On representation varieties of Artin groups, projective arrangements and the fundamental groups of smooth complex algebraic varieties

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Abstract

We prove that for any affine variety S defined over $\mathbb{Q}$ there exist Shephard and Artin groups G such that a Zariski open subset U of S is biregular isomorphic to a Zariski open subset of the character variety $X(G, PO(3)) = \mathrm{Hom}(G, PO(3))//PO(3)$. The subset U contains all real points of S. As an application we construct new examples of finitely-presented groups which are not fundamental groups of smooth complex algebraic varieties.

1 Introduction

The goal of this paper is to understand representation varieties of Artin and Shephard groups and thereby obtain information on Serre's problem of determining which finitely-presented groups are fundamental groups of smooth complex (not necessarily compact) algebraic varieties. The first examples of finitely-presented groups which are not fundamental groups of smooth complex algebraic varieties were given by J. Morgan [Mo1], [Mo2]. We find a new class of such examples which consists of certain Artin and Shephard groups. Since all Artin and Shephard groups have quadratically presented Malcev algebras, Morgan's test does not suffice to distinguish Artin groups from fundamental groups of smooth complex algebraic varieties or even from fundamental groups of compact Kähler manifolds, see §16. Recently Arapura and Nori [AN] have proven that if the fundamental group π of a smooth complex algebraic variety is a solvable subgroup of $GL_n(\mathbb{Q})$ then π is virtually nilpotent. The examples constructed in our paper are not virtually solvable (see Remark 11.1).

Our main results are the following theorems (Artin and Shephard groups are defined in §4 below):

Theorem 1.1 *There are infinitely many distinct Artin groups that are not isomorphic to fundamental groups of smooth complex algebraic varieties.*

Theorem 1.2 *For any affine variety S defined over $\mathbb{Q}$ there are Shephard and Artin groups G such that a Zariski open subset U of S is biregular isomorphic to Zariski open subsets of the character varieties $X(G, PO(3)) = \mathrm{Hom}(G, PO(3))//PO(3)$. The subset U contains all real points of S.*

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The surprising thing about Theorem 1.1 is that Artin groups look very similar to the fundamental groups of smooth complex quasi-projective varieties. For example the free group on n letters is the fundamental group of $\mathbb{C}$ with n points removed and it is the Artin group associated with the graph with n vertices and no edges. On the other extreme, take a finite complete graph where each edge has the label 2. The corresponding Artin group is free Abelian, hence it is the fundamental group of the quasi-projective variety $(\mathbb{C}^{\times})^n$. Yet another example is the braid group which is the Artin group associated with the permutation group S_n. Theorem 1.1 is a consequence of Theorems 1.3, 1.5, 1.6, 1.8, 1.12 and Corollary 1.7 below. The main body of this paper is concerned with a study of the following diagram[1]:

$$\begin{tikzcd}
\boxed{\begin{array}{l}\text{Moduli spaces of arrangements}\\ \text{in the projective plane}\end{array}} \arrow[r, shift left, "\mathrm{T}"] \arrow[d, "Alg"'] & \boxed{\begin{array}{l}\text{Varieties}\\ \text{over } \mathbb{Q}\end{array}} \arrow[l, shift left, "Geo"] \\
\boxed{\begin{array}{l}\text{Character varieties of repre-}\\ \text{sentations of Shephard groups}\\ \text{into } PO(3)\end{array}} \arrow[r, "\Omega"] & \boxed{\begin{array}{l}\text{Character varieties of repre-}\\ \text{sentations of Artin groups into}\\ PO(3)\end{array}}
\end{tikzcd}$$

The arrow T is tautological, the arrow Ω is pull-back of homomorphisms. The arrows *Geo* and *Alg* are defined below. In §§8.1, 8.4 we define *abstract arrangements* A (essentially bipartite graphs Λ) and their projective realizations. The space $R(A)$ of projective realizations of a given abstract arrangement has a canonical structure of a projective variety (i.e. projective scheme, see the preceding footnote) over $\mathbb{Q}$. In fact we refine the notion of projective arrangement to obtain the affine variety $BR_0(A)$ of *finite based realizations*. The variety $BR_0(A)$ injects as an open subvariety into the *moduli space* $\mathcal{M}(A) = BR(A)//PGL(3)$ of the arrangement A. Our version of Mnev's theorem [Mn] is then

Theorem 1.3 *For any affine algebraic variety S defined over $\mathbb{Q}$ there is a marked based abstract arrangement A such that the varieties $BR_0(A)$, S are isomorphic.*

Remark 1.4 *It appears that Mnev's theorem [Mn] implies only that there is a stable homeomorphism between the sets of real points of $BR_0(A)$ and S. In addition Mnev gives only an outline of the proof. For our application to Serre's problem it is critical to prove an isomorphism on the scheme level. An analogue of theorem 1.3 is proven in [KM4].*

The key idea in proving Theorem 1.3 is to construct a cross-section *Geo* to T (over the category of affine varieties) by showing that one can do "algebra via geometry", that is one can describe elementary algebraic operations over any commutative ring using projective arrangements (see §9). This idea actually goes back to the work of von Staudt [St] (the "fundamental theorem of the projective geometry"). The abstract arrangement A corresponding to S under *Geo* depends upon a choice of affine embedding $\tilde{S}$ (i.e. defining equations) for S and upon a choice of particular formulae describing these equations (including the insertion of parentheses). Moreover we obtain an isomorphism $geo : \tilde{S} \to BR_0(A)$ of affine schemes over $\mathbb{Q}$. Thus if $x \in \tilde{S}$ then $\psi = geo(x)$ is a point in $BR_0(A)$ where A is the abstract arrangement corresponding to S under *Geo*.

[1]Here and in what follows we do not assume our varieties are reduced or irreducible, i.e. they are schemes of finite type over the base field.

We next describe the arrow Alg. To an abstract arrangement A we associate a finitely-presented (Shephard) group G_A^s. Then $Alg(BR(A))$ is the affine variety $\mathrm{Hom}(G_A^s, SO(3))$. We have an associated morphism of the varieties

$$alg : BR(A, \mathbb{P}_0^2) \to \mathrm{Hom}(G_A^s, SO(3))$$

which encodes the points and lines of an *anisotropic projective realization* (see §12 for the definition) ψ of the abstract arrangement A into a representation

$$\rho = \rho_\psi : G_A^s \longrightarrow PO(3, \mathbb{C})$$

of the Shephard group G_A^s associated to the abstract arrangement A. A choice of a nondegenerate bilinear form on $\mathbb{C}^3$ determines anisotropic points and lines in $\mathbb{P}^2$ (we choose the bilinear form so that all real points of $\mathbb{P}^2$ are anisotropic). Each anisotropic point P in $\mathbb{P}^2$ determines the Cartan involution σ_P in $PO(3, \mathbb{C})$ around this point or the rotation θ_P of order 3 having this point as the neutral fixed point (i.e. a fixed point where the differential of the rotation has determinant 1). There are two such rotations of order 3, we choose one of them. There is only one vertex v_{11} of Λ (corresponding to a *point* in A) for which we choose an order 3 rotation θ_P around $P = \psi(v_{11})$. Since the realization ψ is *based* (see §8.1), $\psi(v_{11}) = (1:1:1) \in \mathbb{P}^2$ for all ψ, and the choice of rotation is harmless.

Similarly every anisotropic line L uniquely determines the reflection σ_L in $PO(3, \mathbb{C})$ which keeps L pointwise fixed. Finally one can encode the incidence relation between points and lines in $\mathbb{P}^2$ using algebra: two involutions generate the subgroup $\mathbb{Z}/2 \times \mathbb{Z}/2$ in $PO(3, \mathbb{C})$ iff the neutral fixed point of one belongs to the fixed line of the other, rotations of orders 2 and 3 anticommute (i.e. $\sigma\theta\sigma\theta = 1$) iff the neutral fixed point of the rotation of order 3 belongs to the fixed line of the involution, etc. We get a morphism

$$alg : \text{ based anisotropic arrangements } \longrightarrow \text{ representations of } G_A^s$$

$$alg : \psi \mapsto \rho, \rho(g_v) = \sigma_{\psi(v)}, v \in \mathcal{V}(\Lambda) - \{v_{11}\}, \rho(g_{v_{11}}) = \theta_{\psi(v_{11})},$$
$$\rho \in \mathrm{Hom}(G_A^s, PO(3)), \psi \in BR(A)$$

where $\mathcal{V}(\Lambda)$ is the set of vertices of bipartite graph Λ corresponding to A and g_v denotes the generator of Shephard group G_A^s that corresponds to the vertex v of Λ. In the following theorem we shall identify $alg(\psi)$ with its projection to the character variety

$$X(G_A^s, PO(3)) := \mathrm{Hom}(G_A^s, PO(3))//PO(3) \quad .$$

Theorem 1.5 *The mapping* $alg : BR(A, \mathbb{P}_0^2) \to X(G_A^s, PO(3))$ *is a biregular isomorphism onto a Zariski open (and closed) subvariety* $\mathrm{Hom}_f^+(G_A^s, PO(3))//PO(3)$.

The mapping alg has the following important property: Let S be an affine variety defined over $\mathbb{Q}$ and $O \in S$ be a rational point. Then we can choose an arrangement A so that O corresponds to a realization ψ_0 under the mapping $geo : S \to BR_0(A)$ such that the image of the representation $alg(\psi_0)$ is a finite subgroup of $PO(3, \mathbb{C})$ with trivial centralizer.

There is an Artin group G_A^a and a canonical epimorphism $G_A^a \to G_A^s$ associated with the Shephard group G_A^s. It remains to examine the morphism $\omega : \mathrm{Hom}_f^+(G_A^s, PO(3)) \to \mathrm{Hom}(G_A^a, PO(3))$ given by pull-back of homomorphisms.

Theorem 1.6 *Suppose that A is an* **admissible**[2] *based arrangement. Then the restriction of the morphism ω to* $\mathrm{Hom}_f^+(G_A^s, PO(3))$ *is an isomorphism onto a union of Zariski connected components.*

[2]See Section 8.1 for the definition.

Corollary 1.7 *The character variety $X(G^a_A, PO(3))$ inherits all the singularities of the character variety $X(G^s_A, PO(3))$ corresponding to points of $BR(A, \mathbb{P}^2_0)$, whence (since all real points of $BR(A)$ are anisotropic) to all singularities of $BR(A)$ at real points.*

Combining Corollary 1.7 with Theorem 1.3 we obtain

Theorem 1.8 *Let S be an affine algebraic variety defined over $\mathbb{Q}$ and $O \in S$ be a rational point. Then there exists an admissible based arrangement A and a representation ρ_0 : $G^a_A \to PO(3, \mathbb{R})$ with finite image such that the (analytic) germ $(X(G^a_A, PO(3, \mathbb{C})), [\rho_0])$ is isomorphic to (S, O).*

Thus the singularities of representation varieties of Artin groups at representations with finite image are at least as bad as germs of affine varieties defined over $\mathbb{Q}$ at rational points.

As the other corollary we get:

Corollary 1.9 *Suppose that $\Sigma \subset \mathbb{R}^n$ is a compact real algebraic set defined over $\mathbb{Q}$. Then there exist Artin group G^a and Shephard group G^s so that the affine real-algebraic set Σ is algebraically*[3] *isomorphic to a union of components in the affine real-algebraic sets*

$$\mathrm{Hom}(G^a, SO(3, \mathbb{R}))/SO(3, \mathbb{R}), \quad \mathrm{Hom}(G^s, SO(3, \mathbb{R}))/SO(3, \mathbb{R})$$

Since every smooth compact manifold is diffeomorphic to an affine real algebraic set defined over $\mathbb{Q}$ (see [AK]) we obtain:

Corollary 1.10 *For every smooth compact manifold M there exists an Artin group G^a so that the manifold M is diffeomorphic to a union of components (with respect to the Zariski topology) in*

$$\mathrm{Hom}(G^a, SO(3, \mathbb{R}))/SO(3, \mathbb{R}), \quad \mathrm{Hom}(G^s, SO(3, \mathbb{R}))/SO(3, \mathbb{R})$$

On the other hand, if M is a (connected) smooth complex algebraic variety and G is an algebraic Lie group, then singularities of $\mathrm{Hom}(\pi_1(M), G)$ at representations with finite image are severely limited by Theorem 1.12 below. We will need the following

Definition 1.11 *Let X be a real or complex analytic space, $x \in X$ and G a Lie group acting on X. We say that there is a local* **cross-section** *through x to the G-orbits if there is a G-invariant open neighborhood U of x and a closed analytic subspace $S \subset U$ such that the natural map $G \times S \to U$ is an isomorphism of analytic spaces.*

Theorem 1.12 *Suppose M is a smooth connected complex algebraic variety, G is a reductive algebraic Lie group and $\rho : \pi_1(M) \to G$ is a representation with finite image. Then the germ*

$$(\mathrm{Hom}(\pi_1(M), G), \rho)$$

is analytically isomorphic to a quasi-homogeneous cone with generators of weights 1 and 2 and relations of weights 2, 3 and 4. In the case there is a local cross-section through ρ to $Ad(G)$-orbits, then the same conclusion is valid for the quotient germ

$$(\mathrm{Hom}(\pi_1(M), G)//G, [\rho]) \quad .$$

[3] An algebraic isomorphism between two real algebraic sets is a polynomial bijection which has polynomial inverse.

We present two proofs of this result: in §14 we deduce it from a theorem of R. Hain [Hai] and, since Hain's paper is still in preparation, in §15 we also deduce Theorem 1.12 from results of J. Morgan [Mo2] on Sullivan's minimal models of smooth complex algebraic varieties.

Note that in the case when M is a *compact* smooth a compact Kähler manifold then a stronger conclusion may be drawn:

Theorem 1.13 *(W. Goldman, J. Millson [GM], C. Simpson [Si]). Suppose that M is a compact Kähler manifold, G is an algebraic Lie group and $\rho : \pi_1(M) \to G$ is a representation such that the Zariski closure of $\rho(\pi_1(M))$ is a reductive subgroup of G. Then the germ*

$$(\mathrm{Hom}(\pi_1(M), G), \rho)$$

is analytically isomorphic to a (quasi)-homogeneous cone with generators of weight 1 and relations of weight 2 (i.e. a homogeneous quadratic cone). In the case there is a local cross-section through ρ to $Ad(G)$-orbits, then the same conclusion is valid for the quotient germ

$$(\mathrm{Hom}(\pi_1(M), G)//G, [\rho]) \quad .$$

Our proof of Theorem 1.12 is in a sense analogous to the proof of Theorem 1.13 in [GM], [Si]: we construct a differential graded Lie algebra $\mathcal{Q}^\bullet$ which is weakly equivalent to the algebra of bundle-valued differential forms $\mathcal{A}^\bullet(M, adP)$ on M so that $\mathcal{Q}^\bullet$ *controls* a germ which is manifestly a quasi-homogeneous cone with the required weights.

In Figure 16 we describe the graph of an Artin group G^a_A which admits a representation with finite image $alg(\psi_0) = \rho_0 : G^a_A \to PO(3, \mathbb{C})$ such that the germ $(X(G^a_A, PO(3, \mathbb{C})), [\rho_0])$ is isomorphic to the germ at 0 defined by $x^5 = 0$. Thus Theorem 1.12 implies that the group G^a_A is not the fundamental group of a smooth complex algebraic variety.

