GETTING PAID FOR EATING ICE-CREAM

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The beginnings, the Chern-Simons and η invariants. I was fortunate early in my career to have had a number of great mathematicians as teachers, one of them was I.M. Singer, when I was an undergraduate at M.I.T., and another was S.S. Chern, when I was a graduate student at Berkeley. I mention the two of them together because I learned the same point of view from both of them. As a sophomore in the spring of 1966 I had a course in "vector calculus" from Singer. This course was in fact all about differential forms in Euclidean space and concluded with Stokes' Theorem. Singer told us about de Rham's theorem at the end of the course. That was one of the things that made me decide to pursue a career in mathematics . Also I spent my junior year in Paris and took Math Un from Laurent Schwartz at Jussieu. Schwartz was a remarkable teacher and his course played a critical role in my development.

When I arrived at Berkeley in the fall of 1968 there were over 300 graduate students there so it was easy to get lost in the crowd. I had an excellent algebraic topology course from George Cooke (who befriended and helped a number of graduate students including me). We learned about characteristic classes following the Princeton notes of Milnor. However on the basis of my undergraduate education I felt that using only algebraic topology for the development of characteristic classes set aside too many of the powerful tools available for actual computations. In the spring of 1971 I attended Chern's geometry course. It was exactly what I had been waiting for. It was all about computations with connection and curvature forms, and the explicit differential forms in curvature representing the Pontriagin and Chern classes (Chern never called them *Chern* classes, see the next paragraph). We were learning Chern-Weil theory from its creator. I didn't attend classes regularly in Berkeley, but I never missed one of Chern's lectures. One thing that turned out to be very lucky for me was that Chern showed us the construction of the Chern-Simons invariants, brand-new at the time. Of course it wasn't an accident that Chern included the Chern-Simons invariants in his course, I am sure he knew they were going to be important though even he could hardly have guessed how important they have proved to be.

There was a humorous story current among the geometry graduate students in Berkeley at the time. I told this story in an article I wrote for a previous Chern anniversary volume but it bears retelling here. Chern would often teach a wonderful undergraduate geometry course (we all read the course notes). The story was that one undergraduate in filling out his teaching evaluation said about Chern's teaching *"appears to know the subject"*.

Chern always appeared calm, a very welcome quality in that time of chaos and turbulence. The university closed down my first spring (1969) with 3000 national guardsmen on campus, tear gas every day and military helicopters flying overhead because of "Peoples' Park", all sent to us by then Governor Ronald Reagan. Then the next spring the university closed down again, this time because of the Cambodian invasion. It is remarkable that such a large number of my fellow students survived the chaos and went on to very succesful careers the most famous being S.T. Yau and W. Thurston (but there were many others). But we learned from each other and, when classes were held, from the wonderful faculty both senior and junior. Many of those successful students were students of Chern.

However with over 300 graduate students there just weren't enough faculty to go around. I didn't have an official thesis adviser in August 1971 when Jim Simons arrived in Berkeley wanting to compute the Chern Simons invariants for the Lens spaces $L(p, q)$. I had computed all kinds of things for Lens spaces with my friend Tor Skjelbred and had learned about Chern-Simons invariants in Chern's class as I said above so I was ready and made the computation in two days. I still marvel at my good luck and am grateful to Blaine Lawson who arranged my meeting with Jim Simons. Had it not been for Chern's class and meeting Jim Simons I might never have completed a PhD thesis.

After making the computation of Chern-Simons invariants my status changed. For example, I got to attend an incredible Chinese banquet in honor of Jim Simons, when he came to town to talk to Chern, which were given by Chern's admirers in the restaurant community in Chinatown in San Francisco. The dinner had at least ten courses and one course was *flaming baby quail.* More important for my career, I met Michael Atiyah (through Jim Simons) and he inspired me to compute the η-invariant of Atiyah-Patodi-Singer for compact hyperbolic n-manifolds, Annals of Math. 1978,[24]. I was aided in this computation by my friend Takuro Shintani one of the very best Japanese mathematicians at the time, who later discovered the famous "Saito-Shintani" lifting before his premature death. Although my formula for the η-invariant in terms of the value at zero of a "Selberg zeta function" made up from information attached to closed geodesics was beautiful, it gave no information on the critical problem of rationality versus irrationality for the η-invariant (or its reduction modulo $\mathbb{Z}$, the Chern-Simons invariant). It struck me when I left the Institute for Advanced Study to become an assistant professor at Yale in 1974 that rather than try to attack such a subtle question about hyperbolic manifolds it would make more sense to try to compute their Betti numbers about which nothing was known at the time.

Special cycles in locally symmetric spaces. At the time there was a great deal of interest in the Betti numbers of locally symmetric spaces $M = \Gamma\backslash G/K$ in general. Here G is a noncompact semisimple Lie group, K is a maximal compact subgroup (so $D = G/K$ is the symmetric space atached to G) and Γ is a lattice (discrete subgroup with finite-volume quotient) in G. Such spaces are ubiquitous in mathematics. At the time all the work was directed to proving *vanishing* theorems for cohomology using Bochner-type arguments to prove there were no harmonic forms (although eventually representation theory replaced Bochner in the final definitive result of Vogan and Zuckerman). I decided to try to prove *nonvanishing* theorems for homology by looking at the very special submanifolds N in M that were totally-geodesic and trying to prove they were not boundaries. As I said I focused on the case in which M was a compact hyperbolic manifold. I noticed at once that the examples (that occur for all n) constructed by Armand Borel (going back to Siegel) contained totally-geodesic *hypersurfaces* that were fixed-point sets of involutions. George Cooke had pointed out to me that a (locally two-sided) hypersurface N of M was trivial homologically if and only if it separated M (into two parts). I was fortunate that the great Indian mathematician M.S. Raghunathan was visiting Yale. He would invite me up to his office to explain key arguments about discrete groups and on one of those visits he showed me the key idea I needed to prove N didn't separate M. The result was the thirteen-page Annals paper, [23]. I pointed out to him that the hyperbolic manifolds M of Borel contained totally-geodesic submanifolds N of *all* dimensions and they came in *complementary pairs* so we could try to prove the two elements in the pair had nonzero intersection number. Raghunathan then made the brilliant observation that whenever N was the fixed-point set of an involution σ in any locally symmetric space there was always a complementary totally-geodesic submanifold N' which whas the fixed-point set of the product σ' of an appropriate Cartan involution with σ. We gave examples for many $\Gamma\backslash G/K$ as above where the intersection number of N and N' was nonzero and so proved many nonvanishing theorems especially for the orthogonal groups $SO(p,q)$ and the unitary groups $SU(p,q)$. In particular we proved that all the Betti numbers of the Borel examples of compact hyperbolic manifolds were nonzero. Our paper appeared in a special volume published by the Indian Academy of Sciences in memory of Vijay Patodi, one of the coinventors of the η-invariant, see [28]. If the general result were true (and its statement is compatible with all known "vanishing theorems") it would lead to a very large number of nonvanishing results - it is quite easy, given a semisimple Lie group to compute the possible centralizers of involutions.

