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On the first Betti number of a constant negatively curved manifold

By JOHN J. MILLSON

Introduction

In this paper we construct for each n ($n \geqq 3$) families of arithmetic constant negatively curved n-dimensional manifolds with arbitrarily large first Betti number. In fact we show that any constant negatively curved manifold whose fundamental group is an arithmetic group commensurable with the group of units of a quadratic form admits a finite covering with first Betti number not equal to zero; in particular, the examples given by Borel at the end of [4] all admit such coverings.

Previous to this a few examples were known in low dimensions (cf. Vinberg [14]) with examples up to dimension 5 in the compact case. However, his construction uses hyperbolic Coxeter groups which exist only in low dimensions. Rather than attack the problem algebraically by computing the abelianized fundamental group by group theory—an approach that appears hopeless except for the above low dimensional examples, we take a geometric approach and construct explicit nonbounding codimension 1 cycles and then appeal to Poincaré duality.

Although these examples are of obvious geometric interest their main significance is group theoretic. The vanishing theorem of Kajdan [6] has as a consequence that the first Betti number of all compact locally irreducible, locally symmetric spaces of rank greater than 2 vanishes. This was extended by S. P. Wang [15] and Kostant [8] who show that Kajdan's criterion applies to all compact locally irreducible locally symmetric spaces except those associated with SO(n, 1) and SU(n, 1). The results of this paper show that the vanishing theorem does not hold for SO(n, 1). Whether or not it holds for SU(n, 1) remains unsolved.

Our results are of interest in connection with the congruence subgroup problem. Bass, Milnor, Serre [1] show that if an arithmetic group Γ satisfies the congruence subgroup property—that every subgroup of finite index contains a congruence subgroup, then:

$$H^1(\Gamma; \rho) = 0$$

for any finite-dimensional representation ρ of Γ. Since $H^1(\Gamma; 1) \neq 0$ for the

Γ constructed here (1 is the trivial representation) we find that the discrete subgroups $\tilde{\Gamma} \subseteq \mathrm{Spin}(n, 1)$ obtained as the inverse image of $\Gamma \subseteq \mathrm{SO}(n, 1)$ under the covering projection from $\mathrm{Spin}(n, 1)$ to $\mathrm{SO}(n, 1)$ do not satisfy the congruence subgroup property, nor does any group which contains them as a subgroup of finite index. The results of this paper show then that given a quadratic form f over $\mathbf{Q}$ of signature $(n, 1)$ the congruence subgroup property never holds for the integral matrices in $\mathrm{Spin}\, f$. (To treat these examples we are forced to deal with nonuniform Γ; however, this involves only minor technical complications as was pointed out to us by Raghunathan.) Raghunathan [11] has proved that the congruence subgroup property holds for the integral points of all simply-connected, connected, semi-simple $\mathbf{Q}$ groups of $\mathbf{Q}$ rank greater than or equal to 2 (we are in the $\mathbf{Q}$ rank 1 case here).

Lastly in Hotta-Wallach [5] it is shown that the first Betti number of a compact constant negatively curved n-dimensional manifold $n \geqq 4$ is equal to the multiplicity with which a certain non-tempered representation enters into $L^2(\Gamma \backslash \mathrm{SO}(n, 1))$ where Γ is the fundamental group of the manifold. Our results show that this representation does indeed occur for suitable Γ. In fact, in an adelic formulation, we may say that the Hotta-Wallach representation always occurs as the component at the infinite place of an irreducible constituent of the space $L^2(G_{\mathbf{Q}} \backslash G_{\mathbf{A}})$, when $G_{\mathbf{Q}}$ is the restriction of scalars from K to $\mathbf{Q}$ of the automorphism group of a quadratic form defined over a totally real field K, of signature $(n, 1)$ at one real completion and positive definite at all others. Here $\mathbf{A}$ is the adeles of $\mathbf{Q}$ and $G_{\mathbf{A}}$ is the adelization of $G_{\mathbf{Q}}$.

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Results similar to those of this paper have also been obtained by W. Thurston.

1. A geometric idea

In this section we introduce the geometric idea which we use to give examples of constant negatively curved n-manifolds with nonzero first Betti number. The rest of the paper is devoted to realizing the conditions of the Main Lemma.

MAIN LEMMA. *Let M be a connected, orientable n-dimensional Riemannian manifold. Let $\tilde{M} \xrightarrow{\pi} M$ be a finite (Riemannian) covering with a group of covering transformations $\pi_1(\tilde{M}: M)$. Suppose that*

1) *there exists a nontrivial isometric involution σ of $\tilde{M}$;*

2) *there exists a closed, orientable hypersurface F of $\tilde{M}$ contained in the fixed point set of σ;*

3) *there exists $\gamma \in \pi_1(\tilde{M}: M)$ such that*

(a) $\gamma F \cap F = \varnothing$,

(b) $\sigma\gamma F \subset \gamma F$;

then the first Betti number of $\tilde{M}$ is nonzero.

Proof. If X is an oriented hypersurface of a manifold Y then X carries a homology class $[X]$ which will be in $H_{n-1}(Y, \mathbf{R})$, the $n-1$ homology group of Y with arbitrary support. If Y is oriented then there is a simple criterion for deciding whether or not $[X]$ is trivial.