Remark 1.14 *Our convention for Coxeter graphs is different from the standard convention for Dynkin diagrams. Namely, if two vertices are not connected by an edge* **it does not mean that corresponding generators commute**. *If on our diagram an edge has no label, we assume that the edge has the label* 2. *On the diagram for a* **Shephard group** *if a vertex has no label this means that the corresponding generator has* **infinite order**.

There is a local cross-section to the $PO(3)$-orbit through the representation ρ_0 (that appears in Theorem 1.8), hence we apply Theorem 1.12 and conclude that G^a_A is not the fundamental group of a smooth complex algebraic variety. To see that there are infinitely many distinct examples we may proceed as follows.

Take the varieties $V_p := \{x^p = 0\}$, $p \geq 2$ are prime numbers. Clearly V_p is not analytically isomorphic to V_q for $q \neq p$. Thus for all $p \geq 5$ the varieties V_p are not analytically isomorphic to quasi-homogeneous varieties described in the Theorem 1.12. Hence the Artin groups $G^a_{A_p}$ corresponding to V_p are not fundamental groups of smooth complex algebraic varieties. Note that among the groups $G^a_{A_p}$ we have infinitely many ones which are not mutually isomorphic. The reason is that for any finitely-generated group Γ the character variety $X(\Gamma, PO(3, \mathbb{C}))$ has only finite number of isolated singular points whose germs are isomorphic to one of V_p. This proves Theorem 1.1.

In §17 we use the results of §16 to show that for every Artin group Γ and Lie group G the germ $(\mathrm{Hom}(\Gamma, G), \rho)$ is quadratic where ρ is the trivial representation.

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Contents

2 Morphisms of analytic germs

Let $\mathbf{k}$ be either $\mathbb{R}$ or $\mathbb{C}$. Let V be a variety defined over the field $\mathbf{k}$, $o \in V$ be a point and $\widehat{\mathcal{O}_{V,o}}$ the complete local ring. We denote by

$$T_o^m(V) = \mathrm{Hom}_{\mathbf{k}-alg}(\widehat{\mathcal{O}_{V,o}}\ , \mathbf{k}[t]/t^{m+1})$$

the m-th order Zariski tangent space at $o \in V$ and by $\pi_{m,n} : T_o^m \to T_o^n(V)$ the natural projection ($m \geq n \geq 1$). Notice that each space T_o^m has a distinguished point 0, zero. Then the fiber $\pi_{m,m-1}^{-1}(0)$ has a natural structure of $\mathbf{k}$-vector space, which we call the *kernel* of $\pi_{m,m-1}$. We will think of elements in T_o^m as formal curves of degree m which are tangent to V at o up to the order m. If (V,o), (W,p) are two analytic germs as above and $f : (V,o) \to (W,p)$ is a morphism of germs then it induces isomorphisms $D_o^{(n)}f$ between the corresponding towers of Zariski tangent spaces of finite order so that we have commutative diagrams:

$$\begin{array}{ccc} T_o^m(V) & \longrightarrow & T_p^m(W) \\ {\scriptstyle \pi_{mn}}\big\downarrow & & {\scriptstyle \pi_{mn}}\big\downarrow \\ T_o^n(V) & \longrightarrow & T_p^n(W) \end{array}$$

Lemma 2.1 *Suppose that $f : (V,o) \to (W,p)$ is a morphism of analytic germs such that the morphisms $D_o^{(m)}f : T_o^m(V) \to T_p^m(W)$ are bijective maps of $\mathbf{k}$-points for all m. Then f is an isomorphism of germs.*

Proof: Let $R := \widehat{\mathcal{O}_{V,o}}$ and $S := \widehat{\mathcal{O}_{W,p}}$ be the complete local rings and let $\mathfrak{m}_R$ and $\mathfrak{m}_S$ be their maximal ideals. By Artin's theorem (see [GM, Theorem 3.1]) it suffices to prove that the induced map $f^* : S \to R$ is an isomorphism. We observe that $D_o^{(n+1)}f$ induces a $\mathbf{k}$-linear isomorphism from $\ker(\pi_{n+1,n} : T_o^{n+1}(V) \to T_o^n(V))$ to $\ker(\pi_{n+1,n} : T_p^{n+1}(W) \to T_p^n(W))$. It is clear that the above kernels are canonically isomorphic to the dual vector spaces $(\mathfrak{m}_R^{n+1}/\mathfrak{m}_R^{n+2})^*$ and $(\mathfrak{m}_S^{n+1}/\mathfrak{m}_S^{n+2})^*$. Hence f^* induces an isomorphism

$$Gr_n(f^*) : \mathfrak{m}_R^n/\mathfrak{m}_R^{n+1} \longrightarrow \mathfrak{m}_S^n/\mathfrak{m}_S^{n+1}$$

and consequently the induced map $Gr(f^*)$ of the associated graded rings is an isomorphism. The Lemma follows from [AM, Lemma 10.23]. $\square$

Remark 2.2 *Suppose that $(Z,0),(W,0)$ are* **minimal** *germs of varieties in $\mathbb{A}^n$ (i.e. $\mathbf{k}^n$ equals to the both Zariski tangent spaces $T_0(Z)$ and $T_0(W)$), and these germs are analytically isomorphic. Then there is an analytic diffeomorphism $f : \mathbb{A}^n \to \mathbb{A}^n$ defined in a neighborhood of 0 whose restriction to Z induces an isomorphism of germs $(Z,0) \to (W,0)$. See for instance [Di, Proposition 3.16].*

Lemma 2.3 *Suppose (X,x) and (Y,y) are $\mathbf{k}$-analytic germs and $f : (X,x) \to (Y,y)$ is a morphism such that $df_x : T_x(X) \to T_y(Y)$ is an isomorphism. Assume that (X,x) is smooth. Then f is an isomorphism.*

Proof: By Artin's Theorem (see [GM, Theorem 3.1]) it suffices to prove that the induced map of complete local rings $f^* : \widehat{O_{Y,y}} \to \widehat{O_{X,x}}$ is an isomorphism. Let $\{y_1, ..., y_N\}$ be elements of the maximal ideal $\mathfrak{m}_{Y,y}$ such that their images in $\mathfrak{m}_{Y,y}/\mathfrak{m}^2_{Y,y}$ form a $\mathbf{k}$-basis. Put $x_i := f^* y_i$, $1 \le i \le N$. Then the images of $x_1, ..., x_N$ in $\mathfrak{m}_{X,x}/\mathfrak{m}^2_{X,x}$ form a $\mathbf{k}$-basis. By Nakayama's Lemma (see [AM, Proposition 2.8]) the elements $y_1, ..., y_N$ (resp. $x_1, ..., x_N$) generate $\mathfrak{m}_{Y,y}$ (resp. $\mathfrak{m}_{X,x}$) as an *ideal.* Since (X, x) is smooth we can say more: by the formal inverse function theorem

$$\widehat{O_{X,x}} \cong \mathbf{k}[[x_1, ..., x_N]]$$

In particular, the monomials of weight n in $x_1, ..., x_N$ form a basis for the $\mathbf{k}$-vector space $\mathfrak{m}^n_{X,x}/\mathfrak{m}^{n+1}_{X,x}$.

We claim that the monomials $\{m_I\}$ of weight n in $y_1, ..., y_N$ form a basis for the $\mathbf{k}$-vector space $\mathfrak{m}^n_{Y,y}/\mathfrak{m}^{n+1}_{Y,y}$. First, $\{m_I\}$ generate the ideal $\mathfrak{m}^n_{Y,y}$ as an ideal. Hence given $f \in \mathfrak{m}^n_{Y,y}$, there exists $\{f_I\} \subset \widehat{O_{Y,y}}$ such that $f = \sum_I f_I m_I$. But modulo $\mathfrak{m}^{n+1}_{Y,y}$ we have

$$f = \sum_I f_I(0) m_I$$

and we have proved that $\{m_I\}$ span the $\mathbf{k}$-vector space $\mathfrak{m}^n_{Y,y}/\mathfrak{m}^{n+1}_{Y,y}$. However the image of $\{m_I\}$ under f^* is a basis; hence an independent set. Therefore $\{m_I\}$ is also an independent set and the claim is proved. We have proved that

$$Gr(f^*) : Gr(\widehat{O_{Y,y}}) \to Gr(\widehat{O_{X,x}}) = \mathbf{k}[x_1, ..., x_N]$$

is an isomorphism. The lemma follows from Lemma 2.1. $\square$

As an easy corollary we get an alternative proof of the following theorem of A. Weil (see also [LM, Theorem 2.6]).

Theorem 2.4 *Suppose that* Γ *is a finitely generated group,* $\mathbf{k}$ *is either* $\mathbb{R}$ *or* $\mathbb{C}$*,* G *is a reductive algebraic group defined over* $\mathbf{k}$*,* $\rho : \Gamma \to G$ *is a representation. Assume that* $\mathrm{H}^1(\Gamma, ad\rho) = 0$*, i.e.* ρ *is infinitesimally rigid. Then the germ* $(\mathrm{Hom}(\Gamma, G), \rho)$ *is smooth and the* $Ad(G)$*-orbit of* ρ *is open in* $\mathrm{Hom}(\Gamma, G)$ *(in the classical topology).*

Proof: Consider the inclusion morphism

$$\iota : Ad(G)\rho \hookrightarrow \mathrm{Hom}(\Gamma, G)$$

Then for each $g \in G$ the adjoint action of g induces an isomorphism

$$0 = \mathrm{H}^1(\Gamma, ad\rho) \to \mathrm{H}^1(\Gamma, ad(g) \circ ad\rho)$$

Hence for each $\phi \in Ad(G)\rho$ the morphism ι induces isomorphisms of Zariski tangent spaces

$$T_\phi Ad(G)\rho \to Z^1(\Gamma, ad\phi) = T_\phi \mathrm{Hom}(\Gamma, G)$$

The variety $Ad(G)\rho$ is smooth (it is isomorphic to the quotient of G by the centralizer of $\rho(\Gamma)$ in G). Hence by Lemma 2.3, ι is an open isomorphism onto its image. $\square$

Below we give another application. Suppose Γ is a finitely generated group and G is a reductive algebraic group defined over $\mathbf{k}$. Suppose $s \in \Gamma$ is a central element. Let Θ be the subgroup of Γ generated by s, $\Phi := \Gamma/\Theta$ and $\pi : \Gamma \to \Phi$ be the quotient map.

Lemma 2.5 *Suppose* $\rho \in \mathrm{Hom}(\Gamma, G)$ *satisfies:*

1. $\rho(s) = 1$,

2. $\mathrm{H}^0(\Gamma, ad\rho) = 0$.

Then for each cocycle $\sigma \in Z^1(\Gamma, ad\rho)$ we have: $\sigma(s) = 0$. Consequently $\pi^* : \mathrm{H}^1(\Phi, ad\rho) \to \mathrm{H}^1(\Gamma, ad\rho)$ is an isomorphism.

Proof: From $\sigma(sg) = \sigma(gs)$ we deduce

$$\rho(g)\sigma(s) - \sigma(s) = \rho(s)\sigma(g) - \sigma(g) = 0$$

Hence $\sigma(s)$ is fixed by $\rho(\Gamma)$ whence by (1) we have $\sigma(s) = 0$. Thus π^* is onto. But π^* is clearly injective. $\square$

Let $\bar{\rho}$ be the representation of Φ induced by ρ. Then under the assumptions of Lemma 2.5 we have

Lemma 2.6 *If* $\mathrm{Hom}(\Phi, G)$ *is smooth at* $\bar{\rho}$ *then* $\mathrm{Hom}(\Gamma, G)$ *is smooth at* ρ.

This lemma is an immediate consequence of Lemma 2.5 and Lemma 2.3.

In the rest of this section we discuss the following question:

Suppose that $X' \subset X, Y' \subset Y$ are subvarieties in smooth quasi-projective varieties X, Y over $\mathbf{k}$, $\eta : X \to Y$ is a biregular isomorphism which carries X' bijectively to Y'. Does η induce a biregular isomorphism $X' \to Y'$?

Clearly the answer is "yes" if both subvarieties X', Y' are reduced. The simple example:

$$X = Y = \mathbb{C}, \quad X' = \{z^2 = 0\}, \quad Y' = \{z^3 = 0\}, \quad \eta = id$$

shows that in the nonreduced case we need some extra assumptions to get the positive answer. Our goal is to prove that the answer is again positive if we assume that η induces an *analytic* isomorphism between X', Y' (Theorem 2.9).

Let R be a ring, $\mathfrak{m}$ is a maximal ideal in R, $R_{\mathfrak{m}}$ is the localization of R at $\mathfrak{m}$ and $\widehat{R_{\mathfrak{m}}}$ is the completion of R at $\mathfrak{m}$.

Lemma 2.7 *Suppose R is a Noetherian ring and $f \in R$ has the property that its image in $\widehat{R_{\mathfrak{m}}}$ is zero for all maximal ideals $\mathfrak{m}$. Then $f = 0$.*

Proof: By Krull's theorem [AM, Corollary 10.19], the induced map $R_{\mathfrak{m}} \to \widehat{R_{\mathfrak{m}}}$ is an injection. Thus the image of f in $R_{\mathfrak{m}}$ is zero for all maximal ideals $\mathfrak{m}$. Hence for every such $\mathfrak{m}$ there exists $s \notin \mathfrak{m}$ with $sf = 0$. Therefore $Ann(f)$ is contained in no maximal ideal. This implies that $Ann(f) = R$ and $f = 0$. $\square$

Lemma 2.8 *Let $\phi : R \to S$ be a ring homomorphism and $I \subset R, J \subset S$ be ideals. Suppose that for every maximal ideal $\mathfrak{m}$ in S with $\mathfrak{m} \supset J$ we have*

$$\phi(I) \otimes \widehat{S_{\mathfrak{m}}} \subset J \otimes \widehat{S_{\mathfrak{m}}}$$

Then $\phi(I) \subset J$.