I will henceforth refer to the cycles in locally-symmetric spaces carried by submanifolds that are locally the fixed-point sets of involutions as *special cycles.* Steve Kudla and I introduced this terminology in our joint work (to be described next).

Some very special cycles. In what follows, to keep the notation manageable I will not try to explain the work of Kudla-Millson in full generality. I will restrict to a particularly simple class of special cycles which I now describe. Let V be a real (resp complex) m dimensional space equipped with a quadratic (resp. Hermitian) form $(\, , \,)$ of signature (p,q). Furthermore we choose a lattice $\mathcal{L}$ in V and let Γ be a congruence subgroup of the stabilizer of $\mathcal{L}$. As before $M = \Gamma\backslash G/K$. Let X be an oriented positive line in V, that is a one-dimensional subspace of V (with a chosen orientation) such that the restriction of $(\, , \,)$ to D is positive definite. We now realize the symmetric space D of $G = SO(V)$(resp. $SU(V)$) as the space of "negative q-planes", that is, the space of q-planes $Z \subset V$ such that the form $(\, , \,)$ restricted to Z is negative definite. Now define the subsymmetric space D_X of D by

$$D_X = \{Z \in D : Z \text{ is orthogonal to } X\}.$$

Then D_X is the fixed-point set of reflection in X (an involution). Assume further that X is defined over $\mathbb{Q}$ (i.e. has nonzero intersection with the lattice $\mathcal{L}$). Then the image of D_X in M will be a properly embedded submanifold which we will denote C_X. If I had used the Borel examples (using a totally-real number field $\mathbb{K}$ instead of $\mathbb{Q}$) then C_X would have been compact.

In fact the special cycles not only provided nonzero homology classes with trivial coefficients they also gave rise to nonzero homology classes with nontrivial coefficients. Given a flat bundle E over a manifold M one can define homology (and cohomology) with coefficients in E. If M is m-dimensional, compact and oriented and N is an n dimensional compact oriented submanifold of M then N gives rise to a homology class $[N]$ with trivial coefficients called the fundamental class of N. If the restriction of E to N admits a nowhere zero parallel section s, the submanifold N and the section s can be combined to form a cycle with coefficients in E to be denoted $N \otimes s$ and called "a special cycle with coefficients". Raghunathan had previously listed all possible locally symmetric spaces and locally homogeneous flat bundles E over them for which $H_1(M, E)$ (and by duality $H_{m-1}(M, E)$) could be nonzero. Once again M had to be either a hyperbolic or complex hyperbolic manifold and now the coefficient system had to be the space of polynomials of a fixed degree on V (harmonic polynomials in the real hyperbolic case). Kazhdan (see below) had settled the complex hyperbolic case with and without coefficients using theta functions. It turned out the cycles Raghunathan and I had used for trival coefficients could be equipped with coefficients and shown to be nonzero in homology. First [25] I showed that the first homology with harmonic coefficients was nonzero. Later [27], in a paper dedicated to Raghunathan for his sixtieth birthday, I showed that all the homology groups for hyperbolic manifolds with coefficients (all possible degrees and all possible coefficients) that were allowed to be nonzero by the vanishing theorem of Vogan and Zuckerman were in fact nonzero. In case M was a hyperbolic manifold, the totally geodesic hypersurfaces with coefficients in $V = \mathbb{R}^{n,1}$ were tangent to "Thurston bending deformations" and this led me into deformation theory as will be seen below.

Using theta functions to construct the Poincaré duals of special cycles. According to a famous theorem of Kazhdan (sharpened by Kostant) there were only two possible families of irreducible locally symmetric spaces that could have nonzero first Betti number, those associated to $SO(p, 1)$ (quotients of hyperbolic space) and those associated to $SU(n, 1)$ (quotients of complex hyperbolic space). My result described above settled the case of real hyperbolic space and very soon after Kazhdan himself settled the case of complex hyperbolic space by constructing nonzero holomorphic one forms using theta functions "coming from the $\mathfrak{g}, K$ cohomology of the Weil or oscillator representation" (I will go into considerable detail about this shortly). In the spring of 1978, Steve Kudla came to me with the proposal that Kazhdan's method when applied to $SO(p, q)$ and $SU(p, q)$ instead of $SU(n, 1)$ could be used to construct the harmonic forms dual to the special cycles Raghunathan and I had earlier constructed. Furthermore, once this was done it would follow that the remarkable and mysterious result of Hirzebruch and Zagier for the Hilbert modular surfaces ($G = SO(2, 2)$) that the intersection numbers of special cycles were related to the Fourier coefficients of modular forms could be generalized to *all* $SO(p, q)$ and $SU(p, q)$. To explain what Steve had in mind I need to explain an analytic/representation-theoretic process for constructing differential forms on the associated locally symmetric spaces $\Gamma\backslash G/K$ which are Poincaré dual to special cycles.

The general process. The process is a two-stage operation, the first difficult to implement and the second nearly impossible *with the exception of one very special case.*

The first stage. The first stage is *local* and goes back to the invention of Lie algebra cohomology by Chevalley and Eilenberg in 1948, [1], (actually they invented only the *absolute* Lie algebra cohomology $H^{\bullet}(\mathfrak{g},\mathcal{U})$). Let G be a semisimple Lie group with maximal compact subgroup K and $\rho : G \to \mathcal{U}$ be a (possible infinite dimensional) representation of G. Let $\mathfrak{p}$ be the orthogonal complement for the Killing form to the Lie algebra $\mathfrak{k}$ of K in the Lie algebra $\mathfrak{g}$ of G. The first stage is to construct a G-invariant closed differential p-form φ on the symmetric space $D = G/K$ with values in $\mathcal{U}$. We will write the space of all such forms as $A^p(D,\mathcal{U})^G$, here the superscript G denotes the G-invariants The key point is that evaluation at the base-point eK of D induces an isomorphism

$$A^p(D,\mathcal{U})^G \to (\bigwedge^p(\mathfrak{p}^*) \otimes \mathcal{U})^K$$

and furthermore the exterior differential d carries over to the operator:

$$\overline{d} = \sum_{i=1}^{N}[1/2((\epsilon(\omega_i) \circ \mathcal{L}_{x_i}) \otimes 1) + (\epsilon(\omega_i) \otimes \rho(x_i))].$$

Here $\{x_i, 1 \leq i \leq N\}$ is a basis for $\mathfrak{p}$ and $\{\omega_i, 1 \leq i \leq N\}$ is the dual basis. Also $\epsilon(\omega_i)$ denotes left exterior multiplication by ω_i and $\mathcal{L}_{x_i}$ denotes Lie derivative by x_i. Thus to construct closed $\mathcal{U}$-valued forms on D one has to construct closed elements of the above "relative Lie algebra complex with coefficients in $\mathcal{U}$".

