ to obtain $B \in \mathcal{M}$ for which $\int_B f_n = y$; therefore, let $A = B \cap A_n$.)".

• Page 196. After Example 5.5, add:

"Remark. It is not difficult to prove that if $f \in \cap L^p_m[a,b]$ then $f \in L^\infty_m[a,b]$ is and only if $\lim_{p\to\infty} \|f\|_p$ converge (in $\mathbb{R}^+$), in which case $\lim_{p\to\infty} \|f\|_p = \|f\|_\infty < \infty$, cf Problem 5.30.".

• Page 204, Problem 5.33. On line 5, add: "A major result in this area is due to D. Kölzow (1968), e.g. [22, pp.35–37], who proved that whenever a measure space admits R-N then the R-N derivative has the form given in (P5.5).".

• Page 226, line 3. The sentence referring to reference, "[125]", should be changed to: "The best results for compact Hausdorff spces are due to Wells [125]; and Pfanzagl (Manuscripta Math. 4(1971) 91–98) has proved tha analogue of Theorem 6.7 for the case of regular complex measure on any Hausdorff space (cf. Problem 3.4).".

• Page 227, Theorem 6.10. In the statement of the theorem, the symbols, "$1, < p < \infty$", should be changed to "$1 < p < \infty$".

• Page 243, line 10. The sentence after equation (I.2) should be changed to: "Choose a simple function h such that $\|g - h\|_\infty \leq \epsilon/(2K)$ where $K = \max(k, \|f\|_1)$.".

• Page 243, line 15. The sentence, "this is proved in Chapter 6.", should be changed to "this is given in Theorem 6.5.".

• Page 247, line -7. The symbols, "$\{x'_n\}$" should be changed to "$\{x'_\alpha\}$."

• Page 247, Example I.9. Refer to the characterization of regulated functions in Problem 3.16.

• Page 265, line 6. The symbols, "$T^n_\alpha(K)$", should be changed to "$T^{(n)}_\alpha(K)$".

• Page 266, line 11 and 12. In order to prove that μ is positive we assume the opposite and obtain a contradiction using the fact that $\|\mu\|_1 = \mu(1)$; there is no need to refer to "[18, p. 101]".

• Page 266, line -13. The symbols, "$d\mu(x)$", should be changed to "$d\mu(y)$".

• Page 267, reference [23]. The year should be changed to "1966".

• Page 268, reference [28]. The complete reference is: "On differentiation. Acta Sci. Math. Szeged 38 (1976) 239–251".

• Page 268, reference [47]. This reference should be changed to: "de Guzmán, M.: Differentiation of integrals in $\mathbb{R}^n$. Lecture Notes in Mathematics, 481, Berlin - Heidelberg - New York 1975 (Book Review by R. Fefferman BAMS 83 (1977) 196–200)".

ERRATA AND SUPPLEMENTARY NOTES (SECOND LIST, JUNE 1978)

• Page 36, P1.26. Add at the end ", cf., P1.10.".

• Page 55, line -14. Instead of "b) $\iff$ e)" write "b) $\iff$ c)".

• Page 55, line -3. Instead of "holds." write "holds, cf. Problem 2.15.".

• Page 56, line 15. Instead of the sentence, "$\{x : ...$ measurable.", write: "For $\alpha \geq 0$, $\{x : f^2(x) > \alpha\} = \{x : f(x) > \sqrt{\alpha}\} \cup \{x : f(x) < -\sqrt{\alpha}\} \in \mathcal{A}$ implies f^2 is measurable, and for $\alpha < 0$, $\{x : f^2 > \alpha\}$ is the domain of f.".

• Page 56, line -7. Write "map g depicted" instead of "map depicted".

• Page 63, line -13. Add period, ".".

• Page 65. In Problem 2.5 b, omit the Hint.

• Page 68. Before the Hint make the following addition: "Give an example of a Lebesgue measurable function $f : [a, b] \to \mathbb{R}$ for which $f^{-1}(\mathcal{M})$ is not contained in $\mathcal{M}$, cf., Figure 3. Is it true that $f^{-1}(\mathcal{B}) \subset \mathcal{B}$ for every Lebesgue measurable function $f : [a, b] \to \mathbb{R}$?".