SUBLEMMA 1. $[X]$ *is trivial in* $H_{n-1}(Y, \mathbf{R})$ *if and only if* X *separates* Y *into two parts.*

Proof. If X separates Y into two parts then X bounds either part; hence $[X]$ is trivial. To prove the converse we assume that X does not separate. Let $N(X)$ be a tubular neighborhood of X in Y. Then X separates $N(X)$ into two parts. Choose x in one part and y in the other. Since X does not separate Y into two parts we may join x to y by an arc which does not meet X. Following this by a path from y to x which passes through X at one point we obtain a compact one-cycle which has intersection number 1 with X hence $[X]$ is nontrivial as a real class.

Now we are ready to complete the proof of the Main Lemma. It is sufficient to show F does not separate M into two parts (Sublemma 1 tells us then that $[F]$, the real class carried by F, is nontrivial; hence, $b_1(\tilde{M}) \neq 0$ by Poincaré duality).

For the purpose of obtaining a contradiction suppose that F separates $\tilde{M}$ into two parts. Then since σ interchanges the two pieces of $N(F) - F$ (here $N(F)$ is a tubular neighborhood of F), we conclude that σ interchanges the two parts of $\tilde{M} - F$. As a consequence of this there can be no connected submanifold of $\tilde{M} - F$ which is mapped to itself by σ. We now use hypothesis 3) to construct such a submanifold and we will then be able to conclude that F does not separate.

Set $F' = \gamma F$. Then F' is connected because F is. Moreover, since by hypothesis 3(a), $F' \cap F = \varnothing$ we conclude that F' is a connected submanifold of $\tilde{M} - F$. But by hypothesis 3(b), $\sigma(F') \subseteq F'$.

This completes the proof of the Main Lemma.

2. The main construction

In this section we construct compact constant negatively curved n-manifolds admitting an involution τ such that the fixed point set of the involution τ contains a compact constant negatively curved $n-1$-dimensional manifold.

First let us establish some notation. We denote by $O(n, 1)$ the group of matrices which leave invariant the quadratic form f_0 over $\mathbf{R}$ in $n+1$ variables (we assume $n+1 \geqq 4$) given by

$$f_0(X_1, X_2, \cdots, X_{n-1}) = X_1^2 + X_2^2 + \cdots + X_n^2 - X_{n+1}^2 .$$

We denote by $O_0(n, 1)$ the subgroup of $O(n, 1)$ which preserves the half-space $x_{n+1} > 0$. The group $O_0(n, 1)$ has two components corresponding to matrices of determinant $+1$ and -1. The connected subgroup of $O_0(n, 1)$ consisting of the matrices of determinant $+1$ we denote by $\mathrm{SO}_0(n, 1)$. The group $O(n, 1)$ acts isometrically on the Minkowski space; that is, $\mathbf{R}^{n+1}$ with the metric induced by the above quadratic form. $O_0(n, 1)$ stabilizes (and acts transitively on) the set

$$\{(x_1, x_2, \cdots, x_{n+1}) \colon x_1^2 + x_2^2 + \cdots + x_n^2 - x_{n+1}^2 = -1,\ x_{n+1} > 0\} .$$

This set is just the upper sheet of a hyperboloid of revolution. The Minkowski metric induces a Riemannian metric of constant curvature -1 on this hyperboloid. This will be our realization of hyperbolic n-space, denoted X_n henceforth. Since $O_0(n, 1)$ acts transitively on X_n and the isotropy group of $e_{n+1} = (0, 0, \cdots, 0, 1)$ is $O(n)$ we may identify $O_0(n, 1)/O(n)$ with X_n by taking the orbit of e_{n+1}. Similarly we may identify X_n and $\mathrm{SO}_0(n, 1)/\mathrm{SO}(n)$. We denote by X_{n-1} the copy of hyperbolic $n-1$-space corresponding to the intersection of the hyperplane orthogonal to $e_1 = (1, 0, \cdots, 0)$ and X_n.

Now we construct a large family of discrete uniform subgroups of $O_0(n, 1)$. Let K be a totally real number field of degree m over the rational numbers, $\mathcal{O}$ the ring of integers of K and $\sigma_1, \sigma_2, \cdots, \sigma_m$ the imbeddings of K into $\mathbf{R}$. Let

$$f(X_1, X_2, \cdots, X_{n+1}) = a_1 X_1^2 + \cdots + a_n X_n^2 - a_{n+1} X_{n+1}^2$$