Proof: It suffices to prove this when $\phi(I)$ is replaced by an element f of S. Thus we assume that the image of f in $S_{\mathfrak{m}}$ is contained in $J \otimes S_{\mathfrak{m}}$ for all maximal ideals $\mathfrak{m} \subset S$. We want to conclude that $f \in J$. We further simplify the situation by dividing by J. We have an exact sequence

$$0 \to J \to S \to S/J \to 0$$

We use [AM, Proposition 10.15] to conclude that

$$\widehat{S/J_{\mathfrak{m}/J}} \cong S/J \otimes \widehat{S_{\mathfrak{m}}}$$

and [AM, Proposition 10.14] to conclude

$$S/J \otimes \widehat{S_{\mathfrak{m}}} \cong \widehat{S_{\mathfrak{m}}}/(J \otimes \widehat{S_{\mathfrak{m}}})$$

Replace f by its image in S/J. We find that the image of f in all completions of S/J at maximal ideals of S/J is zero. Then f is zero by the Lemma 2.7. $\square$

Theorem 2.9 *Suppose that X, Y are nonsingular (quasi-) projective varieties over $\mathbf{k}$ and $\eta : X \to Y$ is an isomorphism. Let $X' \subset X, Y' \subset Y$ are subvarieties so that: $\eta' := \eta|_{X'} : X' \to Y'$ is a bijection which is an analytic isomorphism. Then $\eta' : X' \to Y'$ is a biregular isomorphism.*

Proof: It is enough to check the assertion on open subsets, so we may as well assume that X, Y are affine with coordinate rings S, R, affine subvarieties Y', X' are given by ideals $I \subset R, J \subset S$. Coordinate rings of Y', X' are $R/I, S/J$. Let $\mathfrak{m}$ be a maximal ideal in R/I, then there is a maximal ideal $\mathfrak{M} \subset R$ such that $\mathfrak{m} = \mathfrak{M}/I$. Thus

$$\widehat{R/I_{\mathfrak{m}}} \cong \frac{\widehat{R_{\mathfrak{M}}}}{I \otimes \widehat{R_{\mathfrak{M}}}}$$

Let $\phi : R \to S$ be the isomorphism induced by $\eta : X \to Y$. Since η' is an analytic isomorphism it induces isomorphisms of all completions

$$\frac{\widehat{R_{\mathfrak{M}}}}{I \otimes \widehat{R_{\mathfrak{M}}}} \longrightarrow \frac{\widehat{R_{\phi(\mathfrak{M})}}}{J \otimes \widehat{R_{\mathfrak{M}}}}$$

Thus the assertion of Theorem follows from Lemma 2.8. $\square$

3 Quasi-homogeneous singularities

Suppose that we have a collection of polynomials $F = (f_1, ..., f_m)$ in $\mathbf{k}^n$, we assume that all these polynomials have trivial linear parts. The polynomial f_j is said to be *weighted homogeneous* if there is a collection of positive integers (weights) $w_1 > 0, ..., w_n > 0$ and a number $u_j \geq 0$ so that

$$f_j((x_1 t^{w_1}), ..., (x_n t^{w_n})) = t^{u_j} f_j(x_1, ..., x_n)$$

for all $t \in \mathbf{k}$. We will call the numbers w_i the *weights of generators* and the numbers u_j the *weights of relations*. Let Y denote the variety given by the system of equations

$$\{f_1 = 0, ..., f_m = 0\}$$

(Note that the germ $(Y, 0)$ is necessarily minimal.) We say that $(Y, 0)$ is a *quasi-homogeneous* if we can choose generators $f_1, ..., f_m$ for its defining ideal such that all the polynomials f_j are weighted homogeneous with the same weights $w_1, ..., w_n$ (we do not require u_j to be equal for distinct $j = 1, ..., m$). In particular, if $(Y, 0)$ is a *quasi-homogeneous* then $(Y, 0)$ is invariant under the $\mathbf{k}^{\times}$-action on $\mathbf{k}^n$ given by the weights $w_1, ..., w_n$.

Remark 3.1 *The variety Y given by a system of quasi-homogeneous equations $\{f_j = 0\}$ is also called a* **quasi-homogeneous** *(or* **weighted homogeneous***) cone.*

We now give an intrinsic characterization of quasi-homogeneous germs. Suppose that $(Y, 0)$ is quasi-homogeneous . Let $S_m \subset \mathbf{k}[x_1, ..., x_n]$ be the subspace of polynomials which are homogeneous of degree m (in the usual sense). We may decompose the subspace S_m into one dimensional eigenspaces under $\mathbf{k}^\times$ (since the multiplicative group of a field is a reductive algebraic group). We obtain a bigrading

$$\mathbf{k}[x_1, ..., x_n] = \oplus_{m,n} S_{m,n}$$

where m is the degree and n is the weight of a polynomial under $\mathbf{k}^\times$ (f transforms to $t^n f$). We obtain a new grading of $\mathbf{k}[x_1, ..., x_n]$ by weight

$$\mathbf{k}[x_1, ..., x_n] = \oplus_{n=1}^{\infty} S'_n$$

where S'_n is the subspace of polynomials of weight n. We let I be the ideal of Y. Then I is invariant under the action of $\mathbf{k}^\times$ (since its generators are). We claim

$$I = \oplus_{n=1}^{\infty} I \cap S'_n$$

This follows by decomposing the action of $\mathbf{k}^\times$ in the finite dimensional subspaces $I \cap \oplus_{m=1}^{N} S_m$. Thus if $f \in I$ we may write

$$f = \sum_{n=1}^{\infty} f_n \ , \quad \text{with} \ \ f_n \in I \cap S'_n$$

(the sum is of course finite). Let $R = \mathbf{k}[Y] = \mathbf{k}[x_1, ...x_n]/I$. Then R is a graded ring, $R = \oplus_{n=1}^{\infty} R_n$ with $R_0 = \mathbf{k}$, $R_n = S'_n/(I \cap S'_n)$.

We let $\widehat{R}$ be the completion of R at $\mathfrak{m}$ where $\mathfrak{m}$ is the ideal of zero, i.e. the ideal generated by $\{x_1, ..., x_n\}$. Hence

$$\widehat{R} \cong \mathbf{k}[[x_1, ..., x_n]]/\widehat{I}$$

where $\widehat{I}$ is the ideal generated by I in $\mathbf{k}[[x_1, ..., x_n]]$. Hence $\widehat{R} = \widehat{O_{Y,o}}$. The ring $\widehat{R}$ is not graded but it has a decreasing filtration $W^\bullet$ such that $W^N \widehat{R}$ is the closure of $\oplus_{n=N}^{\infty} S'_n$. Define $Gr_n^W(\widehat{R}) = W^n(\widehat{R})/W^{n+1}(\widehat{R})$. The filtration $W^\bullet$ satisfies:

(i) $Gr_0^W = \mathbf{k}$.

(ii) $\cap_{n=0}^{\infty} W^n = 0$.

(iii) $\dim_{\mathbf{k}} Gr_n^W(\widehat{R}) < \infty$ for all n.

The inclusion $R \hookrightarrow \widehat{R}$ induces an isomorphism $R \cong Gr^W(\widehat{R})$ and we obtain:

Lemma 3.2 *(a) $\widehat{O_{Y,o}}$ admits a decreasing filtration $W^\bullet$ satisfying the properties (i)—(iii) above.*

(b) There is a monomorphism of filtered rings $Gr^W(\widehat{O_{Y,o}}) \to \widehat{O_{Y,o}}$ with dense image, so $\widehat{O_{Y,o}}$ is the completion of $Gr^W(\widehat{O_{Y,o}})$.

(c) Conversely if $\widehat{O_{Y,o}}$ satisfies (a) and (b) then (Y, o) is quasi-homogeneous .

Proof: It remains to prove (c). Define a $\mathbf{k}^\times$-action on $Gr^W \widehat{O_{Y,o}}$ so that the elements in the n-graded summand have weight n for the action of $\mathbf{k}^\times$. Let $\mathfrak{m}$ be the ideal of o. Choose a basis of eigenvectors under $\mathbf{k}^\times$ action on $\mathfrak{m}/\mathfrak{m}^2$. Lift these vectors to eigenvectors $f_1, ..., f_n$

of $\mathfrak{m}$. Then by a standard argument (see the proof of Lemma 2.3) if we set $\pi(x_i) = f_i$ we obtain a surjection

$$\pi : \mathbf{k}[[x_1, ..., x_n]] \longrightarrow \widehat{O_{Y,o}}$$

which is $\mathbf{k}^\times$-equivariant. Hence the induced map of graded rings

$$\pi' : \mathbf{k}[x_1, ..., x_n] \longrightarrow Gr^W \widehat{O_{Y,o}}$$

is also surjective. Let I be the kernel of π' and let Y be the affine variety corresponding to I. $\square$

Definition 3.3 *We will say that a complete local ring R is* **quasi-homogeneous** *if it satisfies (a) and (b) as in Lemma 3.2. We will say a germ (Y, o) is quasi-homogeneous if the complete local ring $\widehat{O_{Y,o}}$ is quasi-homogeneous .*

Here are several examples. The polynomial $f(x, y, z) = x^2+y^5+z^3$ is quasi-homogeneous with the weights of generators 15, 6 and 10 respectively. The weight of the relation is 30. Let $g(x) = x^n$, then g is quasi-homogeneous for any weight w of the generator and the weight nw of the relation.

Another example is the germ $(V_p, 0) = (\{x^p = 0\}, 0)$, $p \geq 2$ is prime. Let's prove that this germ is not quasi-homogeneous for any weights of relations $< p$. Indeed, suppose Y is a quasi-homogeneous cone whose germ at zero is isomorphic to $(V_p, 0)$. Since we assume that Y is minimal, hence $Y \subset \mathbf{k}$. Polynomials defining Y must be monomials (since Y is quasi-homogeneous). Then the analytical germ $(Y, 0)$ clearly can be defined by a single monomial equation $x^m = 0$. Isomorphism of germs $(Y, 0) \to (V_p, 0)$ induces isomorphisms of finite order tangent spaces, hence $m = p$.

In a certain sense *generic* germs are not quasi-homogeneous. This is discussed in details in [A1], [A2]. Here is one explanation, in the case of germs in the affine plane $\mathbb{A}^2$, one which doesn't require knowledge of singularity theory but is based on 3-dimensional topology. Suppose that $Y \subset \mathbb{A}^2$ is a minimal affine curve (defined over $\mathbb{C}$), which is invariant under weighted action of $\mathbb{C}^\times$ on $\mathbb{A}^2$ (with the weights w_1, w_2). Then the set of complex points

$$Y(\mathbb{C}) \subset \mathbb{C}^2$$

is invariant under the $\mathbb{C}^\times$-action with the weights w_1, w_2. The corresponding weighted action of S^1 preserves a small sphere S^3 around zero and the link $Y_\mathbb{C} \cap S^3 = L$. Thus $S^3 - L$ admits a free S^1-action, therefore $S^3 - L$ is a Seifert manifold. *Generic* singularities do not have such property, see [EN]. For convenience of the reader we describe a way to produce examples of singularities which do not admit $\mathbb{C}^\times$-action. Our discussion follows [EN]. Start with a finite "Piuseaux series"

$$y = x^{q_1/p_1}(a_1 + x^{q_2/p_1p_2}(a_2 + x^{q_3/p_1p_2p_3}(...(a_{s-1} + a_s x^{q_s/p_1...p_s}) \dots)$$

where (p_i, q_i) are pairs of positive coprime integers. The numbers a_j are nonzero integers. Then y, x satisfy some polynomial equation $f(x, y) = 0$ with integer coefficients, the link L of the singularity at zero is an *iterated torus knot*, the number s is the depth of iteration, numbers p_i, q_i describe cabling invariants. The complement $S^3 - L$ is not a Seifert manifold provided that $s \geq 2$. The simplest example is when $s = 2$,

$$y = x^{q_1/p_1}(a_1 + a_2 x^{q_2/p_1p_2})$$

For instance take $a_1 = a_2 = 1$, $p_1 = p_2 = 2$, $q_1 = q_2 = 3$ (the iterated trefoil knot), then

$$y^2 = x^3 + x^9 + 2x^6 \quad .$$

Another example of a singularity which is not quasi-homogeneous is

$$x^2y^2 + x^5 + y^5 = 0$$

see [Di, Page 122].

4 Coxeter, Shephard and Artin groups

Let Λ be a finite graph where two vertices are connected by at most one edge, there are no loops (i.e. no vertex is connected by an edge to itself) and each edge e is assigned an integer $\epsilon(e) \geq 2$. We call Λ a *labelled* graph, let $\mathcal{V}(\Lambda)$ and $\mathcal{E}(\Lambda)$ denote the sets of vertices and edges of Λ. When drawing Λ we will omit labels 2 from the edges (since in our examples most of the labels are 2). Given Λ we construct two finitely-presented groups corresponding to it. The first group G^c_Λ is called the *Coxeter group* with the *Coxeter graph* Λ, the second is the *Artin group* G^a_Λ. The sets of generators for the both groups are $\{g_v, v \in \mathcal{V}(\Lambda)\}$. The relations in G^c_Λ are:

$$g_v^2 = 1, v \in \mathcal{V}(\Lambda),\ (g_v g_w)^{\epsilon(e)} = \mathbf{1}, \text{ over all edges } e = [v, w] \in \mathcal{E}(\Lambda)$$

The relations in G^a_Λ are:

$$\underbrace{g_v g_w g_v g_w \ldots}_{\epsilon \text{ multiples}} = \underbrace{g_w g_v g_w g_v \ldots}_{\epsilon \text{ multiples}}, \quad \epsilon = \epsilon(e), \text{ over all edges } e = [v, w] \in \mathcal{E}(\Lambda)$$

We let $\epsilon(v, w) = \epsilon([v, w])$ if v, w are connected by the edge $[v, w]$ and $\epsilon(v, w) = \infty$ if v, w are not connected by any edge. For instance, if we have an edge $[v, w]$ with the label 4, then the Artin relation is

$$g_v g_w g_v g_w = g_w g_v g_w g_v$$

Note that there is an obvious epimorphism $G^a_\Lambda \longrightarrow G^c_\Lambda$. We call the groups G^c_Λ and G^a_Λ *associated* with each other. Artin groups above appear as generalizations of the Artin braid group. Each Coxeter group G^c_Λ admits a canonical discrete faithful linear representation

$$h : G^c_\Lambda \longrightarrow GL(n, \mathbb{R}) \subset GL(n, \mathbb{C})$$

where n is the number of vertices in Λ. Suppose that the Coxeter group G^c_Λ is finite, then remove from $\mathbb{C}^n$ the collection of fixed points of elements of $h(G^c_\Lambda - \{\mathbf{1}\})$ and denote the resulting complement X_Λ. The group G^c_Λ acts freely on X_Λ and the quotient X_Λ/G^c_Λ is a smooth complex quasi-projective variety with the fundamental group G^a_Λ, see [B] for details. Thus the Artin group associated to a finite Coxeter group is the fundamental group of a smooth complex quasi-projective variety.

The construction of Coxeter and Artin groups can be generalized as follows. Suppose that not only edges of Λ, but also its vertices v_j have labels $\delta_j = \delta(v_j) \in \{0, 2, 3, \ldots\}$. Then take the presentation of the Artin group G^a_Λ and add the relations:

$$g_v^{\delta(v)} = \mathbf{1}, \quad v \in \mathcal{V}(\Lambda)$$

If $\delta(v) = 2$ for all vertices v then we get the Coxeter group, in general the resulting group is called the *Shephard group*, they were introduced by Shephard in [Sh]. Again there is a canonical epimorphism $G^a_\Lambda \to G^s_\Lambda$.

Given a Coxeter, Artin or Shephard group G associated with the graph Λ we define *vertex* subgroups $G_v, v \in \mathcal{V}(\Lambda)$ and *edge* subgroups $G_e = G_{vw}, e = [v, w] \in \mathcal{E}(\Lambda)$ as subgroups generated by the elements: g_v (in the case of the vertex subgroup) and g_v, g_w (in the case of the edge subgroup).

We will use the fact that several of Shephard groups are finite. On Figure 1 we list graphs of the finite Shephard groups that we will use (see [C]).

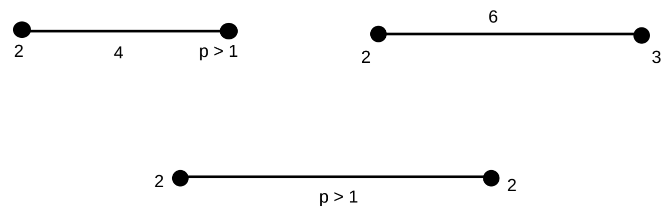

Figure 1: *Graphs for certain finite Shephard groups.*

All Artin groups we consider in this paper are associated to *generalized Cartan matrices* N (see [Lo], [Le]) as follows.

For each pair of distinct vertices v_i, v_j of Λ we have: $\epsilon(v_i, v_j) \in \{2, 4, 6, \infty\}$. Enumerate vertices of Λ from 1 to m. The generalized Cartan matrix N is $m \times m$ matrix with the following entries:

- Diagonal entries n_{ii} of N are equal to 2.