The second stage. The second stage is *global*. Suppose that we have implemented the first stage and found a p- cocycle $\varphi \in (\bigwedge^p(\mathfrak{p}^*)\otimes\mathcal{U})^K$. We let $\widetilde{\varphi}$ denote the corresponding closed form in $A^p(D,\mathcal{U})^G$. However we want more, we want a closed differential form on compact *locally symmetric* spaces $\Gamma\backslash G/K$ for Γ a suitable cocompact discrete subgroup of G. This is the nearly impossible "second stage", to descend $\widetilde{\varphi}$ to the quotient of D by Γ. To this end we now assume that by some miracle (and it almost never happens) that there is an element (distribution) $\alpha \in \mathcal{U}^*$ which is Γ -invariant. Then we may apply α to the values (in $\mathcal{U}$) of $\widetilde{\varphi}$ to form $\langle\alpha, \widetilde{\varphi}\rangle$ (or $\alpha \circ \widetilde{\varphi}$) to obtain a scalar-valued p-form. The key point - because α was Γ -invariant, the resulting scalar form $\langle\alpha, \widetilde{\varphi}\rangle$ *is* Γ*-invariant* and consequently *descends to* M. Diagramatically we have maps of complexes (I will leave out the differentials)

$$(\bigwedge^{\bullet}(\mathfrak{p}^*) \otimes \mathcal{U})^K \cong A^{\bullet}(D,\mathcal{U})^G \subset A^{\bullet}(D,\mathcal{U})^{\Gamma} \xrightarrow{1\otimes\alpha} A^{\bullet}(D,\mathbb{R})^{\Gamma} = A^{\bullet}(M)$$

Implementing the second stage for the oscillator representation. At this stage we will specialize to the *real* case so V is a real vector space of dimension $m = p+q$ equipped with a nondegenerate quadratic form $(\ ,\ )$ of signature (p,q). However the theory works just as well in the complex case.

Thus to implement the second stage we need a representation $\mathcal{U}$ and a lattice Γ in G such that $\mathcal{U}^*$ has a Γ-invariant element. As we have said, very few representations G admit such Γ-invariant linear functionals. The exception is the oscillator or Weil representation. I will not explain the general theory but state only that the oscillator representation induces a representation

$$\varpi : \widetilde{SL}(2,\mathbb{R}) \times O(V) \to Aut(\mathcal{S}(V)).$$

Here $\mathcal{U} = \mathcal{S}(V)$ is the space of Schwartz functions on V. Furthermore $\widetilde{SL}(2,\mathbb{R})$ is the nontrivial two-fold cover of $SL(2,\mathbb{R})$ (the metaplectic group) The group $O(V)$ acts on functions in the obvious way,

$$\varpi(g)(\varphi(v)) = \varphi(g^{-1}v)$$

but the action of $\widetilde{SL}(2,\mathbb{R})$ is more subtle. For example the matrix $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ acts by the Fourier transform with V and V^* identified using $(\ ,\ )$

$$\text{(Fourier transform)} \qquad \varpi(J)(\varphi)(v) = \int_V \varphi(u) \exp\left(-2\pi i(u,v)\right) du.$$

We will often use the symbol G' in place of $\widetilde{SL}(2,\mathbb{R})$.

Remark. The pair $\widetilde{SL}(2,\mathbb{R}) \times O(V)$ is a "reductive dual pair" in Roger Howe's theory. In fact it is a very special subgroup of the big "metaplectic group" $\widetilde{Sp}(\mathbb{R}^2 \otimes V)$ on which ϖ is really defined. Howe's theory greatly extended Weil's work (combining it with the main point of Schur-Weyl theory for decomposing tensor products of $GL(n,\mathbb{C})$ into irreducibles) and made it much more useful to working mathematicians. I should say that my friendship with Roger Howe, beginning my first year as a graduate student in Berkeley when he was my "senior adviser", helped me greatly throughout all my work - as we will see below.

Now, we have chosen a lattice $\mathcal{L}$ in V, accordingly we define the theta distribution Θ by

$$\Theta = \sum_{\ell \in \mathcal{L}} \delta_\ell.$$

Here δ_ℓ is the delta measure concentrated at ℓ. It is obvious that Θ is invariant under Γ (because Γ is contained in the subgroup of G stabilizes $\mathcal{L}$). What is very subtle and was one of André Weil's discoveries is that there is a subgroup of finite index Γ' (e.g. the congruence subgroup of level 4) of $\widetilde{SL}(2,\mathbb{Z})$ (the induced metaplectic cover of $SL(2,\mathbb{Z})$) such that Θ is also invariant under Γ'. This is critical because we want to get modular forms in the metaplectic variable (classically, modular forms of *half-integral weight*). Thus Θ is invariant under $\Gamma' \times \Gamma$ and consequently if we can solve the local problem and get a closed p-form $\widetilde{\varphi}$ on D (which I emphasize will not be easy) then we can apply the theta distribution to obtain a closed p-form θ_φ on M given by

$$\theta_\varphi = \langle \Theta, \widetilde{\varphi} \rangle$$

Implementing the first stage for the oscillator representation. Thus the existence of the remarkable theta-distribution reduces the problem of constructing θ_φ to the local problem of finding *cocycles* $\varphi \in (\bigwedge^p(\mathfrak{p}^* \otimes \mathcal{S}(V))^K$.

Siegel's indefinite theta function. We will start less ambitiously and construct a very special smooth function on M which will be a nonholomorphic modular form of weight $m/2$ in the $SL(2,\mathbb{R})$-variable. It will be the famous Siegel indefinite theta function. To start with we will need an element φ_0 of $\mathcal{S}(V)^K$ called the *Gaussian*. Choose an orthogonal (for $(\ ,\ )$) basis $e_1, \cdots, e_p, e_{p+1}, \cdots, e_{p+q}$ for V so that for $1 \leq \alpha \leq p$ we have $(e_\alpha, e_\alpha) = 1$ and for $p+1 \leq \mu \leq p+q$ we have $(e_\mu, e_\mu) = -1$. We define $(\ ,\ )_0$, the "standard majorant" of $(\ ,\ )$, to be the unique positive definite form such that the above basis is orthonormal (so we change the sign of $(\ ,\ )$ on the e_μ's). We then define the Gaussian $\varphi_0 \in \mathcal{S}(V)$ by

$$\varphi_0(v) = \exp(-\pi(v,v)_0).$$

The Gaussian is clearly invariant under $K = O(p) \times O(q)$ whence $\varphi_0 \in \mathcal{S}(V)^K$ and consequently gives rise to an element $\widetilde{\varphi_0} \in A^0(X, \mathcal{S}(V))^G = C^\infty(X, \mathcal{S}(V))$. It is less obvious but true that it transforms under K' by the character $det^{m/2} : K' \to S^1$ where $K' = \widetilde{SO}(2)$ is the maximal compact subgroup of $\widetilde{SL}(2,\mathbb{R})$. This is more or less the fact that the Gaussian is an eigenfunction of the Fourier transform. Now form

$$\theta_{\varphi_0}(g'K', gK) = \langle \Theta, \varpi(g', g)\varphi_0 \rangle, g' \in G', g \in G.$$