be a diagonal quadratic form defined over $\mathcal{O}$ such that f^{σ_1} has signature $(n, 1)$ and such that all its conjugates f^{σ_i}, $i = 2, 3, \cdots, m$ are positive definite. Let Φ denote the subgroup of $\mathrm{GL}(n+1, \mathcal{O})$ of matrices which preserve f. Φ is called the group of units of f. Now there exists an invertible real matrix g_0 so that $g_0 f g_0^t = f_0$. Hence $g_0^{-1}\Phi g_0$ is a subgroup of $O(n, 1)$. It is in fact a uniform, discrete subgroup. That $g_0^{-1}\Phi g_0$ is uniform follows from the Borel, Harish-Chandra, Mostow, Tamagawa theorem as Φ contains no unipotent elements; see Mostow-Tamagawa [9]. (However the uniformity of unit groups of forms not representing zero rationally was in fact known much earlier—for example, see Siegel [9].) The discreteness of Φ follows from the fact that all conjugates f^{σ_i}, $2 \leqq i \leqq n$ are positive definite, hence, have compact automorphism groups over $\mathbf{R}$. Intersecting $g_0^{-1}\Phi g_0$ with $O_0(n, 1)$, we receive a discrete uniform subgroup which we denote by Γ_n. Now Γ_n contains the involution ι of Minkowski space which is the reflection through the plane $x_1 = 0$. The involution ι acts on X_n and, under the identification of X_n with $O_0(n, 1)/O(n)$ described previously, ι corresponds with the action induced on the homogeneous space by the automorphism of $O_0(n, 1)$ given by conjugation by the diagonal matrix with first diagonal entry -1 and the rest $+1$. For concreteness we give examples of the required quadratic forms. Let $K = \mathbf{Q}(\sqrt{p})$. Define

$$f(X_1, X_2, \cdots, X_{n+1}) = X_1^2 + X_2^2 + \cdots + X_n^2 - \sqrt{p}\, X_n^2 .$$

One then obtains the examples given by Borel [4].

Now Γ_n has many normal subgroups, the congruence subgroups, obtained as follows. Let $\mathfrak{p}$ be a prime ideal in K. Then define

$$\Gamma_n(\mathfrak{p}) = \{\gamma \in \Gamma_n : \gamma = 1 \bmod \mathfrak{p}\} .$$

This notation means of course that the entries in the matrices of $\Gamma_n(\mathfrak{p})$, which are algebraic integers in K, are congruent to $1 \bmod \mathfrak{p}$ if they are on the diagonal and to $0 \bmod \mathfrak{p}$ if they are off the diagonal. Note that ι normalizes $\Gamma_n(\mathfrak{p})$ since $\Gamma_n(\mathfrak{p})$ is normal in Γ_n.

Γ_n may contain elements of finite order; however, for all but finitely many $\mathfrak{p}$; $\Gamma_n(\mathfrak{p})$ contains no elements of finite order and hence acts freely on X_n. For more details see Borel [4]. We obtain then a compact constant negatively curved manifold

$$Y_n(\mathfrak{p}) = \Gamma_n(\mathfrak{p})\backslash O_0(n, 1)/O(n) .$$

Actually, except for those $\mathfrak{p}$ with norm 2, $\Gamma_n(\mathfrak{p}) \subseteqq \mathrm{SO}_0(n, 1)$ hence we may write:

$$Y_n(\mathfrak{p}) = \Gamma_n(\mathfrak{p})\backslash \mathrm{SO}_0(n, 1)/\mathrm{SO}(n) .$$

Note that ι induces an involution of $Y_n(\mathfrak{p})$, which we denote by τ.

We now wish to investigate the fixed-point set of τ. Let H denote the subgroup of $O_0(n, 1)$ which leaves e_1 fixed and define:

$$\Gamma_{n-1} = \Gamma_n \cap H ,$$
$$\Gamma_{n-1}(\mathfrak{p}) = \Gamma_n(\mathfrak{p}) \cap H .$$

H is the centralizer of the involution ι in $O_0(n, 1)$, Γ_{n-1} is the centralizer of ι in Γ_n and $\Gamma_{n-1}(\mathfrak{p})$ the set of points in $\Gamma_n(\mathfrak{p})$ fixed under conjugation by ι. By a well-known result (see Raghunathan [12], pages 23 and 24), Γ_{n-1} and $\Gamma_{n-1}(\mathfrak{p})$ are uniform discrete subgroups of H. We now look for $\mathfrak{p}$ so that if π denotes the projection $X_n \to Y_n(\mathfrak{p})$:

(1) $\Gamma_n(\mathfrak{p})$ is torsion free; hence, $Y_n(\mathfrak{p})$ is a manifold;

(2) $\pi(X_{n-1})$ is a codimension 1 orientable submanifold and $\pi(X_{n-1}) = \Gamma_{n-1}(\mathfrak{p})\backslash X_{n-1}$.

The first condition is easy to arrange as we have stated before and is true for almost every prime $\mathfrak{p}$. The second condition requires a little more work. $\pi(X_{n-1})$ might have "self-intersections". These will correspond to elements $\gamma \in \Gamma_n(\mathfrak{p})$, $\gamma \notin \Gamma_{n-1}(\mathfrak{p})$ so that $\gamma x \in X_{n-1}$ for some $x \in X_{n-1}$. To handle these elements we prove a lemma.

LEMMA 2.1 (H. Jaffe). *Suppose Γ_n acts freely on X_n and is normalized by ι. Suppose there exists $\gamma \in \Gamma_n$ such that $\gamma X_{n-1} \cap X_{n-1} \neq \varnothing$. Then $\gamma \in H \cap \Gamma_n$.*

Proof. Let $\nu = \gamma^{-1}\iota\gamma\iota$. Then $\nu \in \Gamma_n$ and $\nu x = x$, hence $\nu = 1$ and γ and ι commute, so $\gamma \in H \cap \Gamma_n$.