  Now consider off-diagonal entries n_{ij}, n_{ji} of N assuming $i < j$.
- If $\epsilon(v_i, v_j) = 2$ we let $n_{ij} = n_{ji} = 0$.
- If $\epsilon(v_i, v_j) = 4$ we let $n_{ij} = -1, n_{ji} = -2$.
- If $\epsilon(v_i, v_j) = 6$ then $n_{ij} = -1, n_{ji} = -3$.
- Finally, if $\epsilon(v_i, v_j) = \infty$ we let $n_{ij} = n_{ji} = -2$.

Thus for $i < j$ we have: $n_{ij} \leq 0, n_{ij} \geq n_{ji}$ and $\epsilon(v_i, v_j) = 2, 4, 6, \infty$ iff $n_{ij}n_{ji} = 0, 2, 3, 4$.

5 Local deformation theory of representations

Let $\mathbf{k}$ be a field of zero characteristic, define the Artin local $\mathbf{k}$-algebra Π_m as $\mathbf{k}[t]/t^{m+1}$. Take a finitely-generated group Γ and an algebraic Lie group G over $\mathbf{k}$, let $\mathbf{G} := G(\mathbf{k})$ be the set of $\mathbf{k}$-points of G. Then the Lie group $G(\Pi_m)$ splits as the semidirect product $G_0(\Pi_m) \rtimes \mathbf{G}$, where $G_0(\Pi_m)$ is the kernel of the natural projection $p_m : G(\Pi_m) \to \mathbf{G}$ induced by $\Pi_m \to \mathbf{k}$, which is given by $t \mapsto 0$. The group $G_0(\Pi_m)$ is $\mathbf{k}$-unipotent. We consider the germ

$$(\mathrm{Hom}(\Gamma, \mathbf{G}), \rho)$$

for a certain representation $\rho : \Gamma \to \mathbf{G}$. Then m-th order Zariski tangent space $T^m_\rho(\mathrm{Hom}(\Gamma, \mathbf{G}))$ is naturally isomorphic near ρ to the space of homomorphisms

$$Z^1_{(m)}(\Gamma, \mathbf{G}; \rho) := \{\xi : \Gamma \to G(\Pi_m) | p_m \circ \xi = \rho\}$$

(see [GM]). If $m = 1$ then $Z^1_{(m)}(\Gamma, \mathbf{G}; \rho) \cong Z^1(\Gamma, ad \circ \rho)$, where ad is the adjoint action of $\mathbf{G}$ on its Lie algebra.

We call $\xi \in T^m_\rho(\mathrm{Hom}(\Gamma, \mathbf{G}))$ a *trivial deformation* of ρ if there is an element $h = h(\xi) \in G(\Pi_m)$ such that $\xi(g) = h\rho(g)h^{-1}$ for all $g \in \Gamma$. If $m = 1$ then the infinitesimal deformation ξ is trivial iff the corresponding cocycle is a coboundary.

Suppose that G^s is a Shephard group, G^a is the corresponding Artin group, $q : G^a \to G^s$ is the canonical projection. Let V_L denote the set of vertices with nonzero labels in the graph of G^s. Let $\mathbf{G}$ be a group of $\mathbf{k}$-points of an algebraic Lie group. Consider a homomorphism $\rho : G^s \to \mathbf{G}$, let $\tilde{\rho} = \rho \circ q$. The projection q induces an injective morphism of the representation varieties

$$q^* : \mathrm{Hom}(G^s, \mathbf{G}) \longrightarrow \mathrm{Hom}(G^a, \mathbf{G})$$

and injective morphisms of the corresponding finite order Zariski tangent spaces

$$D^{(n)}q^* : T^n_\rho \mathrm{Hom}(G^s, \mathbf{G}) \longrightarrow T^n_{\tilde{\rho}} \mathrm{Hom}(G^a, \mathbf{G})$$

Lemma 5.1 *Suppose that $\xi \in T^n_{\tilde{\rho}}\mathrm{Hom}(G^a, \mathbf{G})$ is an element whose restriction to each cyclic vertex subgroup $G^a_v, v \in V_L$ is a* **trivial deformation** *of $\tilde{\rho}|_{G^a_v}$. Then ξ belongs to the image of $D^{(n)}q^*$.*

Proof: Recall that we identify $T^n_{\tilde{\rho}}\mathrm{Hom}(G^a, \mathbf{G})$ and $T^n_\rho \mathrm{Hom}(G^s, \mathbf{G})$ with spaces of certain representations of G^a, G^s into $G(\Pi_n)$. We have the exact sequence

$$\mathbf{1} \longrightarrow \langle\langle \{g_v^{\delta(v)}, v \in V_L\}\rangle\rangle \longrightarrow G^a \longrightarrow G^s \longrightarrow \mathbf{1}$$

Then ξ belongs to the image of $D^{(n)}q^*$ if and only if $\xi(g_v^{\delta(v)}) = \mathbf{1}$ for all $v \in V_L$. We assume that the restriction of ξ to G^a_v is a trivial deformation, thus there is an element $h := h_v \in \mathbf{G}(\Pi_n)$ such that $\xi(g_v) = h\tilde{\rho}(g_v)h^{-1}$. Since $\tilde{\rho}(g_v^{\delta(v)}) = \mathbf{1}$ we conclude that $\xi(g_v^{\delta(v)}) = \mathbf{1}$ as well. $\square$

Corollary 5.2 *Suppose that $\mathrm{H}^1(G^s, ad \circ \rho) = 0$ (i.e. the representation ρ is infinitesimally rigid) and for each $v \in V_L$ the restriction homomorphism*

$$Res_v : \mathrm{H}^1(G^a, ad \circ \rho) \to \mathrm{H}^1(G^a_v, ad\rho)$$

is zero. Then

$$q^* : (\mathrm{Hom}(G^s, \mathbf{G}), \rho) \longrightarrow (\mathrm{Hom}(G^a, \mathbf{G}), \tilde{\rho})$$

is an analytic isomorphism of germs.

Proof: Let $Z(\rho)$ denote the centralizer of $\rho(G^s)$ if $\mathbf{G}$. Then the representation variety $\mathrm{Hom}(G^s, \mathbf{G})$ is smooth near ρ and is naturally isomorphic to the quotient $\mathbf{G}/Z(\rho)$, see Theorem 2.4. Now the assertion follows from Lemmas 2.3, 5.1. $\square$

One example when the first condition of the corollary are satisfied is the case when the Shephard group G^s is finite.

6 Projective reflections

Fix the bilinear form $\flat = x_1y_1 + x_2y_2 + x_3y_3 = (x_1, x_2, x_3) \cdot (y_1, y_2, y_3)$ on the vector space $V = \mathbb{C}^3$, we shall also use the notation $\langle \cdot, \cdot \rangle$ for $\flat$. Let φ denote the quadratic form corresponding to $\flat$, let $O(3, \mathbb{C})$ be the group of automorphisms of $\flat$. Let $\pi : V \to \mathbb{P}(V)$ denote the quotient map and let $CO(3, \mathbb{C}) := \mathbb{C}^\times \cdot O(3, \mathbb{C}) \subset End(V)$. The inclusion $CO(3, \mathbb{C}) \to End(V)$ induces an embedding $PO(3, \mathbb{C}) \hookrightarrow \mathbb{P}(End\ V)$. Let $p : End(V) \to \mathbb{P}(End\ V)$ be the quotient map. Note that $PSL(2, \mathbb{C}) \cong PO(3, \mathbb{C})$ and $SO(3, \mathbb{R}) \cong PO(3, \mathbb{R})$.

6.1 The correspondence between projective reflections and their fixed points

In this section we study projective properties of elements of order 2 in the group $PO(3,\mathbb{C})$. Consider an element $A \in O(3,\mathbb{C})$ such that the projectivization $p(A)$ is an involution acting on $\mathbb{P}^2(\mathbb{C})$. The fixed-point set of $p(A)$ consists of two components: an isolated point a and a projective line l dual to a (with respect to $\flat$). Our goal is to describe the correspondence $p(A) \leftrightarrow a$ in algebraic terms.

Let $R \subset SO(3,\mathbb{C})$ be the affine subvariety of involutions. Note that $-\mathbf{1} \in O(3,\mathbb{C})$ doesn't belong to R. We leave the proof of the following lemma to the reader.

Lemma 6.1 *$PO(3,\mathbb{C})$ acts transitively by conjugations on $PR(\mathbb{C})$ (the image of $R(\mathbb{C})$ in $PO(3,\mathbb{C})$).*

We now determine $\overline{p(R)}$, the Zariski closure of $p(R)$ in $\mathbb{P}(End\ V)$. We define a morphism $\eta : V \to End(V)$ by $\eta(v)(x) = \varphi(v)x - 2\langle v, x\rangle v$. If v is an anisotropic vector then $\eta(v)$ is a multiple of the reflection through the hyperplane in V orthogonal to v. The reader will verify that η is an $O(3)$-equivariant morphism, i.e. $\eta(gv) = g\eta(v)g^{-1}$, $g \in O(3), v \in V$, and that η induces an equivariant embedding

$$\eta : \mathbb{P}(V) \longrightarrow \mathbb{P}(End\ V)$$

of smooth complex manifolds.

Lemma 6.2 *The image of η is contained in $\overline{p(R)}$.*

Proof: Let V_0 be the complement in V of $X = \{v \in V : \varphi(v) = 0\}$. Then $\mathbb{P}_0(V) := \pi(V_0)$ is Zariski dense in $\mathbb{P}(V)$. But also $\eta(V_0) \subset \mathbb{C}^\times R$. Hence $\overline{\mathbb{P}_0(V)} \subset \overline{p(R)}$. $\square$

We now consider $\mathbb{P}(V)$ and $\mathbb{P}(End(V)$ as varieties over $\mathbb{Q}$.

Lemma 6.3 *The morphism η induces an isomorphism of varieties $\mathbb{P}(V) \cong \overline{p(R)}$.*

Proof: Since $\eta : \pi(V_0) \to p(R)$ is equivariant it is easy to verify that it is onto (in fact a bijection). Hence $\eta(\pi(V_0)) = p(R)$ and accordingly $\overline{\eta\pi(V_0)} = \overline{p(R)} = \eta(\mathbb{P}(V))$. But we have seen that the morphism $\eta : \mathbb{P}(V) \to \eta(\mathbb{P}(V))$ is an analytic isomorphism of smooth compact complex manifolds. Hence (by the GAGA-principle) it is an isomorphism of projective varieties. $\square$

Let $N := p^{-1}(\overline{p(R)}) - p^{-1}(p(R))$. Then N may be described as follows. The bilinear form $\flat$ induces an isomorphism $\tilde{\flat} : V \otimes V \to V^* \otimes V = End(V)$. Then $N = \tilde{\flat}(X \otimes X)$. We note that $n_v := \tilde{\flat}(v \otimes v)$ is then given by $n_v(x) = \langle v, x\rangle v$. Hence the set of real points $N(\mathbb{R})$ is empty.

Let $Q \subset End(V)$ be the affine cone defined by $Q := p^{-1}(\overline{p(R)})$. Hence $N \subset Q$. We define $Q_0 := Q - N$. Then η induces a commutative diagram

$$\begin{array}{ccc} V & \longrightarrow & Q \\ \uparrow & & \uparrow \\ V_0 & \longrightarrow & Q_0 \end{array}$$

Remark 6.4 *It can be shown that the cone $Q \subset End(V)$ is defined by the equations:*

1. $XX^\top = X^\top X$,

2. $XX^\top E_{ij} = E_{ij}XX^\top$, $1 \le i, j \le 3$,

3. $X^2 E_{ij} = E_{ij}X^2$, $1 \le i, j \le 3$.

Here E_{ij} is the matrix with 1 in the (ij)-th position and 0 elsewhere. The equations (1) and (2) define the closure $\overline{PO(3)} \subset \mathbb{P}(End\ V)$. We will not need the explicit equations for $\overline{p(R)}$ in what follows.

We let $\zeta : PQ \to \mathbb{P}^2$ be the inverse of η and abbreviate $\mathbb{P}(V_0)$ to $\mathbb{P}^2_0$. Note that $\zeta : PR = PQ_0 \to \mathbb{P}^2_0$ assigns to each projective reflection its neutral (isolated) fixed-point. Thus we have described the correspondence $p(A) \leftrightarrow a$ algebraically. Note that we have $\mathbb{P}^2(\mathbb{R}) = \mathbb{P}^2_0(\mathbb{R})$ and $PQ(\mathbb{R}) = PQ_0(\mathbb{R})$. Let $\widehat{\flat} : V \to V^*$ be the isomorphism induced by $\flat$. Define $(\mathbb{P}^2_0)^\vee$ by $(\mathbb{P}^2_0)^\vee = \widehat{\flat}(\mathbb{P}^2_0)$. Hence the space of *anisotropic lines* $(\mathbb{P}^2_0)^\vee$ is the space of lines dual to the set of *anisotropic points* $\mathbb{P}^2_0$.

6.2 Fixed points

Suppose that $g \in PO(3, \mathbb{C})$ is a nontrivial element. In this section we discuss the fixed-point set for the action of g on $\mathbb{P}^2(\mathbb{C})$.

Definition 6.5 *A fixed point x for the action of g on $\mathbb{P}^2(\mathbb{C})$ is called* **neutral** *if the determinant of the differential of g at x is equal to 1.*

There are two classes of nontrivial elements $g \in PO(3, \mathbb{C})$: (a) g is unipotent, (b) g is semi-simple.

Case (a). In this case g has a single fixed point $a \in \mathbb{P}^2$, the point a belongs to an invariant projective line $L \subset \mathbb{P}^2$. If we choose coordinates on L such that $a = \infty$, then g acts on $L - \{a\}$ as a translation. The flag (L, a) is determined by the element g uniquely. On the other hand, the flag (L, a) uniquely determines the 1-parameter maximal unipotent subgroup in $PO(3)$ which contains g. Finally, the fixed point a of g uniquely determines the line L. It is easy to see that a is the *neutral* fixed point of g.

The collection $P \subset \mathbb{P}^2$ of fixed points of all unipotent elements in $PO(3, \mathbb{C})$ is the projectivization of the cone $\{\overrightarrow{x} \in \mathbb{C}^3 : \varphi(\overrightarrow{x}) = 0\}$.

Case (b). Each semisimple element of $O(3)$ is conjugate (in $GL(3, \mathbb{C})$) to

$$A = \begin{pmatrix} \pm\lambda & 0 & 0 \\ 0 & \lambda^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(up to $\pm$). We have two possible cases depending on whether or not $A^2 = \mathbf{1}$. If $A^2 \neq \mathbf{1}$ then A has three distinct complex eigenvectors, one of them $\overrightarrow{f}$ is fixed by A. Thus, in the case the transformation $p(A)$ has three fixed points on $\mathbb{P}^2(\mathbb{C})$, one of them $f = p(\overrightarrow{f})$ is the *neutral* fixed point. The maximal torus in $PO(3, \mathbb{C})$ containing $p(A)$ is uniquely determined by the (neutral) fixed point f in the both real and complex cases.

Finally, consider the case $A^2 = \mathbf{1}$. Then $p(A)$ has one isolated fixed point f on $\mathbb{P}^2(\mathbb{C})$ (which is *neutral*) and the fixed projective line disjoint from f.