Put $M' = \Gamma' \backslash G'/K'$ so M' is a cover of the modular curve. Since Θ is invariant under $\Gamma' \times \Gamma$ the function $\theta_{\varphi_0}(g'K', gK)$ is a (nonholomorphic) modular form on M' and a smooth function on M. This is Siegel's famous indefinite theta function. Siegel showed that the integral of θ_{φ_0} over M was a (nonholomorphic) Eisenstein series whose n-th Fourier coefficients a_n was the "representation number of n by $(\ ,\ )$ *modulo* Γ" i.e. the cardinality of the quotient set of the integral vectors of length n by Γ

$$\mathcal{S}_n(\Gamma) = \{v \in \mathcal{L} : (v,v) = n\}/\Gamma.$$

It is critical here that the element $\varphi_0 \in \mathcal{S}(V)$ transforms nicely under K' as well as K. Thus we have $\varphi_0 \in (\mathbb{C}[m/2] \otimes \mathcal{S}(V))^{K' \times K}$ where $\mathbb{C}[m/2]$ denotes the one dimensional representation of K' given by $det^{m/2}$. The representation det gives rise to a locally homogeneous line bundle $\mathbb{L}$ over the $M' = \Gamma' \backslash X'$ and $\theta_{\varphi_0} \in C^\infty(M' \times M, \mathbb{L}^{m/2} \boxtimes \mathbb{R})$.

A "Chern convention" reapplied. The convention of using "early Greek letters α and β for the coordinates associated to the basis vectors of with positive self inner product and "late" Greek letters μ and ν for the the coordinates associated to the basis vectors of with negative self inner product and their associated coordinates, turned out to be very useful for the computations that we made later. I found it by adapting a convention I learned in Chern's class - he used the early Greek letters α and β as subscripts for the vectors in a frame field that were tangent to a submanifold of

Euclidean space and the later Greek letters μ and ν for the vectors that were normal. Although this may seem rather insignificant, in fact as we all know, having the right notation is often the key to being able to make a hard computation.

The Poincaré duals of the special cycles C_X and the Howe operator.

Real life. This is where matters stood in the summer of 1978 when Steve Kudla and I began work on the problem of constructing the Poincaré dual of C_X for $SO(p,q)$ and $SU(p,q)$. That summer was the best time of my life. I was working on the most exciting project of my life and in the beginning of July of that year I met my wife Gretchen at the Strawberry Canyon swimming pool, the single best day of my life. I cannot imagine how I would have got through the next thirty years without her help and companionship. Her understanding and feel for art and beauty was an awakening for me, for example I learned to love Mark Rothko's huge squares which I would have considered absurd without her guidance. We also have a large painting of a coffee cup by the California artist Joe Goode over our fireplace. This painting is now precious to me.

In addition, that year I had just received a Sloan Fellowship and a tenure job at the University of Toronto. The circumstances under which I got the Sloan fellowship once again bring out how lucky I have been and how much I owe to my teachers and friends. It was fortunate for me that Singer was on the committe to award the Sloan Fellowships because when Jim Simons told him I had been nominated he said "no, I didn't see Millson's name". Singer investigated and it turned out that a letter had arrived eliminating somebody else because he was ineligible (I forgot why, perhaps because he was already a full professor) and that letter had been mistakenly placed in my file. So I had been eliminated that year and probably for all future years. I should also say that during the time I was in Yale I was about to be deported (I was Canadian and needed a work permit) in spite of the efforts of a number of mathematicians, including Chern. But Singer knew who to write to (George Hammond, the foreign secretary of the National Academy) and I got my work permit in a few weeks.

Now back to mathematics. In the fall of 1978 I left for Oxford using my Sloan Fellowship. I worked ten hours a day (with breaks for meals and afternoon tea in the Mathematics Institute) for three months on the project and combining what I did there with what Steve did on his end in Maryland we were able to handle the case of $SO(n,1)$. We found the one-cocycle for $G = SO(n,1)$

$$\varphi = \sum_{\alpha=1}^{n} \omega_{\alpha,n+1} \otimes x_\alpha \varphi_0 \in (\mathfrak{p}^* \otimes \mathcal{S}(V))^{SO(n)}.$$

Here the coordinates are associated to the basis we introduced in subsection , here $q = 1$, and the $\omega_{\alpha,n+1}$ are the horizontal Maurer Cartan forms. One of the main points was that *intersection numbers of cycles in M with certain "composite" special cycles C_n were Fourier coefficients of modular forms.* We recall the set $\mathcal{S}_n(\Gamma)$ of Γ orbits of vectors v satifying $(v,v) = n$. Then we define the composite special cycle

C_n by

$$C_n = \sum_{x \in \mathcal{S}_n(\Gamma)} C_x.$$

Here we have abused notation slightly: by C_x we mean what we have called previously C_X where X is the line through x oriented by x.

Now we can state the main theorem of [20]. Given any one-cycle C in M the closed one form $\theta_\varphi(g'K', gK)$ could be integrated over C and the result was a *holomorphic* modular form of weight $(n+1)/2$ on M' which accordingly had a Fourier expansion. In order to write it in the usual way we will take K' to be the stabilizer of i in the upper half plane $\mathbb{H}$ whence $G'/K' = \mathbb{H}$ and $\tau = u + iv$ are the usual coordinate in $\mathbb{H}$. Then we proved [20]

Theorem.

$$\int_C \theta_\varphi(\tau) = \sum_{n=1}^{\infty} C_n \bullet C \ q^n \tag{0.1}$$

Here as usual $q = \exp 2\pi i \tau$.

This paper was an analogue of the famous paper of Hirzebruch and Zagier [6] which had proved an analogous result for special cycles in the Hilbert modular curves and was the main motivation behind our project.

One may formulate this theorem in a better way by thinking of the cohomology class $[\theta_\varphi(\tau)]$ as a holomorphic modular form with values in the first cohomology group $H^1(M)$ of the hyperbolic manifold M. We could then rewrite it as

Theorem.

$$[\theta_\varphi(\tau)] = \sum_{n=1}^{\infty} PD[C_n] \ q^n \tag{0.2}$$

The paper [20] was a blend of our skills. Steve taught me how to do analytic computations with theta functions e.g. how to take their Fourier coefficients. I provided the fact that to find the a Poincaré dual to C one had only to find a " Thom form ", that is, a *closed* form supported in tubular neighborhood of C with integral one on each normal fiber. I had learned about the Thom class from George Cooke, once again information George provided me with in my first years in Berkeley was critical in my later research.