Thus, somewhat surprisingly, condition (1) implies condition (2).

In order to ensure that $\pi(X_{n-1})$ is orientable it is enough to ensure that $\Gamma_{n-1}(\mathfrak{p})$ is contained within the connected component of the identity, $\mathrm{SO}_0(n-1, 1) \subseteqq H$. But this is immediate if the norm of $\mathfrak{p}$ is not 2.

Finally then we obtain for every prime $\mathfrak{p}$ not of norm 2 and such that $\Gamma_n(\mathfrak{p})$ is torsion free, a constant negatively curved manifold $Y_n(\mathfrak{p}) = \Gamma_n(\mathfrak{p})\backslash\mathrm{SO}_0(n, 1)/\mathrm{SO}(n)$ admitting an involution τ whose fixed-point set contains the constant negatively curved orientable codimension-1 submanifold

$$Y_{n-1}(\mathfrak{p}) = \Gamma_{n-1}(\mathfrak{p})\backslash\mathrm{SO}_0(n-1, 1)/\mathrm{SO}(n-1) .$$

From now on whenever we write $Y_n(\mathfrak{p})$ it will be assumed that $\mathfrak{p}$ is chosen to fulfil these conditions.

We now show by using the Main Lemma that there are coverings of $Y_n(\mathfrak{p})$ with nonzero first Betti number obtained by passing to deeper congruence subgroups.

For this purpose we define the further congruence subgroups $\Gamma_n(\mathfrak{p}, \mathfrak{p}')$ and $\Gamma_{n-1}(\mathfrak{p}, \mathfrak{p}')$ for $\mathfrak{p}$ and $\mathfrak{p}'$ primes in K and norm $\mathfrak{p}' \neq 2$ by:

$$\Gamma_n(\mathfrak{p}, \mathfrak{p}') = \{\gamma \in \Gamma_n : \gamma \equiv 1 \bmod \mathfrak{p} \ \text{ and } \ \gamma \equiv 1 \bmod \mathfrak{p}'\} ,$$
$$\Gamma_{n-1}(\mathfrak{p}, \mathfrak{p}') = \{\gamma \in \Gamma_{n-1} : \gamma \equiv 1 \bmod \mathfrak{p} \ \text{ and } \ \gamma \equiv 1 \bmod \mathfrak{p}'\} .$$

As before we receive manifolds

$$Y_n(\mathfrak{p}, \mathfrak{p}') = \Gamma_n(\mathfrak{p}, \mathfrak{p}')\backslash SO_0(n, 1)/SO(n) ,$$
$$Y_{n-1}(\mathfrak{p}, \mathfrak{p}') = \Gamma_{n-1}(\mathfrak{p}, \mathfrak{p}')\backslash SO_0(n-1, 1)/SO(n-1) .$$

Moreover ι induces an involution of $Y_n(\mathfrak{p}, \mathfrak{p}')$ which we call τ'. We also receive two regular coverings π_n and π_{n-1}:

$$\frac{\Gamma_n(\mathfrak{p})}{\Gamma_n(\mathfrak{p}, \mathfrak{p}')} \longrightarrow Y_n(\mathfrak{p}, \mathfrak{p}') \xrightarrow{\pi_n} Y_n(\mathfrak{p}) , \tag{1}$$

$$\frac{\Gamma_{n-1}(\mathfrak{p})}{\Gamma_{n-1}(\mathfrak{p}, \mathfrak{p}')} \longrightarrow Y_{n-1}(\mathfrak{p}, \mathfrak{p}') \xrightarrow{\pi_{n-1}} Y_{n-1}(\mathfrak{p}) . \tag{2}$$

We put $\Psi_n = \Gamma_n(\mathfrak{p})/\Gamma_n(\mathfrak{p}, \mathfrak{p}')$ and $\Psi_{n-1} = \Gamma_{n-1}(\mathfrak{p})/\Gamma_{n-1}(\mathfrak{p}, \mathfrak{p}')$.

Now we are in the situation of the Main Lemma. We have a Riemannian covering $Y_n(\mathfrak{p}, \mathfrak{p}') \xrightarrow{\pi_n} Y_n(\mathfrak{p})$, an involution τ' of $Y_n(\mathfrak{p}, \mathfrak{p}')$ and a closed orientable hypersurface $Y_{n-1}(\mathfrak{p}, \mathfrak{p}')$ contained in the fixed point set of τ'. We now must show that hypothesis 3) of the Main Lemma is satisfied.

Ψ_n is the group of deck transformations of the covering (1). We must find $\eta \in \Psi_n$ so that

(a) $\eta Y_{n-1}(\mathfrak{p}, \mathfrak{p}') \cap Y_{n-1}(\mathfrak{p}, \mathfrak{p}') = \varnothing$,

(b) $\tau'\eta Y_{n-1}(\mathfrak{p}, \mathfrak{p}') \subseteq \eta Y_{n-1}(\mathfrak{p}, \mathfrak{p}')$.