6.3 Commuting and anticommuting elements

Suppose that α, α' are involutions in $PO(3, \mathbb{C})$, thus they have isolated fixed points λ, λ' and fixed projective lines $\Lambda, \Lambda' \subset \mathbb{P}^2(\mathbb{C})$.

Lemma 6.6 *The elements α, α' commute if and only if either:*

(1) $\alpha = \alpha'$ and $\lambda = \lambda', \Lambda = \Lambda'$, or (2) $\lambda \in \Lambda'$, $\lambda' \in \Lambda$ and Λ intersects Λ' orthogonally (with respect to the quadratic form φ).

Proof: Proof is obvious and is left to the reader. $\square$

Suppose that α is an involution in $PO(3, \mathbb{C})$ and $\beta \in PO(3, \mathbb{C}) - \{\mathbf{1}\}$. We say that the elements α, β *anticommute* if $\alpha\beta\alpha\beta = \mathbf{1}$ (i.e. $\alpha\beta\alpha^{-1} = \beta^{-1}$) and $\alpha \neq \beta$.

Lemma 6.7 *The elements α, β anticommute iff the neutral fixed point μ of β belongs to the fixed projective line Λ of α.*

Proof: If α, β anticommute then μ must belong to the fixed-point set of α. If μ is the neutral point of α then α and β commute (and $\beta^2 \neq \mathbf{1}$) or $\alpha = \beta$ which contradicts our assumptions. $\square$

Remark 6.8 *Suppose that α, β are anticommuting elements as above. Then they satisfy the Artin relation*

$$\alpha\beta\alpha\beta = \beta\alpha\beta\alpha$$

(since both right- and left-hand side are equal to $\mathbf{1}$).

7 Representation theory of elementary Artin and Shephard groups

In this section we consider mostly representation varieties of certain *elementary* Artin and Shephard groups (their graphs have only two vertices and one edge). The section is rather technical, its material will be needed in Section 12.

We will denote the action of elements γ of $SO(3)$ on vectors ξ in the Lie algebra $so(3, \mathbb{C})$ by $\gamma\xi := ad(\gamma)\xi$.

7.1 Central quotients of elementary Artin and Shephard groups

Let G_n^a be an *elementary Artin group*: its graph Λ_n has 2 vertices and one edge with the even label $n = 2m$. We abbreviate g_v to a and g_w to b. Set $c = ab, z := c^m$. Then

$$az = a(ab)^m = a(ba)^m = (ab)^m a = za$$

Similarly $bz = zb$. This proves

Lemma 7.1 *The element z is central in G_n^a.*

We set $s := z^2$ and $\overline{G}_n^a := G_n^a / \langle s \rangle$. It is clear that we get a short exact sequence:

$$\mathbf{1} \to \mathbb{Z}_2 \to \overline{G}_n^a \to S := \langle a, b, c | ab = c, c^m = 1 \rangle \cong \mathbb{Z} * \mathbb{Z}_m \to \mathbf{1}$$

Put the label 2 on the vertex v of Λ_n, let G_n^s be the corresponding Shephard group. Set

$$\overline{G}_n^s := G_s^n / \langle z^2 \rangle$$

Then we get a short exact sequence:

$$\mathbf{1} \to \mathbb{Z}_2 \to \overline{G}_n^s \to \langle a, b, c | ab = c, c^m = 1, a^2 = 1 \rangle \cong \mathbb{Z}_2 * \mathbb{Z}_m \to \mathbf{1}$$

Notice that the groups $\overline{G}_n^s$ and $\overline{G}_n^a$ are virtually free, hence their 2-nd cohomologies vanish. Therefore both $\bar{G} := \overline{G}_n^s$ and $\bar{G} := \overline{G}_n^a$ have smooth representation varieties $\mathrm{Hom}(\bar{G}, SO(3))$.

We begin with a partial classification of real representations of $G := G_n^a$ into $SO(3)$.

Definition 7.2 *A representation $\rho : G \to SO(3)$ is called* **dihedral** *if $\rho(G)$ has an invariant line but no invariant nonzero vector. If $\rho(G)$ fixes a nonzero vector and consequently is conjugate to a subgroup of $SO(2)$, then we will say that ρ is* **toral**. *The reducible representations ρ split into two subclasses: (a) ρ is* **central** *if $\rho(z) \neq \mathbf{1}$, (b) ρ is* **noncentral** *otherwise.*

We will use the notations $\alpha := \rho(a), \beta := \rho(b), \gamma := \rho(c)$ (as well as notations a, b, c, z, s for the elements of G^a, G_s and S for the quotient of $\overline{G}_n^s$) throughout this section and sections 7.2, 7.5, 7.6.

Remark 7.3 *Let $\rho : G^a \to SO(3)$ be a real representation. Then ρ is central dihedral if and only if both α, β have order 2 and their fixed subspaces in $\mathbb{R}^3$ are distinct.*

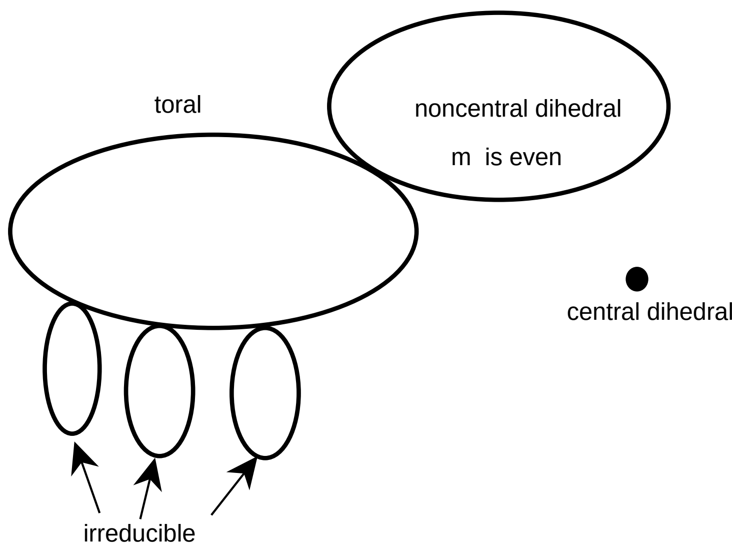

Figure 2: *The space of conjugacy classes of $SO(3, \mathbb{R})$ representations of the group G_n^s.*

Clearly each reducible real representation ρ is either dihedral or toral. Suppose that $\rho : G_n^a \to SO(3)$ is a real representation which factors through $\rho^s : G_n^s \to SO(3)$ and $\alpha \neq \mathbf{1}$ (i.e has order 2). The following theorem is a direct application of Lemmas 2.5, 2.6, the above short exact sequences for the groups $G_n^a, G_n^s, \overline{G}_n^a, \overline{G}_n^s$ and the associated Serre-Hohschild spectral sequences, we leave computations to the reader.

Theorem 7.4 *Consider a real representation $\rho : G_n^a \to SO(3)$ which factors through G_n^s. Then:*

1. $\rho(s) = \mathbf{1}$.
2. *Suppose that $\rho : G_n^a \to SO(3)$ is* **central dihedral**. *Then*
$$\mathrm{H}^1(G_n^a, ad\rho) = \mathrm{H}^1(G_n^s, ad\rho^s) = 0$$
and hence the representations ρ, ρ^s are infinitesimally rigid.
3. *Suppose that ρ is either* **central toral**, *or* **noncentral dihedral** *or* **irreducible**. *Then*
$$\mathrm{H}^1(G_n^a, ad\rho) \cong \mathbb{R}^2, \quad \mathrm{H}^1(G_n^s, ad\rho^s) \cong \mathbb{R}$$

and both germs

$$(\mathrm{Hom}(G_n^a, SO(3)), \rho), (\mathrm{Hom}(G_n^s, SO(3)), \rho^s)$$

are smooth. In the central toral case the embedding $SO(2) \hookrightarrow SO(3)$ induces isomorphisms of germs

$$(X(G_n^a, SO(2), [\rho]) \to (X(G_n^a, SO(3)), [\rho]),$$
$$(X(G_n^s, SO(2)), [\rho^s]) \to (X(G_n^s, SO(3)), [\rho^s])$$

4. *Suppose that ρ is* **noncentral toral** *(and nontrivial). Then both germs*

$$(\mathrm{Hom}(G_n^a, SO(3)), \rho), (\mathrm{Hom}(G_n^s, SO(3)), \rho^s)$$

are singular and

$$\mathrm{H}^1(G_n^a, ad\rho) \cong \mathbb{R}^3, \mathrm{H}^1(G_n^s, ad\rho^s) \cong \mathbb{R}^2 \quad .$$

On Figure 2 we give a schematic picture of the space of conjugacy classes of $SO(3, \mathbb{R})$ representations of the group G_n^s such that $\alpha \neq \mathbf{1}$, this space is a graph with two components.

7.2 2-generated Abelian group

Let Γ be the graph with two vertices v, w connected by the edge e with the label 2 (Figure 3). The corresponding Artin group G^a is free Abelian group on two generators $a = g_v, b = g_w$. Take two anti-commuting involutions $\alpha, \beta \in PO(3)$ and the homomorphism $\rho_0 : G^a \to PO(3, \mathbb{C})$ given by $\rho_0 : a \mapsto \alpha$, $\rho_0 : b \mapsto \beta$. Hence ρ_0 is **central dihedral**. The following results are special cases of Theorem 7.4, however because of their importance we give separate proofs.

Lemma 7.5 *The representation ρ_0 is infinitesimally rigid, i.e. $\mathrm{H}^1(G^a, ad\rho_0) = 0$. The point $\rho_0 \in \mathrm{Hom}(G^a, PO(3, \mathbb{C}))$ is nonsingular.*

Proof: $G^a = \pi_1(T^2)$ (where T^2 is the 2-dimensional torus) and the group $PO(3)$ is reductive, hence Poincare duality gives an isomorphism $\mathrm{H}^0(G^a, ad\rho_0) \cong \mathrm{H}^2(G^a, ad\rho_0)$. The centralizer of $\rho_0(G^a)$ in $PO(3, \mathbb{C})$ is trivial. Therefore,

$$0 = \mathrm{H}^0(G^a, ad\rho_0) \cong \mathrm{H}^2(G^a, ad\rho_0)$$

Since the Euler characteristic of T^2 (and hence of G^a) equals zero, we conclude that $0 = \mathrm{H}^1(G^a, ad\rho_0)$. The second assertion of Lemma follows from Theorem 2.4. $\square$

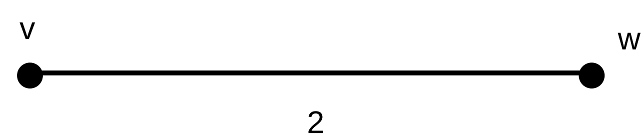

Figure 3: *Graph Γ for 2-generated free Abelian group.*

Note that the associated Coxeter group G^c is the finite group $\mathbb{Z}/2 \times \mathbb{Z}/2$. Let ρ_0^c denote the homomorphism $G^c \to PO(3, \mathbb{C})$ corresponding to ρ_0.

Corollary 7.6 *We have natural isomorphism between germs of representation varieties:*

$$(\mathrm{Hom}(G^c, PO(3, \mathbb{C})), \rho_0^c) \cong (\mathrm{Hom}(G^a, PO(3, \mathbb{C})), \rho_0)$$

given by composing homomorphisms $G^c \to PO(3, \mathbb{C})$ with the projection $G^a \to G^c$.

Proof: Follows from the Lemma 7.5 and Corollary 5.2. $\square$

Now we consider the global structure of $\mathrm{Hom}(G^a, PO(3))$.

Lemma 7.7 *The variety* $\mathrm{Hom}(G^a, PO(3,\mathbb{C}))$ *is the disjoint union of two Zariski closed subsets:*

(1) Toral representations $S_1 := \{\rho : \dim \mathrm{H}^0(G^a, ad\rho) \geq 1\}$*;*

(2) The orbit $S_2 := Ad(PO(3,\mathbb{C}))\rho_0$ *of the (unique up to conjugation) central dihedral representation* ρ_0*.*

Proof: First we verify that $\mathrm{Hom}(G^a, PO(3,\mathbb{C}))$ is the union $S_1 \cup S_2$. Let $A \subset PO(3,\mathbb{C})$ be an abelian subgroup, $\bar{A}$ is the Zariski closure of A. If A is infinite then the abelian group $\bar{A}$ is the finite extension of a 1-dimensional connected abelian Lie subgroup $\bar{A}^0$ of $PO(3,\mathbb{C})$. Hence $\bar{A}^0$ is either a maximal torus or a maximal unipotent subgroup of $PO(3,\mathbb{C})$: both are maximal abelian subgroups in $PO(3,\mathbb{C})$, thus $A \subset \bar{A}^0$. We apply this to the group $A = \rho(G^a)$ and conclude that in this case $\dim \mathrm{H}^0(G^a, ad\rho) = 1$.

Now we consider the case when A is finite, hence, after conjugation, we get: $A \subset SO(3,\mathbb{R})$. It follows from the classification of finite subgroups in $SO(3,\mathbb{R})$ that A either has an invariant vector in $\mathbb{R}^3 - \{0\}$ (which again means that $\dim \mathrm{H}^0(G^a, ad\rho) \geq 1$) or $A \cong \mathbb{Z}/2 \times \mathbb{Z}/2$ is generated by two involutions with orthogonal axes. Hence $S_1 \cup S_2 = \mathrm{Hom}(G^a, PO(3,\mathbb{C}))$.

It's clear that S_1 and S_2 are disjoint. Since the representation ρ_0 is locally rigid it follows that the orbit $Ad(PO(3,\mathbb{C}))\rho_0$ is open (see Theorem 2.4). The representation ρ_0 is *stable* (see [JM, Theorem 1.1]) since $\rho_0(G^a) = \mathbb{Z}/2 \times \mathbb{Z}/2$ is not contained in a proper parabolic subgroup of $PO(3,\mathbb{C})$. Hence the orbit $Ad(PO(3,\mathbb{C}))\rho_0$ is closed. $\square$

7.3 Finite elementary Shephard groups

Take Γ be any of the labelled graphs from the Figure 1. The corresponding Shephard group $G := G^s_\Gamma$ is finite, hence we have

Proposition 7.8 *All representations of* G *to* $PO(3)$ *are infinitesimally rigid and* $\mathrm{Hom}(G, PO(3))$ *is smooth.*

We will need a slight modification of the above proposition. Let L be a reductive algebraic group over $\mathbb{R}$ with the Lie algebra $\mathcal{L}$, G be a finitely-generated group such that all representations $\rho : G \to L(\mathbb{C})$ are infinitesimally rigid, pick elements $a \in G$ and $\alpha \in L(\mathbb{C})$ and consider the subvariety

$$F = F_{a,\alpha}(G, L) = \{\rho : G \to L | \rho(a) = \alpha\} \subset \mathrm{Hom}(G, L)$$

Proposition 7.9 *The subvariety* F *is smooth.*

Proof: The space $\mathrm{Hom}(G, L(\mathbb{C}))$ is the union of $L(\mathbb{C})$-orbits of representations $\rho_j, 1 \leq j \leq m$. If $\alpha \neq \rho(a)$ for all $\rho \in \mathrm{Hom}(G, L(\mathbb{C}))$ then there is nothing to prove. Otherwise we can assume that $\alpha = \rho_j(a)$, $j \in J \subset \{1, ..., m\}$. Since the representations ρ_j are locally rigid we get

$$F(\mathbb{C}) \cong \bigcup_{j \in J} Z_{L(\mathbb{C})}(\langle \alpha \rangle)/Z_{L(\mathbb{C})}(\rho_j(G))$$

where $Z_{L(\mathbb{C})}(H)$ denotes the centralizer of a subgroup $H \subset L(\mathbb{C})$. It is enough to verify smoothness of F at the representations $\rho_j, j \in J$. Consider the Zariski tangent space $T = T_{\rho_j} F(\mathbb{C})$. It is naturally isomorphic to

$$\{\xi \in Z^1(G, \mathcal{L}_{ad\rho_j}) : \xi(a) = 0\}$$

However infinitesimal rigidity of ρ_j implies that[4] $T \cong \mathcal{L}^{\langle a \rangle}/\mathcal{L}^G$, where G and $\langle a \rangle$ act on $\mathcal{L}$ via the adjoint representation $ad\rho_j$. Hence the dimension of $F(\mathbb{C})$ (as a complex manifold) at ρ_j is equal to the dimension of its Zariski tangent space at ρ_j, which implies that F is smooth at ρ_j. $\square$

As a particular case we let a be one of the generators of $G = G^s_\Gamma$, pick an element $\alpha \in PO(3, \mathbb{C})$ and consider the subvariety

$$F_{a,\alpha}(G, PO(3)) = \{\rho : G \to PO(3, \mathbb{C}) | \rho(a) = \alpha\} \subset \mathrm{Hom}(G, PO(3, \mathbb{C}))$$

Corollary 7.10 *Suppose that $G = G^s_\Gamma$ is a Shephard group as above, a is one of the generators of G, $\alpha \in PO(3, \mathbb{C})$. Then the $F_{a,\alpha}(G, PO(3))$ is smooth.*

7.4 The infinite cyclic group

Consider the infinite cyclic group $G = \langle b \rangle$ and a representation $\rho : \langle b \rangle \to PO(3, \mathbb{C})$ so that $\rho(b) = \beta$ is a nontrivial semisimple element with the neutral fixed point B.