However Steve and I were unable find the q-cocycle φ_q that we needed in $(\bigwedge^q(\mathfrak{p}^*) \otimes \mathcal{S}(V))^{(SO(p)\times SO(q))}$ for the case of $SO(p,q)$ with $q > 1$. The reason for our difficulty was explained to me by Jim Simons. For the case of $SO(p,1)$ the normal bundle of D_X in D was flat. This made the construction of the Thom form for C_X trivial (it was essentially the connection form). However this was no longer true for $q > 1$. The closed q-cocycle φ_q would necessarily involve a complicated combination of many terms involving curvature and connection forms as in Chern's famous construction of the "transgression" in his proof of the Gauss-Bonnet theorem, a pivotal moment in modern mathematics. Once again I would like to tell another story about Chern. Robert Greene told me that, when he asked Chern how he managed to find the intricate formulas involved, Chern said that he "could just see them". Given Chern's modesty this was just a statement of fact, evidence of Chern's remarkable algebraic insight. I was always reassured that Chern - like one of my other heroes Armand Borel, did not seem lightening fast although both could do extraordinarily difficult algebraic computations. And, like Armand Borel, he saw which computations were important.

At this moment when we had hit a road-block, Roger Howe intervened. He produced a vector valued differential operator $\mathcal{D}$ made up of sums of products of the form $(\frac{\partial}{\partial x_\alpha} - x_\alpha) \otimes \epsilon(\omega_{\alpha,\mu})$ such that

$$\varphi_q = \mathcal{D}(\varphi_0) \in (\mathcal{S}(V) \otimes \bigwedge^q(\mathfrak{p}^*))^{(SO(p)\times SO(q))}$$

was exactly the q-cocycle we needed.

In 1979 I took up my tenure job at the University of Toronto but I missed California from my days in Berkeley, so in 1980 I moved to UCLA. In 1981 I won the Coxeter James Prize from the Canadian Mathematical Society. "The prize is awarded to young mathematicians in recognition of outstanding contribution to mathematical research" (from Wikipedia).

In fact many more ideas were needed but in [21] we finished the job for *compact* quotients of the symmetric spaces associated to $SO(p,q)$ and $SU(p,q)$. However the basic cases [6] that motivated our entire project, the Hilbert modular surfaces are *noncompact* quotients of $SO(2,2)$ so the job wasn't done. For almost ten years I have been engaged in a joint project with Jens Funke to finish the job. Very recently we have managed to do it [5]. The key point was that the Kudla-Millson 2-form θ_{φ_2} restricted to the Borel-Serre boundary of the Borel-Serre compactification was exact with primitive another theta function θ_ω. This result was hard-won, necessitating going back and redoing the previous theory *with coefficients added* [3] then developing a general theory of restriction to the Borel-Serre boundary [4]. There is still much to be done in the finite-volume noncompact case.

0.1. **Applications of the theory.** There have already been two applications (that I am aware of) of this theory. I will be slightly imprecise in what follows. The first is a paper of Hoffman and He [7] depending on earlier work of Weissauer [29]. They prove that for the case $G = SO(3,2)$ and $\Gamma =$ the subgroup of integral points, then $H^2(M,\mathbb{R})$ is spanned by the two-forms θ_φ and the Kähler form. Hence $H_2(M,\mathbb{R})$ is spanned by the special cycles $\{C_X\}$. Based on a recent exchange of e-mails with Nicolas Bergeron I now believe it may be possible to prove the analogous result for $H_{m-1}(M,\mathbb{R})$ where M is one of the Borel examples of compact hyperbolic m-manifolds, that is, that the next-to-top homology is spanned by totally-geodesic hypersurfaces.

The second application comes from Ai-Ko Liu, [22], and seems quite remarkable. First, the results of [21] and my later work with Jens Funke produce a vast number of formal power series with positive integer coefficients that are modular forms. My hope is that some of these same power series will reappear as generating functions for numbers arising from counting problems in other fields. This is just what Ai-Ko Liu found for "virtual Gromov Witten invariants for the cosmic string".

The " cosmic string" is a family of $K3$ surfaces over $\mathbb{P}^1$. This family gives rise to a period map $f : \mathbb{P}^1 \to M$ where M is the locally symmetric space associated to $SO(Q)$ where Q is the quadratic form over $\mathbb{Z}$ which is the orthogonal sum of two hyperbolic planes and two copies of the negative of the quadratic space E_8 whence on the real group level $G = SO(2,18)$. The period map gives a (compact) two cycle C whence according to [21] we have

$$\int_C \theta_\varphi = \sum_{n=0}^{\infty} C_n \bullet C \; q^n.$$

It follows (because the left-hand side is such) that the right-hand side is a homolomorphic modular form of weight 10. But Liu proves that the right-hand side is the generating function for his "virtual numbers of nodal rational curves". Liu asserts these numbers are related to the usual virtual Gromov-Witten invariants. Thus, in a fashion I do not yet understand, Liu has proved that the generating function for these numbers is modular using [21]. It seems in this case one should use vectors x satisfying $(x,x) = -n$ in the definition of C_n. His proof uses six of his previous papers (one joint). It is one of my top priorities to try to understand what is happening here.

There is one more piece of related work I would like to quote. In fact it is vastly different (arithmetic) but it is in the spirit of the above work. It is very deep work done by Steve Kudla with his collaborators, beginning with Steve Rallis, then Michael Rapoport and Tonghai Yang. Now the special cycles must be replaced by their "arithmetic versions" and again the generating functions for intersection numbers is a modular form. Some of this work may be found in [16].

DEFORMATION THEORY

As I mentioned at the end of my discussion of special cycles with coefficients the hypersurfaces with coefficients in $\mathbb{R}^{n,1}$ for the Borel examples G in $SO(n,1)$ turned out to be tangent to the "Thurston bending deformations". By the Mostow (or even Weil/Calabi) rigidity theorems if $\Gamma \subset SO(n,1)$ is a cocompact lattice then Γ is rigid. But considered as a discrete subgroup (of infinite covolume) in $SO(n+1,1)$, using the inclusion $SO(n,1) \subset SO(n+1,1)$ as the stablilizer of the first standard basis vector e_1, the discrete group Γ is not necessarily rigid. The Zariski tangent space at the composed representation $\rho : \Gamma \to SO(n+1,1)$ to the space of representations $Hom(\Gamma, SO(n+1,1))$ is $Z^1(\Gamma, Ad\rho)$ (the space of one-cocycles). Here $Ad\rho$ denotes the flat bundle (local system) over $M = \Gamma\backslash SO(n,1)/SO(n)$ associated to the composition $Ad \circ \rho$ where Ad denotes the action of $SO(n+1,1)$ on its Lie algebra.