LEMMA 2.2. *Suppose $\eta \in \Psi_n - \Psi_{n-1}$, then η satisfies* (a).

Proof. For the purpose of contradiction assume there exist $x, y \in Y_{n-1}(\mathfrak{p}, \mathfrak{p}')$ so that $\eta x = y$. Since $Y_{n-1}(\mathfrak{p}, \mathfrak{p}') = \Gamma_{n-1}(\mathfrak{p}, \mathfrak{p}')\backslash X_{n-1}$ there exist $\gamma_1, \gamma_2 \in \Gamma_{n-1}(\mathfrak{p}, \mathfrak{p}')$, $\tilde{x}, \tilde{y} \in X_{n-1}$ so that

$$\gamma\gamma_1\tilde{x} = \gamma_2\tilde{y} , \quad \gamma_2^{-1}\gamma\gamma_1\tilde{x} = \tilde{y}$$

where γ is a representative for η in $\Gamma_n(\mathfrak{p})$. By Lemma 2.1 it follows that $\gamma_2^{-1}\gamma\gamma_1 \in \Gamma_{n-1}(\mathfrak{p})$, hence $\gamma \in \Gamma_{n-1}(\mathfrak{p})$ and $\eta \in \Psi_{n-1}$.

We see then that if we choose $\eta \in \Psi_n - \Psi_{n-1}$ then (a) will be satisfied. (b) will require more work. We first note that the operation of conjugation by ι normalizes $\Gamma_n(\mathfrak{p}, \mathfrak{p}')$ and centralizes $\Gamma_{n-1}(\mathfrak{p})$; hence it induces an involution α on Ψ_n centralizing Ψ_{n-1} and an involution β of the coset space Ψ_n/Ψ_{n-1}. For each $\eta \in \Psi_n$ let $\bar{\eta}$ denote the class of η modulo Ψ_{n-1}.

The key to obtaining (b) is the following lemma.

LEMMA 2.3. *Suppose $\eta \in \Psi_n$ has the property that*

$$\beta(\bar{\eta}) = \bar{\eta} .$$

Then η satisfies (b).

Proof. Suppose $\eta\Psi_{n-1}$ is a coset which is fixed under β. Then $\alpha(\eta) = \eta\eta_1$, where $\eta_1 \in \Psi_{n-1}$. Now let $y \in \eta Y_{n-1}(\mathfrak{p}, \mathfrak{p}')$ so $y = \eta x$, $x \in Y_{n-1}(\mathfrak{p}, \mathfrak{p}')$. Then $\tau' y = \tau' \eta x = \alpha(\eta) x = \eta\eta_1 x$. But η_1 preserves $Y_{n-1}(\mathfrak{p}, \mathfrak{p}')$ hence $\tau' y \in \eta Y_{n-1}(\mathfrak{p}, \mathfrak{p}')$.

Remark 2.1. We have shown that $\tau' \mid \eta Y_{n-1}(\mathfrak{p}, \mathfrak{p}') = \eta\eta_1\eta^{-1}$, where $\eta_1 \in \Psi_{n-1}$. We will need this result in Chapter 4.

Thus to apply the Main Lemma to conclude that the first Betti number of $Y_n(\mathfrak{p}, \mathfrak{p}')$ is nonzero we have to prove the existence of a nontrivial (that is, not defined by the identity) coset in Ψ_n/Ψ_{n-1} which is fixed by β. This will be accomplished in Section 3.

3. The existence of a fixed coset

The main result of this section, which will be established by using some results from the theory of quadratic forms, is the following proposition.

PROPOSITION 3.1. *For all but finitely many primes the coset space* Ψ_n/Ψ_{n-1} *is the unit sphere of a quadratic form (derived from f) over a finite field or a two-fold unramified cover of that sphere.*

We now explain why this allows us to deduce the existence of a nontrivial fixed coset in Ψ_n/Ψ_{n-1}. Note that if the characteristic of the finite field is not 2 then the unit sphere for any quadratic form has an even number of points. This is true because in that case if x is on the sphere then $-x$ is also, and $x \neq -x$. From Proposition 3.1 it now follows that the coset space Ψ_n/Ψ_{n-1} has an even number of points, hence, that the number of fixed points of β is even. But we know that β leaves fixed the identity coset; hence, there is another fixed coset.

The rest of this section is devoted to proving Proposition 3.1. It is pure algebra. In fact Proposition 3.1 will follow from the following Proposition 3.2. Before stating it we establish some notation.

Let $\mathfrak{p}_1, \mathfrak{p}_2, \cdots, \mathfrak{p}_r$ be a collection of prime ideals in K. Then define

$$\Gamma_n(\mathfrak{p}_1, \mathfrak{p}_2, \cdots, \mathfrak{p}_r) = \{\gamma \in \Gamma_n \colon \gamma \equiv 1 \bmod \mathfrak{p}_i,\ 1 \leqq i \leqq r\} .$$

We will also need to define a subgroup of the orthogonal group.