Note that the Lie algebra $so(3, \mathbb{C})$ has $ad\rho(b)$-invariant splitting $V^\beta \oplus (V^\beta)^\perp$ where V^β consists of vectors fixed by $ad\beta$. The action of b on $(V^\beta)^\perp$ has no nonzero invariant subspaces. Thus $\mathrm{H}^1(G, (V^\beta)^\perp) = 0$ and

$$\mathrm{H}^1(G, so(3, \mathbb{C})) \cong \mathrm{H}^1(G, V^\beta) \oplus \mathrm{H}^1(G, (V^\beta)^\perp) \cong \mathrm{H}^1(G, V^\beta)$$

This proves the following

Proposition 7.11 *Any cocycle $\sigma \in Z^1(G, ad\rho)$ has the form $\sigma(b) = \tau + \beta\xi - \xi$ where $\beta\tau = \tau$. The element τ depends only on the cohomology class of σ.*

Remark 7.12 *The vector τ measures the infinitesimal change of the rotation angle of β and ξ measures the infinitesimal motion of the fixed point B.*

7.5 An elementary Shephard group with the edge-label 4.

Consider the graph with two vertices v, w connected by the edge with the label 4, we put the label 2 on the vertex v, see Figure 4. The corresponding Shephard group G^s has the presentation $\langle a, b | a^2 = \mathbf{1}, (ab)^2 = (ba)^2 \rangle$.

Figure 4: *Graph for a Shephard group*

Consider a representation $\rho : G^s \to SO(3, \mathbb{R})$, $\rho : a \mapsto \alpha$, $\rho : b \mapsto \beta$, where we choose α to be an involution, $\beta \neq \mathbf{1}$ is an element such that α, β anticommute, i.e. $\alpha\beta\alpha\beta = \mathbf{1}$ and

[4] Recall that $\mathcal{L}^H$ denotes the subspace of H-invariant vectors for a group H acting on $\mathcal{L}$.

$\alpha \neq \beta$. Hence ρ is noncentral dihedral (note that we have $\rho(z) = \rho(c^2) = \mathbf{1}$). We have $S \cong \mathbb{Z}/2 * \mathbb{Z}/2$. The fixed line V^β of β is orthogonal to the fixed lines V^α and V^γ of α and γ. Moreover $V^\alpha \neq V^\gamma$, hence $so(3) = V^\alpha + V^\beta + V^\gamma$. Note also that α operates by $-\mathbf{1}$ on V^β. We already know that $\mathrm{H}^1(G^s, ad\rho)$ is 1-dimensional (Theorem 7.4). Below is description of a canonical form for cocycles representing cohomology classes.

Proposition 7.13 *Let $[\sigma] \in \mathrm{H}^1(G, ad\rho)$. Then $[\sigma]$ can be represented by a cocycle σ satisfying:*

$$\sigma(a) = 0, \quad \sigma(b) = \tau, \quad \textit{where} \quad \tau \in V^\beta$$

The vector τ depends only on the cohomology class of the restriction $\sigma|_{\langle b\rangle}$.

Proof: Since $\rho(z) = \mathbf{1}$ and $\rho(G^s)$ has no nonzero fixed vectors we have $\sigma(c^2) = 0$ by Lemma 2.5. Hence we may write $\sigma(c) = \xi - \gamma\xi$. We may subtract off a coboundary to arrange that $\sigma(a) = 0$. If we replace σ by $\tilde{\sigma} = \sigma - \delta_v$ with $v \in V^\alpha$ then $\tilde{\sigma}(a) = 0$ and $\tilde{\sigma}(c) = \xi - v - \gamma(\xi - v)$. Hence we may choose σ so that $\xi \in V^\beta + V^\gamma$. But if we replace σ by $\xi - w$ with $w \in V^\gamma$ this does not change the cocycle σ. Hence we may assume $\xi \in V^\beta$. Therefore $\sigma(c) = \xi - \gamma\xi$, $\xi \in V^\beta$ and

$$\sigma(b) = \sigma(ac) = \alpha\sigma(c) = \alpha\xi - \alpha\gamma\xi$$

On the other hand, $\alpha\xi = -\xi$ (since $\xi \in V^\beta$) and $\alpha\gamma\xi = \beta\xi = \xi$. We obtain $\sigma(b) = -2\xi = \tau \in V^\beta$. This proves existence. Uniqueness of τ follows from Proposition 7.11. $\square$

Remark 7.14 *Note that the cocycles σ described in the above Proposition are integrable, they correspond to deformations ρ_t of the representation ρ which fix $\rho(a) = \alpha$ and change the element $\rho(b) = \beta$ within the corresponding 1-parameter subgroup in $SO(3, \mathbb{R})$. For such representations the elements $\rho(a), \rho_t(b)$ anticommute.*

7.6 Elementary Artin group with the edge-label 6.

Now consider 2-generated Artin group G^a given by the relation $(ab)^3 = (ba)^3$, see Figure 5. Take a representation $\rho : G^a \to SO(3, \mathbb{R})$ which maps a and b to elements α, β so that:

- $\alpha^2 = 1, \beta^2 \neq \mathbf{1}$,
- the product $\gamma = \alpha\beta$ has the order 3,
- $[\alpha, \beta] \neq \mathbf{1}$.

In particular ρ is irreducible, hence, according to Theorem 7.4, $\mathrm{H}^1(G^a, ad\rho)$ is 2-dimensional.

Figure 5: *Graph for the Artin group G^a*

Lemma 7.15 *Under the above conditions we have: $so(3) = V^\alpha + V^\beta + V^\gamma$.*

Proof: Let $A, B, C \in \mathbb{P}^2$ be the neutral fixed points of α, β and γ. Since the representation is irreducible all these points are distinct. Notice that $\beta = \alpha\gamma$. Suppose that there is a projective line $L \subset \mathbb{P}^2$ which contains all three fixed points. This line is invariant under α and $\gamma(L) \cap L = C$ since γ has the order 3. Thus $\gamma(B) \notin L$ and $\alpha\gamma(B) \notin L$. This implies that $\alpha\gamma(B) = \beta(B) \neq B$, which is a contradiction. $\square$

Lemma 7.16 *Let $\sigma \in Z^1(G^a, ad\rho)$ be a cocycle such that $\sigma(a) = 0$ and $\sigma(b) = \tau$ where $\beta\tau = \tau$. Then $\sigma = 0$.*

Proof: We again use Lemma 2.5 to conclude that $\sigma(z) = \sigma(c^3) = 0$. Hence $\sigma(c) = \xi - \gamma\xi$. If $\sigma \neq 0$ then $\xi \neq 0$. Using Lemma 7.15 and arguing as in the previous section we may assume that $\xi \in V^\beta$. Thus

$$\sigma(c) = \xi - \alpha\xi$$

hence

$$\sigma(b) = \sigma(ac) = \alpha\xi - \xi$$

We deduce $\alpha\xi = \xi + \tau \in V^\beta$ and $0 \neq \xi \in V^\beta$. Hence α carries V^β into itself and ρ is dihedral, a contradiction. $\square$

7.7 A non-elementary Shephard group.

Now suppose that we have a group G^s with the presentation:

$$\langle a_1, a_2, a_3, b | a_j^2 = \mathbf{1}, \quad j = 1, 2, 3 \; ; \quad (a_i b)^2 = (ba_i)^2, i = 1, 2; \quad (a_3 b)^3 = (ba_3)^3 \rangle$$

See the graph on the Figure 6.

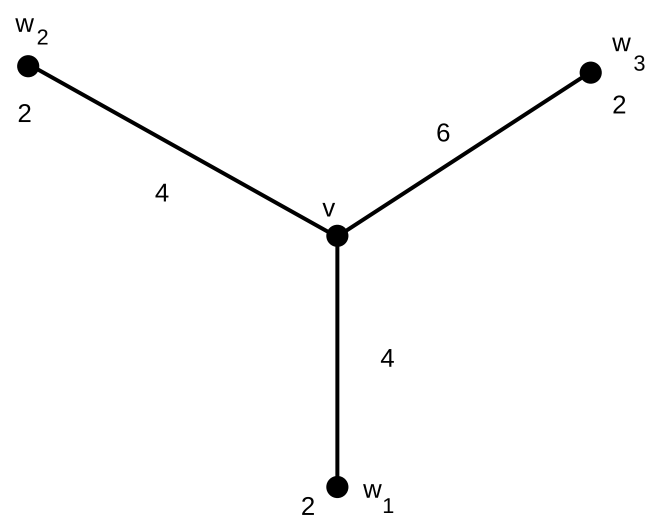

Figure 6: *Graph for the nonelementary Shephard group G^s*

Consider a representation $\rho_0 : G^s \to SO(3, \mathbb{R})$ so that

1. $\alpha_j := \rho_0(a_j) \neq \mathbf{1}$, $1 \leq j \leq 3$;
2. The group generated by α_3 and $\beta := \rho_0(b)$ has no fixed point in $\mathbb{P}^2$;
3. the neutral fixed points of the elements $\alpha_1, \alpha_2, \beta$ do not belong to a common projective line in $\mathbb{P}^2$;

4. $(\alpha_3\beta)^3 = \mathbf{1}$.

Take the subvariety $W \subset \mathrm{Hom}(G, PO(3,\mathbb{C}))$ which contains ρ_0 and consists of homomorphisms that are constant on the generators $a_j, 1 \le j \le 3$.

Lemma 7.17 *The point ρ_0 is an isolated reduced point in W.*

Proof: Notice that our assumptions also imply that $\beta^2 \neq \mathbf{1}$ and $[\alpha_3, \beta] \neq \mathbf{1}$. Take a cocycle $\sigma \in Z^1(G^s, ad\rho_0)$ which is tangent to the variety W. Hence $\sigma(a_j) = 0, 1 \le j \le 3$. According to Proposition 7.11 the value of σ on the generator b equals $\tau + b\xi - \xi$. On the other hand, by Proposition 7.13 we can find coboundaries $\delta_{\theta_j} \in B^1(\langle a_j, b\rangle, ad\rho_0)$, $j = 1, 2$, so that:

$$\sigma_j := \sigma - \delta_{\theta_j}, \quad \sigma_j(a_j) = 0, \quad \sigma_j(b) = \tau$$

Here and below $\langle a_j, b\rangle$ denotes the subgroup of G^s generated by a_j and b. Notice that τ does not depend on j (see Proposition 7.13). The coboundary δ_{θ_j} is given by

$$\delta_{\theta_j}(x) = \rho_0(x)\theta_j - \theta_j, \quad x \in \langle a_j, b\rangle, \quad \theta_j \in so(3)$$

Thus

$$\alpha_j\theta_j = \theta_j, \quad \beta(\theta_1 - \theta_2) = \theta_1 - \theta_2$$

Note however that the condition (3) on the representation ρ_0 implies that the (1-dimensional) fixed-point sets for the adjoint actions of $\rho_0(a_1), \rho_0(a_2), \rho_0(b)$ on $so(3)$ are linearly independent. Therefore we conclude that $\theta_j = 0$, $j = 1, 2$, thus $\sigma(b) = \tau$ and by Lemma 7.16 we have $\sigma(b) = 0$. Hence $\sigma = 0$ and the Zariski tangent space to ρ_0 in W is zero. $\square$

7.8 Nondegenerate representations

Let Γ be a labelled graph where all vertices and edges have nonzero even labels. Let G^s denote the Shephard group corresponding to the graph Γ. The following technical definition will be used in Section 12.1.

Definition 7.18 *A representation $\rho : G^s \to PO(3,\mathbb{C})$ will be called* **nondegenerate on the edge** $e = [v, w] \subset \Gamma$ *if the element $\rho(g_v g_w)$ has the order*

$$\begin{cases} \epsilon(e) \quad , & \text{if } \delta(v) = \delta(w) = 2 \\ \epsilon(e)/2 \quad , & \text{otherwise} \end{cases}$$

A representation ρ will be called **nondegenerate on the vertex** $v \in V(\Gamma)$ *if $\rho(g_v) \neq \mathbf{1}$. A representation ρ will be called* **nondegenerate** *if it is nondegenerate on each edge and each vertex of Γ. Let $\mathrm{Hom}_f(G^s, PO(3,\mathbb{C}))$ denote the space of all nondegenerate representations.*

Proposition 7.19 *Suppose that for each edge $e \subset \Gamma$ the corresponding edge subgroup $G_e \subset G^s$ is finite. Then $\mathrm{Hom}_f(G^s, PO(3,\mathbb{C}))$ is Zariski open and closed in $\mathrm{Hom}(G^s, PO(3,\mathbb{C}))$.*

Proof: Since each $G_v, G_e \subset G^s$ is finite, $\mathrm{Hom}(G_v, PO(3,\mathbb{C}))$, $\mathrm{Hom}(G_e, PO(3,\mathbb{C}))$ are disjoint unions of finite numbers of $PO(3,\mathbb{C})$-orbits of rigid representations. Since each orbit is Zariski open the proposition follows in the case $G^s = G_v, G_e$. Let

$$Res_e : \mathrm{Hom}(G^s, PO(3,\mathbb{C})) \to \mathrm{Hom}(G_e, PO(3,\mathbb{C})),$$

$$Res_v : \mathrm{Hom}(G^s, PO(3,\mathbb{C})) \to \mathrm{Hom}(G_v, PO(3,\mathbb{C}))$$

be the restriction morphisms. Then $\mathrm{Hom}_f(G^s, PO(3,\mathbb{C})) =$

$$\bigcap_{v \in \mathcal{V}(\Gamma)} Res_v^{-1}\mathrm{Hom}_f(G_v, PO(3,\mathbb{C})) \cap \bigcap_{e \in \mathcal{E}(\Gamma)} Res_e^{-1}\mathrm{Hom}_f(G_e, PO(3,\mathbb{C}))$$

and the proposition follows. $\square$

8 Arrangements

8.1 Abstract arrangements

An *abstract arrangement* A is a disjoint union of two finite sets $A = \mathcal{P} \sqcup \mathcal{L}$, with the set of "points" $\mathcal{P} = \{v_1\ , v_2\ , \cdots\}$ and the set of "lines" $\mathcal{L} = (l_1\ , l_2\ , \cdots)$ together with the *incidence relation* $\iota = \iota_A \subset \mathcal{P} \times \mathcal{L}$; $\iota(v, l)$ is interpreted to mean "the point v lies on the line l ". We may represent the arrangement A by a bipartite graph, $\Gamma = \Gamma_A$: vertices of Γ are elements of A, two vertices are connected by an edge if and only if the corresponding elements of A are incident (Γ is also called the *Hasse diagram* of the arrangement A).