Thurston had constructed "bending deformations"associated to totally geodesic hypersurfaces $C_X \subset M = \Gamma\backslash SO(n,1)/SO(n)$. Dennis Johson and I showed, [9], that if we attached the coefficient $e_1 \wedge x \in \bigwedge^2(\mathbb{R}^{n+1,1}) = \mathfrak{so}(n+1,1)$ to the totally-geodesic hypersurface C_X then the Poincaré dual of $[C_X \otimes e_1 \wedge x]$ was the derivative of the Thurston bending deformation (recall X is the line through the vector x which satisfies $(x,x) > 0$). This was hardly a surprise but needed to be checked. What was a surprise was that if we chose two intersecting hypersurfaces C_{X_1} and C_{X_2} in M(so one could bend on either one) then the intersection class $[C_{X_1} \otimes e_1 \wedge x_1] \bullet [C_{X_2} \otimes e_1 \wedge x_2]$, a codimension two special cycle equipped with the coefficient $x_1 \wedge x_2$, represented the

Poincaré dual of the first obstruction to finding a curve of representations tangent to the sum of the two infinitesimal bendings. Furthermore we could prove this class was nonzero once $n > 3$ and consequently *the variety* $Hom(\Gamma, SO(n+1,1))$ *can be singular if* $n > 3$. This was surprising, the corresponding varieties $Hom(\Gamma, SO(n+1,1))$ is always smooth if $n = 2$ hence all infinitesimal deformations are integrable i.e. tangent to curves. It is unknown whether the corresponding varieties are smooth for $n = 3$.

For the complex hyperbolic case one can try the same thing. Namely take a co-compact lattice $\Gamma \subset SU(n,1)$ and try to deform it in $SU(n+1,1)$. Once again there are infinitesimal deformations which are essentially the Kazhdan classes (but no totally geodesic hypersurfaces). Bill Goldman and I proved that (essentially) *all* the infinitesimal deformations were obstructed again by the first obstruction. We then went on to prove that if Γ was the fundamental group of a compact Kähler manifolds and G was a compact Lie group and $\rho \in Hom(\Gamma, G)$ then if the *first* obstruction to integrating an infinitesimal deformation vanished then the infinitesimal deformation was integrable, i.e. tangent to a curve of representations - equivalently the tangent cone of the real analytic germ $(Hom(\Gamma, G), \rho)$ was quadratic cut out by the first obstruction. We conjectured that the analytic local ring of the germ was quadratically presented but we couldn't prove it. That was how matters stood in the spring of 1986 when one of the most remarkable events in my career ocurred. On April 24 I received a letter from Pierre Deligne explaining how to solve the problem and *much more*. In his letter Deligne explained a general principle for deformation theory that postulated that deformation problems in characteristic zero should be "controlled" by differential graded Lie algebras in much the same way that Sullivan had discovered that the de Rham algebra "controlled" the rational homotopy of a space. I can't say it better than Deligne so I will quote from his letter (which may be found on my web page)

The philosophy, which I had not realized before reading your paper, seems the following: in characteristic zero, a deformation problem is controlled by a differential graded Lie algebra ($\mathcal{L}^\bullet$) *with quasi-isomorphic DG Lie algebras giving the same deformation theory. If the DG Lie algebra controlling a problem is "formal",i.e. quasi-isomorphic to* $\bigoplus H^\bullet$*, then, the versal (formal) deformation space is that of* $\bigoplus H^\bullet$ *, i.e. is the (completion at* 0*) of the subscheme if* H^1 *defined by the equations* $[u,u] = 0$.

I was stunned at the magnitude of the idea and the extraordinary generosity of Deligne that he had handed this to me to work out. I went to Maryland for two years to work out the details of the letter (which was five pages) with Bill Goldman. The result was the forty-seven page paper [2]. Eventually that paper (and by extension Deligne's letter) led to my invitation to speak at the International Congress of Mathematicians in Kyoto in 1990. The text of my address is on my web page, [26].

I enjoyed the two year stay at Maryland to such an extent that when the chairman made me a very generous offer I moved to Maryland in 1989. It was hard to leave Los Angeles which I loved but it is now clear to me that I made the right move. For a start I met Misha Kapovich there in 1991.

Configurations of elementary geometric objects - my work with Misha Kapovich. Actually I met Misha Kapovich when he introduced himself to me in

Kyoto in 1990. I already knew about him because Misha Gromov had mentioned him to me as a top young Russian mathematician (at that time, Kapovich was in Novosibirsk). It was my good fortune that he came to Maryland in 1991. We have written seventeen papers together with the eighteenth on its way.

Linkages in the plane and space and line arrangements in the projective plane. We began work trying to decide the following question - suppose $\Gamma \subset SO(3,1)$ is a cocompact lattice. Is it true that the space of representations $Hom(\Gamma, SO(4,1))$ is smooth? I have mentioned above Dennis Johnson and I were unable to resolve this question - our work on nonintegrability of sums of infinitesimal bendings didn't apply for such a small Lorentz group. Misha and I never settled that question but our work naturally evolved into studying a "high-school geometry problem". What does the space of n-gons with fixed side-lengths in the plane (up to orientation-preserving congruence) look like? We settled the question as to when the space is connected - the space is disconnected if and only if there are "three long sides" (so the n-gon seen from a distance looks like a triangle) and otherwise the space is connected. In this case you get two components. The point is that two triangles with the same side-lengths are *not* (orientation-preserving) congruent contrary to what I was told in high-school so the moduli space consists of *two points.* The general connectivity result was not previously known in spite of a huge amount of previous work on these spaces. We also computed the topology of the spaces for small n. For example for a regular pentagon the moduli space is a surface of genus four, (this was already known) and did some other things.

Next we realized that if we asked the same question, only for n-gons in three-space, we got a much more interesting answer. We discovered that the moduli space was now a "Kähler space" and if the side-lengths were integers, it was a projective algebraic variety, the Geometric Invariant Theory quotient of n weighted points on the line under the group of projective automorphisms,[11]. It was stunning to me to find that another mathematician A. Klyachko, following a very different path, proved the same result.

We next gave a new twist to a famous result of Mnev that you "get anything as the moduli space of an arrangement of lines in the projective plane" to give examples, [12], of Artin groups that were not fundamental groups of smooth complex quasi-projective varieties (in particular smooth affine varieties). It is a problem of Serre to characterize the groups that are such fundamental groups. This paper was given a "featured review" by AMS Mathematical Reviews. We used the same techniques to prove, [13], that given any compact smooth manifold M there was a planar linkage L such that the configuration space of L was a disjoint union of a number of copies of M. This result was announced by Thurston in the 1990's and he gave several lectures on it but never wrote up a proof. I have often heard the result stated without the "disjoint union" condition. This is false. Every configuration space of a planar linkage admits a nontrivial involution, change of orientation, and there are manifolds M that admit no nontrivial involution. Thus such an M cannot be realized as a configuration space. The actual (more general) result that we proved was: given any compact real algebraic set V there is a planar linkage L such that the configuration space of L is a disjoint union of a number of copies of V. We then appealed to the work of Nash, completed by Tognoli, that every compact smooth manifold could be realized as a real algebraic set.