Definition. Given a quadratic form f on a finite dimensional vector space over a field K, we define $O'(f)$ to be the kernel of the homomorphism:

$$O(f) \xrightarrow{\ \theta\ } \frac{K^*}{(K^*)^2}$$

where K^* is the multiplicative group of the field and θ is the spinorial norm—see O'Meara [10], page 137. θ is defined as follows. Given $u \in O(f)$ write u as a product of reflections $u = r_{x_1} r_{x_2} \cdots r_{x_n}$ where r_{x_i} is reflection in x_i (and consequently x_i is anisotropic). This can always be done—see

O'Meara [10], Theorem 43.3. Then

$$\theta(u) = f(x_1)f(x_2) \cdots f(x_n) \bmod (K^*)^2 .$$

We also define

$$\Phi'_n = \{\gamma \in \Phi_n : \theta(\gamma) = 1,\ \det \gamma = 1,\ \langle \gamma e_{n+1}, e_{n+1} \rangle < 0\} ,$$
$$\Gamma'_n = \{\gamma \in \Gamma_n : \theta(\gamma) = 1\} .$$

Finally, for a quadratic form f on a finite dimensional vector space V, we define the unit sphere of f, denoted $S(f)$ by

$$S(f) = \{v \in V : f(v) = 1) .$$

LEMMA 3.1. *Suppose K is finite and that the dimension of V is at least 3. Then $O'(f)$ acts transitively on $S(f)$.*

Proof. It follows from Witt's Theorem that $\mathrm{SO}(f)$ acts transitively on $S(f)$. Now let $x, y \in S(f)$ be given. Choose $T \in \mathrm{SO}(f)$ so that $Tx = y$. Now we claim that we can choose $R \in \mathrm{SO}(f)$ so that:

$$Rx = x ,$$
$$\theta(R) = \theta(T) .$$

Then $TRx = y$ and $TR \in O'(f)$. Of course the existence of such an R would follow if we knew that θ was onto K^* for any quadratic form in two or more variables, for in this case we would define R to leave x fixed and to be any R' with $\theta(R') = \theta(T)$ on the orthogonal complement of x. To prove that θ is onto we note that this would follow immediately from the statement that $f: V \to K$ is onto. But it is well-known, see O'Meara [10], page 157, that f is onto if f is a quadratic form over a finite field in two or more variables.

PROPOSITION 3.2. *Suppose $\mathfrak{p}_1, \mathfrak{p}_2, \cdots, \mathfrak{p}_r$ do not divide the discriminant of f. Then f reduced modulo $\mathfrak{p}_i$ gives rise to a nondegenerate form taking values in the finite field $\mathcal{O}/\mathfrak{p}_i$. We denote this form by $\bar{f}_i$. Then we have an exact sequence*

$$1 \longrightarrow \Gamma'_n(\mathfrak{p}_1, \mathfrak{p}_2, \cdots, \mathfrak{p}_r) \longrightarrow \Phi'_n \longrightarrow O'(\bar{f}_1) \times \cdots \times O'(\bar{f}_r) \longrightarrow 1 .$$

Proof. We apply the Strong Approximation Theorem, Kneser [7], to $O'(f)$. This is permissible because $O'(f)$ is the image of $\mathrm{Spin} f$, a simply-connected group, in $\mathrm{SO}(f)$. Because the set of real points of $O'(f)$ is not compact, the strong approximation theorem allows us to conclude that the set of K-rational points of $O'(f)$ is dense in the finite adeles of $O'(f)$. Intersecting with the integral adeles of $O'(f)$, an open subset of the finite adeles, we find that Φ'_n is dense in the integral adeles of $O'(f)$. But for any $a_i \in O'(\bar{f}_i)$, $1 \leqq i \leqq r$, the set of finite integral adeles g satisfying $g \equiv a_i \bmod \mathfrak{p}_i$, $1 \leqq i \leqq r$, is an open subset, hence, contains a point of Φ'_n proving Proposition 3.2.

Of course we also obtain the corresponding exact sequence:

$$1 \longrightarrow \Gamma'_{n-1}(\mathfrak{p}_1, \mathfrak{p}_2, \cdots, \mathfrak{p}_r) \longrightarrow \Phi'_{n-1} \longrightarrow O'(\bar{g}_1) \times \cdots \times O'(\bar{g}_r) \longrightarrow 1 .$$

Here $\bar{g}_j$ is the form induced by $\bar{f}_j$ on the vectors with first coordinate zero—we remind the reader that f is assumed to be diagonal.

We also obtain exact sequences:

$$1 \longrightarrow \Gamma'_n(\mathfrak{p}_1, \mathfrak{p}_2, \cdots, \mathfrak{p}_r) \longrightarrow \Gamma'_n(\mathfrak{p}_1) \longrightarrow O'(\bar{f}_2) \times \cdots \times O'(\bar{f}_r) \longrightarrow 1 ,$$
$$1 \longrightarrow \Gamma'_{n-1}(\mathfrak{p}_1, \mathfrak{p}_2, \cdots, \mathfrak{p}_r) \longrightarrow \Gamma'_{n-1}(\mathfrak{p}_1) \longrightarrow O'(\bar{g}_2) \times \cdots \times O'(\bar{g}_r) \longrightarrow 1 .$$

COROLLARY.