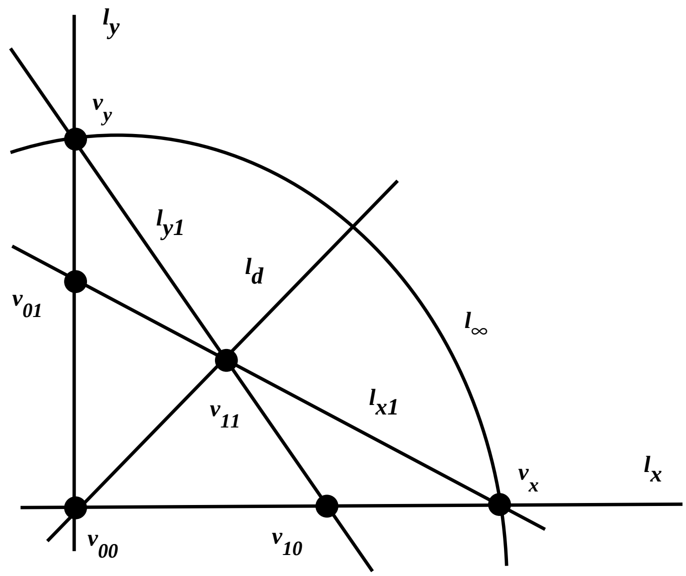

Figure 7: *Standard triangle.*

Convention 8.1 *When drawing an arrangement we shall draw* **points** *as* **solid points** *and* **lines** *as* **lines**. *If $\iota(v, l)$ then we shall draw the point v on the line l.*

An example of an abstract arrangement is the *standard triangle* T described on the Figure 7 (this is a triangle with the complete set of bisectors):

$$T = \{v_{00}, v_x, v_y, v_{1,0}, v_{01}; l_x, l_y, l_\infty, l_d, l_{y1}, l_{x1} : \iota(v_{00}, l_x), \iota(v_{01}, l_y), \iota(v_x, l_x), \iota(v_x, l_\infty),$$

$$\iota(v_y, l_y), \iota(v_y, l_\infty), \iota(v_{11}, l_d), \iota(v_{00}, l_d), \iota(v_{00}, l_y), \iota(v_{10}, l_{y1}), \iota(v_{01}, l_{x1}),$$

$$\iota(v_{10}, l_x), \iota(v_y, l_{y1}), \iota(v_x, l_{x1}), \iota(v_{11}, l_{y1}), \iota(v_{11}, l_{x1})\}$$

Here is another example of arrangement (Figure 8), we take $A = \{v_1, v_2; l_1, l_2\}$ with the incidence relation:

$$\iota(v_1, l_1), \iota(v_1, l_2), \iota(v_2, l_1), \iota(v_2, l_2) \quad .$$

Suppose that $(A, \iota_A), (B, \iota_B)$ are abstract arrangements, $\phi : B \to A$ is a map which sends points to points and lines to lines. We say that ϕ is a *morphism* of arrangements if $\iota_B(x, y)$ implies $\iota_A(\phi(x), \phi(y))$. A *monomorphism* of arrangements is an injective morphism. An *isomorphism* of arrangements is an invertible morphism. Suppose that $\phi : B \to A$ is a monomorphism of arrangements, we call the image $\phi(B)$ a *subarrangement* in A. Note that if we work with the corresponding bipartite graphs Γ_A, Γ_B, then morphisms $A \to B$ are morphisms of these bipartite graphs which send points to points, lines to lines.

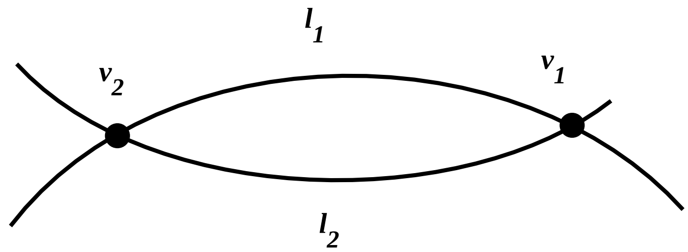

Figure 8: *Bigon.*

Definition 8.2 *An abstract* **based arrangement** *A is an arrangement together with a monomorphism of the standard triangle $T \hookrightarrow A$ that we call the* **canonical embedding**.

An arrangement A is called *admissible* if it satisfies the axiom:

(A1) *Every element of A is incident to at least two distinct elements.* (I.e. every point belongs to at least two lines and every line contains at least two points.)[5]

Suppose that A, B, C is a triple of arrangements and $\phi : C \to A, \psi : C \to B$ are monomorphisms. We define the *fiber sum* $A \times_C B$ as follows. First we take the disjoint union of the arrangements A and B. Then identify in $A \cup B$ the elements $\phi(c), \psi(c)$ for all $c \in C$. If A, B are based arrangements, C is as above, then their join $A *_C B$ is defined as $A \times_{T \sqcup C} B$, where T is the standard triangle with canonical embedding into A, B. If C is an arrangement which consists of a single *point* c and $\phi(c) = a \in A, \psi(c) = b \in B$, then we use the notation:

$$A *_{a \equiv b} B := A *_C B$$

8.2 Fiber products

We remind the reader of the definition of the *fiber product* of varieties (recall that our varieties are neither reduced nor irreducible). Let $f : X \to Z, g : Y \to Z$ be morphisms. Then the *fiber product* $X \times_Z Y$ of X and Y with respect to Z is a variety $X \times_Z Y$ together with canonical morphisms $\Pi_X : X \times_Z Y \to X$ and $\Pi_Y : X \times_Z Y \to Y$ such that the following diagram is commutative

$$\begin{array}{ccc} X \times_Z Y & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Z \end{array}$$

These data satisfy the property that given a variety W and a commutative diagram

$$\begin{array}{ccc} W & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Z \end{array}$$

we obtain a commutative diagram

$$\begin{tikzcd} W \arrow[rr] \arrow[dr] \arrow[dd] & & X \arrow[dd] \\ & X \times_Z Y \arrow[ur] \arrow[dl] & \\ Y \arrow[rr] & & Z \end{tikzcd}$$

[5] This axiom will be needed only in the Section 12.3.

The fiber product of quasi-projective (resp. projective) varieties is again a quasi-projective (resp. projective).

We recall how to describe graphs of morphisms via fiber products. Suppose that $f : X = \mathbb{A}^n \to \mathbb{A}^m = Z$ is a morphism. Let $g : Y = \mathbb{A}^m \to \mathbb{A}^m$ be the identity morphism. Define Γ_f, graph of f, to be the subvariety

$$\Gamma_f = X \times_Y Y = \{(x, y) \in X \times Y : f(x) = g(y) = y\}$$

Clearly $\pi_X : \Gamma_f \to X$ is an isomorphism of affine varieties.

Let $f : X = \mathbb{A}^n \to Y = \mathbb{A}^m$ be a morphism, Γ_f its graph and $\pi_X : \Gamma_f \cong \mathbb{A}^n$ be the canonical projection. We split $\mathbb{A}^n$ as $\mathbb{A}^{n-k} \times \mathbb{A}^k$, so $x \in \mathbb{A}^n$ is written as $x = (x', x'')$, $x' \in \mathbb{A}^{n-k}, x'' \in \mathbb{A}^k$. We obtain projections

$$\Pi'_X : \Gamma_f \to \mathbb{A}^{n-k} \quad \text{and} \quad \Pi''_X : \Gamma_f \to \mathbb{A}^k$$

defined by $\Pi'_X(x) = x'$ and $\Pi''_X(x) = x''$. Now let $g : \mathbb{A}^s \to \mathbb{A}^{n-k}$ be a morphism with graph Γ_g, let $y \in \mathbb{A}^s$ denote the variable in this space. Using the morphism $\Gamma_g \to \mathbb{A}^{n-k}$ (the second projection) and $\Pi'_X : \Gamma_f \to \mathbb{A}^{n-k}$ we form the fiber product

$$\Gamma_g \times_{\mathbb{A}^{n-k}} \Gamma_f \quad .$$

Now let $h : \mathbb{A}^s \times \mathbb{A}^k \to \mathbb{A}^m$ be the morphism given by

$$h(y, x'') = f(g(y), x'')$$

Lemma 8.3 *The projection map*

$$p : (\mathbb{A}^s \times \mathbb{A}^{n-k}) \times (\mathbb{A}^{n-k} \times \mathbb{A}^k) \times \mathbb{A}^m \to \mathbb{A}^s \times \mathbb{A}^k \times \mathbb{A}^m = T$$

given by

$$p((y, x'), (u', u''), z) = (y, u'', z)$$

induces an isomorphism

$$\Gamma_g \times_{\mathbb{A}^{n-k}} \Gamma_f \longrightarrow \Gamma_h \quad .$$

Proof: Obvious. $\square$

Corollary 8.4 *The morphism*

$$q : \Gamma_g \times_{\mathbb{A}^{n-k}} \Gamma_f \longrightarrow \mathbb{A}^s \times \mathbb{A}^k$$

(given by the restriction of p to $\Gamma_g \times_{\mathbb{A}^{n-k}} \Gamma_f$ and the the projection on the first and second factor of T) is an isomorphism.

Corollary 8.5 *Suppose that $k = 0$, then the composition of q with the projection on the first factor*

$$r : \Gamma_g \times_{\mathbb{A}^{n-k}} \Gamma_f \longrightarrow \mathbb{A}^s$$

is an isomorphism.

8.3 Intersection operations in the projective plane

Let $\mathcal{R}$ be a commutative ring. We recall that the projective space $\mathbb{P}(M)$ for a projective $\mathcal{R}$-module M of rank n is defined by

$$\mathbb{P}(M) := \{V \subset M : V \quad \text{is a projective submodule of rank 1 such that } M/V \text{ is projective}\}$$

We then define $\mathbb{P}^n(\mathcal{R})$ and $\mathbb{P}^n(\mathcal{R})^\vee$ by

$$\mathbb{P}^n(\mathcal{R}) := \mathbb{P}(\mathcal{R}^{n+1}), \quad \mathbb{P}^n(\mathcal{R})^\vee := \mathbb{P}(\mathrm{Hom}_{\mathcal{R}}(\mathcal{R}^{n+1}, \mathcal{R}))$$

We refer to [DG, §1.3.4, §1.3.9], to see that this is consistent with the usual definition of $\mathbb{P}^n$.

Note that an element $\alpha \in \mathbb{P}^2(\mathcal{R})^\vee$ gives rise to a projective $\mathcal{R}$-submodule $L \subset \mathcal{R}^3$ of rank 2, $L := \ker(\alpha)$, such that $\mathcal{R}^3/L$ is projective. We will call both α and L *lines* in $\mathbb{P}^2(\mathcal{R})$. We say that a point $V \in \mathbb{P}(M)$ belongs to a line L (corresponding to $\alpha \in \mathbb{P}^2(\mathcal{R})^\vee$) if and only if $V \subset L$; equivalently $\alpha(V) = 0$. Suppose $V \in \mathbb{P}^2(\mathcal{R})$ corresponds to a rank one *free* submodule of $\mathcal{R}^3$ with the basis $u = (x, y, z)$, then we will write $V := [x : y : z]$ (these are the homogeneous coordinates of V).

We now show how to do projective geometry over $\mathcal{R}$. We define two elements $Span(u_1)$, $Span(u_2) \in \mathbb{P}^2(\mathcal{R})$ to be *independent* if the submodule

$$L = Span\{u_1, u_2\} := \mathcal{R}u_1 + \mathcal{R}u_2$$

is a projective summand of $\mathcal{R}^3$ of rank 2. In this case we will also say that u_1, u_2 are independent.

Lemma 8.6 *If $Span(u_1)$ and $Span(u_2)$ are independent over $\mathcal{R}$ then $Span\{u_1, u_2\}$ is the unique projective summand of $\mathcal{R}^3$ containing u_1 and u_2.*

Proof: Let M denote $Span\{u_1, u_2\}$. Suppose that N is a projective summand containing u_1 and u_2. Then N contains M. We want to prove that $M = N$. We may assume that $\mathcal{R}$ is local whence M and N are free. Let $\{v_1, v_2\}$ be a basis for N. Note that $\{u_1, u_2\}$ is a basis for M. Write

$$u_1 = av_1 + cv_2, \quad u_2 = bv_1 + dv_2$$

Let $\mathbf{k}$ be the residue field of $\mathcal{R}$. The images of u_1 and u_2 in $\mathbf{k}^3$ are independent so the image of $ad - bc$ in k is nonzero. Hence $ad - bc$ is a unit in $\mathcal{R}$. $\square$

Thus two independent points $U_1 = Span(u_1), U_2 = Span(u_2) \in \mathbb{P}^2(\mathcal{R})$ belong to the unique line $L = Span\{u_1, u_2\}$ in $\mathbb{P}^2(\mathcal{R})$. We shall use the notation

$$L := U_1 \bullet U_2$$

for the line L through the points U_1, U_2. If $u \in \mathcal{R}^3$ we let $u^\vee$ denote the element of $(\mathcal{R}^3)^\vee$ given by $u^\vee(v) := u \cdot v = \sum_{i=1}^3 u_i v_i$. We have the following sufficient condition for independence:

Lemma 8.7 *Suppose that there exists $u_3 = (x_3, y_3, z_3) \in \mathcal{R}^3$ such that $(u_1 \times u_2) \cdot u_3 = 1$. Then u_1, u_2 are independent, moreover*

$$Span\{u_1, u_2\} = \ker(u_1 \times u_2)^\vee$$

Proof: The determinant of the matrix with the columns u_1, u_2, u_3 equals 1, whence $\{u_1, u_2, u_3\}$ is a basis for $\mathcal{R}^3$. Furthermore, suppose $v = au_1 + bu_2 + cu_3$. Then $(u_1 \times u_2) \cdot v = c$, so $c = 0$ if and only if $v \in \ker(u_1 \times u_2)^\vee$. $\square$

Remark 8.8 *We observe that u_3 as above always exists (and hence u_1, u_2 are independent) if one of the coordinates of $u_1 \times u_2$ is a unit in $\mathcal{R}$. In this case we will say that u_1 and u_2 satisfy the* **cross-product test** *for independence.*

Lemma 8.9 *If $u_1, u_2 \in \mathcal{R}^3$ above satisfy the* **cross-product test**, *then $V_1 = Span(u_1)$, $V_2 = Span(u_2)$ can be joined by the unique projective line $V_1 \bullet V_2$ in $\mathbb{P}^2(\mathcal{R})$ corresponding to $(u_1 \times u_2)^\vee$.*

Dual to the correspondence $\bullet : \mathbb{P}^2 \times \mathbb{P}^2 \to (\mathbb{P}^2)^\vee$ there is an operation of intersection of lines in $\mathbb{P}^2$. Namely, if λ, μ are lines in $\mathbb{P}^2$ such that $\lambda^\vee, \mu^\vee$ are independent points in $\mathbb{P}^2$, then we let $(\lambda \bullet \mu)^\vee = \lambda^\vee \bullet \mu^\vee$. Clearly $\lambda \bullet \mu = \ker(\lambda) \cap \ker(\mu)$. Suppose $L_j \in (\mathbb{P}^2)^\vee$ correspond to rank one free modules with bases $\sigma_j = (\alpha_j, \beta_j, \gamma_j)$, $(j = 1, 2)$. We will write $L_j = [\alpha_j : \beta_j : \gamma_j]$. We have

Lemma 8.10 *If σ_1, σ_2 above satisfy the cross-product test then L_1, L_2 intersect in the unique point $L_1 \bullet L_2$ with the homogeneous coordinates $[\sigma_1 \times \sigma_2]$.*

The *incidence variety* $\mathcal{I} \subset \mathbb{P}^2 \times (\mathbb{P}^2)^\vee$ is given by the equations:

$$\{(p, l) \in \mathbb{P}^2 \times (\mathbb{P}^2)^\vee | l(p) = 0\}$$

Let x, y, z be the coordinate functions on $\mathbb{C}^3$ relative to the standard basis and α, β, γ be the coordinate functions on $(\mathbb{C}^3)^*$ relative to the basis dual to $\overrightarrow{e_1}, \overrightarrow{e_2}, \overrightarrow{e_3}$. Then the homogeneous coordinate ring of $\mathcal{I}$ is isomorphic to

$$\frac{\mathbb{C}[x, y, z; \alpha, \beta, \gamma]}{(x\alpha + y\beta + z\gamma)}$$

For a general commutative ring $\mathcal{R}$ the set of $\mathcal{R}$-points $I(\mathcal{R}) \subset \mathbb{P}^2(\mathcal{R}) \times \mathbb{P}^2(\mathcal{R})^\vee$ consists of pairs (V, α) such that $V \subset L = \ker(\alpha)$.