The generalized triangle inequalities. We then moved into new territory beginning with three papers with Bernhard Leeb. In a symmetric space $D = G/K$ of non-compact type of rank one the G-orbits of geodesic segments are determined by their length. This is no longer true if the rank is $\ell > 1$. In this case the G-orbits of line segments are parametrized by points in Δ, the positive Weyl chamber in $\mathfrak{a}$, the Lie algebra of a maximal $\mathbb{R}$-split Cartan subgroup. Indeed, suppose $\overline{xy}$ is such a geodesic segment. Then we can move x to the basepoint o of D (the point fixed by K). Now y is moved to a new point which we will represent as a coset gK. But by the "Cartan decomposition" we have

$$G = K(\exp \Delta)K$$

so we can write $KgK = KhK$ with $h = \exp \alpha \in \exp \Delta$. It is part of the Cartan decomposition that α is unique. We define the Δ-distance between x and y by

$$d_\Delta(x, y) := \alpha.$$

Now we have the problem of finding the triangle inequalities:

Problem. Give conditions on a triple of "side-lengths" $\alpha, \beta, \gamma \in \Delta$ that are necessary and sufficient in order that there exist a geodesic triangle with vertices $x, y, z \in D$ with

$$d_\Delta(x, y) = \alpha, d_\Delta(y, z) = \beta \text{ and } d_\Delta(z, y) = \gamma.$$

This problem (in another guise, the "eigenvalues of a sum" problem, was solved by Klyachko for the case of $G = SL(n, \mathbb{C})$ who showed that system of inequalities (already known to be necessary) based on the Schubert calculus of the flag manifolds G/P of G was sufficient. We solved the problem for the general real case for all G in [17] based on the mod 2 Schubert calculus. In the special case where G is complex we were able to use the usual (integral) Schubert calculus on the flag manifolds G/P. A different solution for the general real case (based on the solution for the complex case) had beeen given by O'Shea and Sjamaar. The inequalities were improved by Belkale and Kumar and finally the Belkale-Kumar improved inequalities were proved to be the irredundant subsystem of the original triangle inequalities by Ressayre. In [18] we found that the same system worked for Euclidean buildings - in this paper we proved that the system of triangle inequalities depended only the Weyl group as a group of transformations of $\mathfrak{a}$. A number of things we proved concerning the triangle inequalities we later found had either been proved earlier or simultaneously by someone else using a variety of different methods but this latter invariance result seems to have been done only by us.

Saturation. Now assume that G is the group of real points of a reductive algebraic group $\underline{G}$ defined over $\mathbb{Q}$, that we have chosen a maximal split torus $\underline{T}$ defined over $\mathbb{Q}$ and that α, β and γ are integral (coweights of $\underline{G}$ in the Lie algebra $\mathfrak{a}$ of $\underline{T}$). We will assume $\underline{G}$ has trivial center. Choose a p-adic field $\mathbb{Q}_p$, let G_p denote the group of p-adic points,$\underline{G}(\mathbb{Q}_p)$, and put $K_p = \underline{G}(\mathbb{Z}_p)$. Then the set of double cosets $K_p \backslash G_p / K_p$ can be given the structure of a commutative ring, the spherical Hecke ring, $\mathcal{H}(G_p, K_p)$, which has a basis $\{e_\alpha\}$ of double cosets indexed by the (dominant) coweights $\{\alpha\} \subset \Delta$. The same set indexes the basis $\{e'_\alpha\}$ (of irreducible characters) for the representation ring $Rep(G^\vee)$ of the Langlands' dual $G^\vee = \underline{G}^\vee(\mathbb{C})$. We will use $m_{\alpha,\beta,\gamma}$, resp. $n_{\alpha,\beta,\gamma}$ to denote the coefficient of the identity 1 in the triple product $e_\alpha \bullet e_\beta \bullet e_\gamma$ in $\mathcal{H}(G,K)$ resp. $e'_\alpha \bullet e'_\beta \bullet e'_\gamma$ in $Rep(G^\vee)$. The two rings are isomorphic by the Satake isomorphism but the isomorphism does not carry the first basis to the second. So the "same " ring has two different bases indexed by the same set. Thus comparing the two sets of structure constants relative to the two bases is a natural problem.

In [19] we addressed the connection between the triangle inequalities for G, the nonvanishing of the structure constant $m_{\alpha,\beta,\gamma}$ and the nonvanishing of the structure constants $n_{\alpha,\beta,\gamma}$. We found the following:

$$n_{\alpha,\beta,\gamma} \neq 0 \Rightarrow m_{\alpha,\beta,\gamma} \neq 0 \Rightarrow \alpha,\beta,\gamma \text{ is a solution of the triangle inequalities.}$$

The first implication was new based on a fundamental change of basis formula of Lusztig(see the discussion in [19]), the composed implication was well-known. The arrows cannot be reversed in general, however, it follows from general theory that there exists $N > 0$ such we have

$$n_{N\alpha,N\beta,N\gamma} \neq 0 \Leftarrow m_{N\alpha,N\beta,N\gamma} \neq 0 \Leftarrow \alpha,\beta,\gamma \text{ is a solution of the triangle inequalities.}$$

However the general theory does not give a formula (or even an upper bound) for N. It is a remarkable theorem of Knutson and Tao, the "Saturation Theorem" that for $G^\vee = GL(n,\mathbb{C})$ one can take $N = 1$. Here and in what follows we assume the condition that the sum $\alpha + \beta + \gamma$ is in the root lattice for $G^\vee$. This is necessary in order that the above structure constants are nonzero. In [14] we proved that if k_R was the LCM of the coefficients of the highest root θ written in terms of the simple roots then

$$n_{k_R^2\alpha,k_R^2\beta,k_R^2\gamma} \neq 0 \Leftarrow m_{k_R\alpha,k_R\beta,k_R\gamma} \neq 0 \Leftarrow \alpha,\beta,\gamma \text{ is a solution of the triangle inequalities.}$$

This is the only general result known at this point, however, it is clear that k_R is far from the best possible constant. For example in [15] we prove that $N = 1$ for $\underline{G} = PO(8)$ whence $\underline{G}^\vee = Spin(8)$. We conjecture that $N = 1$ whenever $\underline{G}$ is simply-laced (all roots have the same length) and $N = 2$ otherwise.

HMSV and the projective moduli of ordered points on the line. My work with Kapovich led me into a classical problem in the projective invariant theory of points on the projective line and an extensive collaboration with my student Ben Howard, Andrew Snowden (an undergraduate at Maryland and a PhD from Princeton) and Ravi Vakil. I can do no better than to quote the abstract of our paper [8].

The ring of projective invariants of n ordered points on the projective line is one of the most basic and earliest studied examples in Geometric Invariant Theory. It is a remarkable fact and the point of this paper that unlike its close relative the ring of invariants of n unordered points this ring can be completely and simply described. In 1894 Kempe found generators for this ring, thereby proving the First Fundamental Theorem for it (in the terminology introduced by Weyl). In this paper we compute the relations among Kempe's invariants, thereby proving the Second Fundamental

Theorem (again in the terminology of Weyl), and completing the description of the ring 115 years later.