(a) $$\frac{\Gamma'_n(\mathfrak{p}_1)}{\Gamma'_n(\mathfrak{p}_1, \mathfrak{p}_2)} \cong O'(\bar{f}_2) ;$$

(b) $$\frac{\Gamma'_{n-1}(\mathfrak{p}_1)}{\Gamma'_{n-1}(\mathfrak{p}_1, \mathfrak{p}_2)} \cong O'(\bar{g}_2) .$$

Combining (a) *and* (b) *with Lemma* 3.1 *we obtain*:

(c) $$\frac{O'(\bar{f}_2)}{O'(\bar{g}_2)} \cong S(\bar{f}_2) .$$

Now we are ready to prove Proposition 3.1. First we make a slight change of notation. The prime ideals $\mathfrak{p}$ and $\mathfrak{p}'$ of Section 2 will now be denoted $\mathfrak{p}_1$ and $\mathfrak{p}_2$ respectively. Now we have inclusions:

$$O'(\bar{f}_2) \subseteqq \Psi_n \subseteqq \mathrm{SO}(\bar{f}_2) ,$$
$$O'(\bar{g}_2) \subseteqq \Psi_{n-1} \subseteqq \mathrm{SO}(\bar{g}_2) ,$$

and if we have $\Psi_{n-1} = \mathrm{SO}(\bar{g}_2)$ then we must have $\Psi_n = \mathrm{SO}(f_2)$. Hence noting that $\mathrm{SO}(\bar{f}_2)/\mathrm{SO}(\bar{g}_2) = O'(\bar{f}_2)/O'(\bar{g}_2) = S(\bar{f}_2)$ we see that there are two possibilities for (Ψ_n/Ψ_{n-1}). Either $\Psi_n/\Psi_{n-1} = S(\bar{f}_2)$ or $\Psi_n/\Psi_{n-1} = \mathrm{SO}(\bar{f}_2)/O'(\bar{g}_2)$. But we have a map

$$\frac{\mathrm{SO}(\bar{g}_2)}{O'(\bar{g}_2)} \longrightarrow \frac{\mathrm{SO}(\bar{f}_2)}{O'(\bar{g}_2)} \longrightarrow \frac{\mathrm{SO}(\bar{f}_2)}{\mathrm{SO}({}_2\bar{g})}$$

and $\mathrm{SO}(\bar{g}_2)/O'(\bar{g}_2) = \mathbf{Z}_2$ so in this case Ψ_n/Ψ_{n-1} is a two-fold cover of $S(\bar{f}_2)$. This completes the proof of Proposition 3.1.

4. Some generalizations

At this point there is a gap between what we have proved and the results we have claimed in the introduction. We have proved that the unit group of any diagonal form over a totally real field, which is of signature $(n, 1)$ at one real place and positive definite at all others has a subgroup of finite index with first Betti number nonzero.

We first note that the results extend immediately to diagonal forms over Q of signature $(n, 1)$—in this case $X_n(\mathfrak{p})$ and $X_{n-1}(\mathfrak{p})$ need not be compact. Thus the group SO$(n, 1; \mathbf{Z})$, the group of units of the form f given by:

$$f(x_1, X_2, \cdots, X_{n+1}) = X_1^2 + X_2^2 + \cdots + X_n^2 - X_{n+1}^2$$

admits congruence subgroups with first Betti number nonzero.

Next we note that we may extend our results to nondiagonal forms as follows. Let f be a nondiagonal form over a totally real number field K and Φ its unit group. Since f may be diagonalized over K there exists an isometry ι with entries in K so that $\iota^2 = 1$ and ι has the right centralizer in the group of real points. Then $\Gamma = \iota\Phi\iota \cap \Phi$ has finite index in Φ, see Borel [3], page 50. Clearly Γ is normalized by ι. Now we pass to congruence subgroups being careful to choose primes $\mathfrak{p}$ which do not divide the denominators of the entries of ι. Then $\Gamma(\mathfrak{p})$ is also normalized by ι and we can proceed as in Section 2 and Section 3.

At this point then we have proved all the results stated in the introduction except the assertion that the first Betti number can be made arbitrarily large. In the compact case this follows from a general result of Borel and Serre, (see Borel [2]), which states that if a unitary representation $\pi \neq 1$ of a non-compact simple Lie group G occurs in $L^2(\Gamma\backslash G)$ with Γ arithmetic and uniform then one may make the multiplicity of π arbitrarily large by passing to congruence subgroups. That the first Betti number can be made arbitrarily large in both the compact and non-compact cases can be deduced by the methods of this paper as follows. First one extends the Main Lemma by changing hypothesis 3) to

3′) There exist $\gamma_1, \gamma_2, \cdots, \gamma_r \in \pi_1(\tilde{M}: M)$ so that

(a) $\gamma_i F \cap F = \varnothing$ and $\gamma_i F \cap \gamma_j F = \varnothing$, $\quad 1 \leqq i, j \leqq r$,

(b) $\sigma\gamma_i F \subseteqq \gamma_i F$, $\quad 1 \leqq i \leqq r$,

(c) σ preserves the orientation of $\gamma_i F$, $\quad 1 \leqq i \leqq r$,

and concludes that $b_1(\tilde{M}) \geqq r$. To prove this extension of the Main Lemma one has only to observe that by (c) σ reverses the orientation of the normal bundle of $\gamma_i F$ in $\tilde{M}$, hence, interchanges the components of $\tilde{M} - \bigcup_1^r \gamma_i F$. But F is fixed so there can be only one component. The rest of the proof is a routine application of Poincaré duality.