After the Hint add: "Thus, Lebesgue measurable functions f are characterized by the condition, $f^{-1}(\mathcal{B}) \subset \mathcal{M}$, as Example 2.11 shows."

• Page 74, line 1 to end of the proof. Replace the proof by the following:

"Proof: i) We must prove that $\sup s_Q = \inf S_Q$, and begin by noting that $-\infty < \sup s_Q, \inf S_Q < \infty$; in fact, $\alpha(b - a) \leq s_Q \leq S_Q \leq \beta(b - a)$ where $\alpha \leq f(x) \leq \beta$. We shall verify that $|\sup s_Q - \int S_Q| < \epsilon$ for each $\epsilon > 0$.

ii) Given $Q' \subset Q$, we check that $s_{Q'} \leq s_Q$ and $S_{Q'} \geq S_Q$. Without loss of generality take $Q' : \alpha = y_0 < ... < y_n = \beta$ and let $Q = Q' \cup \{y\}$ where $y_{j-1} < y < y_j$. Define $B_j = \{x : y \leq f(x) < y_j\}$ and $C_j = \{x : y_{j-1} \leq f(x) < y\}$ so that

$$y_j mB_j + ymC_j \leq y_j(mB_j + mC_j) = y_j mA_j;$$

consequently, $S_Q \leq S_{Q'}$, with a similar calculation for s_Q.

iii) Therefore, since the sup and inf are finite, there are partitions Q_1' and Q_2' such that, whenever $Q_k' \subset Q_k$, $k = 1, 2$,

$$|\sup s_Q - s_{Q_1}| < \epsilon/3, \qquad |\inf S_Q - S_{Q_2}| < \epsilon/3,$$

and if $\{y'_{i,k}\} = Q'_k$, $k = 1, 2$, then $y'_{i,k} - y'_{i-1,k} < \epsilon/(3(b - a))$. Let $\overline{Q} = Q_1 \cup Q_2 = \{\overline{y}_i\}$. Then $|\sup s_Q - \inf S_Q| < \epsilon$ because of the triangle inequality and the fact that

$$0 \leq S_{\overline{Q}} - s_{\overline{Q}} = \sum_0^n (\overline{y}_{i+1} - \overline{y}_i) mA_i < \frac{\epsilon}{3(b - a)} \sum_0^n mA_i = \epsilon/3.$$

q.e.d."

• Page 78, line -1 and -2. Begin both parts with the hypothesis: "Assume $f \in L^1_\mu(X)$.".

- Page 80, line -3. Complete the absolute value.

- Page 148, line 2. Write "$G(x) = s + \int_a^s g$" instead of "$G(x) = s + \int_a^s f$".

- Page 162. Add Problem 4.45:

"4.45. Let $f : [a, b] \to \mathbb{R}$ be a differentiable function and suppose $f' \in BV[a, b]$. Prove that $f' \in C[a, b]$.

(Hint. Use the intermediate value theorem for derivatives: let $f : [a, b] \to \mathbb{R}$ be a differentiable function and suppose $f'(a) < \lambda < f'(b)$; then there is $x \in (a, b)$ for which $f'(x) = \lambda$)"

- Page 230, line -2. Write "f^{-1} and f" instead of "f and f^{-1}".

- Page 234. Replace the last sentence of paragraph 5 (i.e. lines -9, -8) by: "In both cases each injection is continuous (continuous functions are defined in Appendix I.4)."

- Page 240, line -1. Write "a_n" instead of "a".

- Page 241, line 1. Write "$\cup_n \cap_m I_{m,n}$" instead of "$\cap_m I_{m,n}$".

ERRATA AND SUPPLEMENTARY NOTES (THIRD LIST, FALL 1978)

- Page 16, line -5. Replace the display "$x \notin C ... < 3^n$" by:

$$\text{"}x \notin C \quad \text{if} \quad x \in \left(\frac{3k+1}{3^n}, \frac{3k+2}{3^n} \right), \quad k \geq 0 \quad \text{and} \quad n \geq 1.\text{"}$$

- Pages 156–157. P4.24 is false as it stands. Replace it with "Problem 4.45" (p.162) which I list on my "Errata and supplementary notes (2nd list, June 1978)".