Conclusion. In my career I have written (to date) 65 papers of which 57 are joint. I have enjoyed collaborations immensely especially those with Steve Kudla (seven papers), Bill Goldman (five papers) and Misha Kapovich (seventeen papers) and now with Jens Funke (five papers). I have had 36 consecutive years of grant support (counting my four years at Yale when I was a part of Dan Mostow's grant and my NSERC in Canada) and I won a Sloan Fellowship in 1978, the Coxeter-James prize in 1981, spoke at the ICM in Kyoto in 1990 and was honored to speak at the sixtieth birthday conferences for three great mathematicians, Armand Borel in 1983, Dan Mostow in 1984 and M.S. Raghunathan in 2001. Best of all, my friends organized a sixty-second birthday conference for me in March, 2008, at a time when I needed support. Also I would like to thank Lizhen Ji and S.T. Yau for inviting me to write this article for the Chern one hundredth anniversary volume.

The last three years have been very difficult for me. My wife Gretchen became ill from a degenerative brain disorder and died on June 24, 2009 and I developed some physical problems. Without the help of my friends I would never have got through those years and this last year. In particular my continuing collaborations in mathematics were a great source of comfort.

What mathematics has meant to me was best expressed by Armand Borel who in turn was quoting a baseball star who managed to say it better and more simply than any of us could. At the dinner concluding his sixtieth birthday conference at the Institute for Advanced Study in 1983 Borel gave a concluding speech. He finished by saying that what mathematics meant to him had already been said by Willie Mays, the great center-fielder for the New York/San Francisco Giants, in another context. Mays, on being asked how he felt about being a professional baseball player, said "*getting paid for playing baseball was like getting paid for eating ice-cream*".

References

[1] C. Chevalley and S. Eilenberg, *Cohomology theory of Lie groups and Lie algebras*, Transactions of the AMS **63**(1948), 85-124.

[2] W. M. Goldman and J. Millson, *The deformation theory of representations of fundamental groups of compact Kahler manifolds*, Publ. Math. IHES **67** (1988), pp. 49-96.

[3] J. Funke and J. Millson, *Cycles with local coefficients for orthogonal groups and vector-valued Siegel modular forms*, American J. Math. **128** (2006), 899-948.

[4] J. Funke and J. Millson, *Boundary behavior of special cohomology classes arising from the Weil representation*, preprint.

[5] J. Funke and J. Millson, *On a theorem of Hirzebruch and Zagier*, in preparation.

[6] F. Hirzebruch and D. Zagier, *Intersection numbers of curves on Hilbert modular surfaces and modular forms of Nebentypus*, Inv. Math. **36** (1976), 57-113.

[7] J. W. Hoffman and H. He, *Picard groups of Siegel modular threefolds and the theta lifting*, preprint.

[8] B. Howard, J. Millson, A. Snowden and R. Vakil, *The ideal of relations for the ring of invariants of n points on the line*, submitted to the Journal of the European Mathematical Society.

[9] D. Johnson and J. Millson, *Deformation spaces associated to compact hyperbolic manifolds*, Discrete Groups in Geometry and Analysis, Papers in Honor of G.D. Mostow on His 60th Birthday, Progress in Mathematics **67** (1987), Birkhauser, pp. 48-106 .

[10] M. Kapovich and J. Millson, *On the moduli space of polygons in the Euclidean plane*, J. Diff. Geom. **42** (1995), pp. 430-464.

[11] M. Kapovich and J. Millson, *The symplectic geometry of polygons in Euclidean space*, J. Diff. Geom. **44** (1996), pp. 479-513

[12] M. Kapovich and J. Millson, *On representation varieties of Artin groups, projective arrangements and the fundamental groups of smooth complex algebraic varieties*, Publ. Math. IHES **88** (1998), pp. 5-95.

[13] M. Kapovich and J. Millson, *Universality theorems for configuration spaces of planar linkages*, Topology **41** (2002), pp. 1051-1107.

[14] M. Kapovich and J. Millson, *A path model for geodesics in Euclidean buildings and applications to representation theory*, Groups, Geometry and Dynamics **2** (2008),pp. 405–480.

[15] M. Kapovich, S. Kumar and J. Millson, *The eigencone and saturation for Spin(8)*, Pure and Applied Mathematics Quarterly **5** (2009)(papers dedicated to F. Hirzebruch), pp. 755–780.

[16] S. Kudla, M. Rapoport and T. Yang, *Modular Forms and Special Cycles on Shimura Curves*, Annals of Mathematics Studies **161** (2006), Princeton University Press.

[17] B. Leeb, M. Kapovich and J. Millson, *Convex functions on symmetric spaces, side lengths of polygons and the stability inequalities for weighted configurations at infinity*, Journal of Diff. Geom. **81** (2009), pp. 297–354.

[18] B. Leeb, M. Kapovich and J. Millson, *Polygons in buildings and their refined side lengths*, Geometry And Functional Analysis, **19** (2009), pp. 1081-1100.

[19] B. Leeb, M. Kapovich and J. Millson, *The generalized triangle inequalities in symmetric spaces and buildings with applications to algebra*, Memoirs of the AMS **192**, **896** (2008).

[20] S. Kudla and J. Millson, *Geodesic cycles and the Weil representation I; quotients of hyperbolic space and Siegel modular forms*, Compositio Math. **45** (1982), pp. 207-271.

[21] S. Kudla and J. Millson, *Intersection numbers of cycles on locally symmetric spaces and Fourier coefficients of holomorphic modular forms in several complex variables*, IHES Pub. **71** (1990), 121-172.

[22] Ai-Ko Liu, *Cosmic String, Harvey-Moore Conjecture and Family Seiberg Witten Theorem* , preprint (2004).

[23] J. Millson, *On the first Betti number of a constant negatively curved manifold*, Ann. of Math **104** (1976), pp. 235-247.

[24] J. Millson, *Closed geodesics and the η-invariant*, Ann. of Math.**108** (1978), pp. 1-39.

[25] J. Millson, *A remark on Raghunathan's vanishing theorem*, Topology **24** (1985), pp. 495-498.

[26] J. Millson, *Rational homotopy theory and deformation problems from algebraic geometry*, Proceedings of I.C.M. 90, (1990) Vol. I, pp. 549-558.

[27] J. Millson, *The cohomology with local coefficients of a compact hyperbolic manifold*, Procedings of the International Conference in Algebraic Groups and Arithmetic, Mumbai 2001 (Papers dedicated to M.S. Raghunathan) Published for the Tata Inst. Fund. Res. by the Narosa Publishing House and distributed by the A.M.S., pp. 41-62. This paper appeared in 2004.

[28] J. Millson and M. S. Raghunathan, *Geometric construction of cohomology for arithmetic groups I*, Patodi memorial issue Geometry and Analysis, Indian Acad. of Sciences (1979), pp. 103-123.

[29] R. Weissauer, *The Picard group of Siegel modular three-folds*, J. Reine Angew. Math., **430** (1992), pp. 179-211.