Then in the notation of Section 4 one considers the coverings:

$$Y_n(\mathfrak{p}_1, \mathfrak{p}_2, \cdots, \mathfrak{p}_r) \longrightarrow Y_n(\mathfrak{p}_1) ,$$
$$Y_{n-1}(\mathfrak{p}_1, \mathfrak{p}_2, \cdots, \mathfrak{p}_r) \longrightarrow Y_{n-1}(\mathfrak{p}_1) ,$$

with covering transformation groups

$$\Psi_n = \frac{\Gamma_n(\mathfrak{p}_1)}{\Gamma_n(\mathfrak{p}_1, \mathfrak{p}_2, \cdots, \mathfrak{p}_r)},$$
$$\Psi_{n-1} = \frac{\Gamma_{n-1}(\mathfrak{p}_1)}{\Gamma_{n-1}(\mathfrak{p}_1, \mathfrak{p}_2, \cdots, \mathfrak{p}_r)}.$$

One then generalizes the results of Chapter 2 to obtain that if there are N nontrivial fixed cosets for the action of β on Ψ_n/Ψ_{n-1} then $b_1(Y_n(\mathfrak{p}_1, \mathfrak{p}_2, \cdots, \mathfrak{p}_r)) \geqq N$. In order to apply the extended Main Lemma (see above) one will need Remark 2.1 which guarantees that condition (c) is fulfilled. Finally one generalizes the results of Chapter 3 by noting that Ψ_n/Ψ_{n-1} is a product of $r-1$ spheres over finite fields (or two-fold covers thereof) and consequently that β has at least 2 fixed points in each factor, hence:

$$* \qquad b_1(Y_n(\mathfrak{p}_1, \mathfrak{p}_2, \cdots, \mathfrak{p}_r)) \geqq 2^{r-1} - 1 .$$

We conclude with two explicit examples. The estimate * is valid provided

(1) The norm of $\mathfrak{p}_i$ is not 2, $1 \leqq i \leqq r$.

(2) $\Gamma_n(\mathfrak{p}_1)$ is torsion-free.

(3) $\mathfrak{p}_i$ does not divide the discriminant of f, $1 \leqq i \leqq r$.

It is more convenient to choose a *pair* of primes $\mathfrak{p}_1$ and $\mathfrak{p}_2$ so that $\Gamma_n(\mathfrak{p}_1, \mathfrak{p}_2)$ is torsion free. The estimate * then becomes:

$$** \qquad b_1(Y_n(\mathfrak{p}_1, \mathfrak{p}_2, \cdots, \mathfrak{p}_r)) \geqq 2^{r-2} - 1 .$$

For our first example we choose $\Phi_n = SO(n, 1; \mathbf{Z})$. Then the pair of primes $\{3, 5\}$ has the property that $\Gamma_n(3, 5)$ is torsion free and we obtain the following explicit result. Let $p_1, p_2, \cdots, p_r$ be any r odd rational primes (distinct from 3 and 5). Then

$$b_1(\Gamma_n(3, 5, p_1, p_2, \cdots, p_r)) \geqq 2^r - 1 .$$

Of course by the Chinese remainder theorem we have

$$\Gamma_n(3, 5, p_1, p_2, \cdots, p_r) = \Gamma_n(a) = \{\gamma \in \Phi_n : \gamma \equiv 1 \bmod a\}$$

where $a = 3 \cdot 5 \cdot p_1 \cdot p_2 \cdots p_r$.

In our first example Γ_n was not uniform. We now give an example with Γ_n uniform. Let $K = \mathbf{Q}(\sqrt{3})$ and f_n be the quadratic form defined over K by

$$f_n(X_1, X_2, \cdots, X_{n+1}) = X_1^2 + X_2 + \cdots + X_n^2 - \sqrt{3}X_{n+1}^2 .$$

Then if $\mathcal{O}$ denotes the integers of $\mathbf{Q}(\sqrt{3})$, 5 and 11 are a pair of primes in $\mathcal{O}$ with the property that $\Gamma_n(5, 11)$ is torsion free and we obtain the following explicit result. Let $p_1, p_2, \cdots, p_r$ be any r primes with norm not equal to 2 or -3 from $\mathcal{O}$ (for example odd rational primes congruent to 2 modulo 3) distinct from 5 and 11. Then

$$b_1(\Gamma_n(5, 11, p_1, p_2, \cdots, p_r)) \geqq 2^r - 1 .$$

YALE UNIVERSITY
THE INSTITUTE FOR ADVANCED STUDY, PRINCETON, N.J.

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