Pick a point $t \in \mathbb{P}^2(\mathbb{C})$. The *relative* incidence variety $\mathcal{I}(t) \subset (\mathbb{P}^2)^\vee(\mathbb{C})$ is given by the equation:

$$\{l \in (\mathbb{P}^2)^\vee(\mathbb{C}) | l(t) = 0\}$$

By dualizing we define the *relative* incidence variety $\mathcal{I}(l)$ for any element $l \in (\mathbb{P}^2)^\vee(\mathbb{C})$. We define *anisotropic* incidence varieties $\mathcal{I}_0$, $\mathcal{I}_0(t)$ and $\mathcal{I}_0(l)$ by intersecting with $\mathbb{P}^2_0 \times (\mathbb{P}^2_0)^\vee$. The proof of the following lemma is a straightforward calculation and we leave it to the reader.

Lemma 8.11 *For any t and l the varieties $\mathcal{I}$, $\mathcal{I}(t)$, $\mathcal{I}(l)$ are smooth.*

Notation 8.12 *We will make the following convention about inhomogeneous coordinates of points in $\mathbb{P}^2$: if $q = [x : y : 1]$ then we let $q := (x, y)$, if $q = [0 : 1 : 0]$ we let $q := (0, \infty)$, if $q = [1 : 0 : 0]$ we let $q := (\infty, 0)$ and if $q = [1 : 1 : 0]$ we let $q := (\infty, \infty)$.*

8.4 Projective arrangements

A *geometric realization* of the abstract arrangement $A = \mathcal{P} \sqcup \mathcal{L}$ is a map

$$\phi : \mathcal{P} \sqcup \mathcal{L} \to \mathbb{P}^2(\mathbb{C}) \sqcup (\mathbb{P}^2(\mathbb{C}))^\vee$$

which sends *points* to points and *lines* to lines. This map must satisfy the following condition:

$$\iota(v, l) \Rightarrow \quad [\phi(v) \in \phi(l) \iff \quad \phi(l)^\vee \cdot \phi(v) = 0\ ] \tag{1}$$

The image of ϕ is a *projective arrangement* in $\mathbb{P}^2$. Usually we shall denote *lines* of arrangements A by uncapitalized letters (l, m, etc.), and their images under geometric realizations by the corresponding capital letters (L, M, etc.).

Example 8.13 *Consider the standard triangle T. Define a geometric realization ϕ_T of T in $\mathbb{P}^2$ so that:*

$$\phi_T(v_{00}) = (0,0), \phi_T(v_x) = (\infty, 0), \phi_T(v_y) = (0, \infty), \phi_T(v_{11}) = (1,1)$$

This realization uniquely extends to the rest of T. See Figure 9. We call ϕ_T the standard *realization of T.*

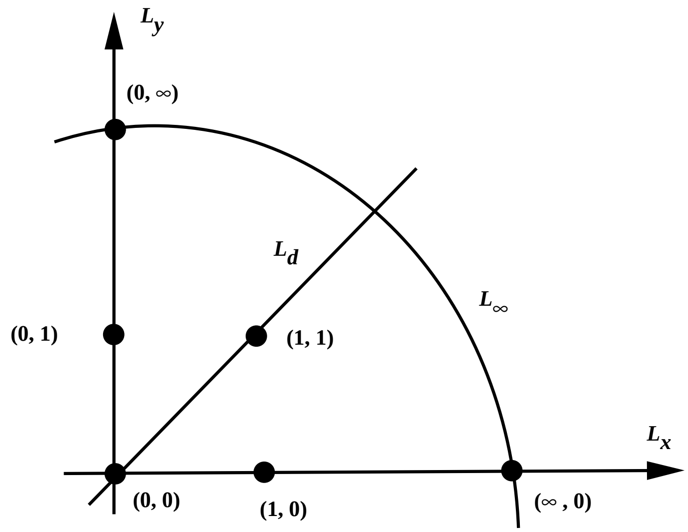

Figure 9: *Standard realization of the standard triangle.*

The *configuration space* of an abstract arrangement A is the space $R(A, \mathbb{P}^2(\mathbb{C}))$ of all geometric realizations of C. The space $R(A, \mathbb{P}^2(\mathbb{C}))$ is the set of $\mathbb{C}$-points of a projective variety $R(A)$ defined over $\mathbb{Z}$ with equations determined by the condition (1).

Remark 8.14 *We will consider $R(A)$ as a variety over $\mathbb{Q}$.*

We now give a concrete description of the homogeneous coordinate ring of $R(A)$ and the functor of points $R(A)(\bullet)$. Recall that x, y, z are the coordinate functions on $\mathbb{C}^3$ relative to the standard basis and α, β, γ are the coordinate functions on $(\mathbb{C}^3)^*$ relative to the basis dual to $\overrightarrow{e_1}, \overrightarrow{e_2}, \overrightarrow{e_3}$. Now let Γ be the (bipartite) graph corresponding to the abstract arrangement A. We let $\{v_1, ..., v_m\}$ denote the vertices of Γ corresponding to the *points* of A and let $\ell_1, ..., \ell_n$ denote the vertices of Γ corresponding to the *lines* of A. Let P be the polynomial ring defined by

$$P := \mathbb{C}[x_1, y_1, z_1, ..., x_m, y_m, z_m; \alpha_1, \beta_1, \gamma_1, ..., \alpha_n, \beta_n, \gamma_n]$$

If v_i and ℓ_j are incident we impose the relation $x_i\alpha_j + y_i\beta_j + z_i\gamma_j = 0$. The resulting quotient of P is the homogeneous coordinate ring of $R(A)$ which we denote $\mathbb{C}[R(A)]$. We next describe the functor of points $\mathcal{R} \to R(A)(\mathcal{R})$ where $\mathcal{R}$ is a commutative ring. It follows from the description of $\mathbb{P}^2(\mathcal{R})$ in §8.3 that the set $R(A)(\mathcal{R})$ is described by

Lemma 8.15 *Let $\mathcal{R}$ be a commutative ring. Then an $\mathcal{R}$-point $\psi \in R(A)(\mathcal{R})$ consists of an assignment of a* **point** *$V_i \in \mathbb{P}^2(\mathcal{R})$ for each* **point***-vertex $v_i \in \Gamma$ and a* **line** *$L_j \in \mathbb{P}^2(\mathcal{R})^\vee$ for each* **line***-vertex $\ell_j \in \Gamma$ such that $\iota(v_i, \ell_j)$ implies $V_i \subset L_j$.*

A *based realization* is a realization ϕ of a based abstract arrangement such that the restriction of ϕ to the canonically embedded triangle T is the *standard* realization ϕ_T.

The space of based realizations of an arrangement A will be denoted by $BR(A, \mathbb{P}^2(\mathbb{C}))$. It also is the set of $\mathbb{C}$-points of a projective variety $BR(A)$ defined over $\mathbb{Q}$. Suppose that A is a based arrangement which has distinguished set of *points* $\nu = \{v_1, ..., v_n\}$ which lie on the *line* $l_x \in T$. We call A_ν a *marked* arrangement. A *morphism of marked arrangements* is a mapping $h : A_\mu \to B_\nu$ such that $h(\mu) \subset \nu$. Note that fiber sum of two marked arrangements has the natural structure of a marked arrangement. If $\nu = \{v_1, ..., v_n\}$ is a marking we define the space of *finite realizations* as

$$BR_0(A_\nu) = \{\psi \in BR(A) : \quad \psi(v_j) \notin L_\infty \ , \quad v_j \in \nu\}$$

Clearly $BR_0(A)$ is a quasi-projective subvariety in $BR(A)$.

Lemma 8.16 *The maps R, BR and BR_0 define contravariant functors from the category of abstract arrangements, based abstract arrangements and marked arrangements to the category of projective and quasi-projective varieties defined over $\mathbb{Q}$.*

Proof: Obvious. $\square$

We leave the description of the coordinate rings and functors of points of $BR(C)$ and $BR_0(C)$ to the reader.

Theorem 8.17 *Under the functors R, BR and BR_0 the operation of fiber sum of arrangements corresponds to the operation of fiber product of projective varieties and quasi-projective varieties.*

Proof: Note that fiber sum of arrangements fits into the following commutative diagram of monomorphisms of arrangements:

$$\begin{tikzcd} & A \arrow[dr] & \\ C \arrow[ur] \arrow[r] \arrow[dr] & A \times_C B & \\ & B \arrow[ur] & \end{tikzcd}$$

We shall consider the case of the functor BR, the other two cases are similar. It follows from the previous lemma that we have commutative diagram of morphisms of varieties:

$$\begin{tikzcd} & BR(A) \arrow[dr] & \\ BR(A \times_C B) \arrow[ur] \arrow[rr] \arrow[dr] & & BR(C) \\ & BR(B) \arrow[ur] & \end{tikzcd}$$

Thus, by the universal property we get a morphism

$$\lambda : BR(A \times_C B) \to BR(A) \times_{BR(C)} BR(B)$$

To see that λ is an isomorphism we have only to check that it induces bijections of $\mathcal{R}$-points, for each commutative ring $\mathcal{R}$ (see [EH, Proposition IV-2]). This is clear by Lemma 8.15. $\square$

8.5 The moduli space of a projective arrangement

In this section we will construct a distinguished Mumford quotient $R(A, \mathbb{P}^2(\mathbb{C}))//PGL(3, \mathbb{C})$ which we will refer to as the *moduli space* $\mathcal{M}(A, \mathbb{P}^2(\mathbb{C}))$ for a based arrangement A. Since the equations defining $R(A, \mathbb{P}^2(\mathbb{C}))$ are invariant under $PGL(3, \mathbb{C})$ it suffices to construct a Mumford quotient of

$$(\mathbb{P}^2)^m \times ((\mathbb{P}^2)^\vee)^n$$

where m is the number of *points* in A and n is the number of *lines*. The *Mumford quotient* $R(A, \mathbb{P}^2(\mathbb{C}))//PGL(3, \mathbb{C})$ of $R(A, \mathbb{P}^2(\mathbb{C}))$ will be then the subvariety of the quotient variety cut out by the incidence equations. In the next section we will identify $\mathcal{M}(A, \mathbb{P}^2(\mathbb{C}))$ with $BR(A, \mathbb{P}^2(\mathbb{C}))$ which will be seen to be a cross-section to the action of $PGL(3, \mathbb{C})$ on an open subvariety of $R(A, \mathbb{P}^2(\mathbb{C}))$.

To construct a (weighted) Mumford quotient of $(\mathbb{P}^2(\mathbb{C}))^m \times ((\mathbb{P}^2(\mathbb{C}))^\vee)^n$ we need a projective embedding. Such an embedding corresponds to a choice of polarizing line bundle over each factor of the product. Since the group of isomorphism classes of line bundles on $\mathbb{P}^2(\mathbb{C})$ is infinite cyclic this amounts to assigning a positive integer weight to each factor (i.e. to each vertex of the graph Γ of A). It will be more convenient to assign positive rational weights to each vertex, then the integer weights are obtained by clearing the denominators. We choose a small positive rational number ϵ and assign the weight $\frac{1}{4} - \epsilon$ to each of the four *point vertices* v_{00}, v_x, v_y, v_{11} of $T \subset A$ and the weight ϵ to all other vertices. Let $W := \{v_{00}, v_x, v_y, v_{11}\}$.

Note that all semistable configurations for the four-point case are stable, see below. Thus it is clear that if ϵ is small enough the calculation of stable and semistable points will reduce to the corresponding calculation for $PGL(3, \mathbb{C})$ acting on the product of four copies of $\mathbb{P}^2(\mathbb{C})$ corresponding to the four *point*-vertices described above. This calculation is well-known (see [N]). A configuration is stable (resp. semistable) iff less than (resp. no more than) 1/3 of the total weight is concentrated on any point and less than 2/3 (resp. no more than 2/3) of the total weight is concentrated on any line. We obtain

Lemma 8.18 *All semistable configurations in $R(A, \mathbb{P}^2(\mathbb{C}))$ are stable. A realization $\psi : A \to \mathbb{P}^2(\mathbb{C}) \sqcup (\mathbb{P}^2(\mathbb{C}))^\vee$ is stable if and only if no three points of $\psi(W)$ lie on the same projective line in $\mathbb{P}^2(\mathbb{C})$.*

8.6 The moduli space of the standard triangle

As an important example we consider the configuration space and moduli space of the standard triangle T. First of all we note that $R(T, \mathbb{P}^2(\mathbb{C}))$ is not irreducible. Here is the reason. Let $\phi_T \in R(T, \mathbb{P}^2(\mathbb{C}))$ be the standard realization. Then all $\phi \in R(T, \mathbb{P}^2(\mathbb{C}))$ nearby are equivalent to ϕ_T under some projective transformation. However there are other (degenerate) realizations $\psi_d \in R(T, \mathbb{P}^2(\mathbb{C}))$. Namely, send all the *points* of T to the origin $(0,0) \in \mathbb{A}^2 \subset \mathbb{P}^2$. The triangle T has 6 *lines*, which can be sent to any 6 lines in $\mathbb{P}^2$ passing through $(0,0)$. This gives us a 6-parameter family F of degenerate realizations (which is the product of six copies of $\mathbb{P}^1$). After we mod out by the stabilizer of $(0,0)$ in $PGL(3, \mathbb{C})$ we get 3-dimensional quotient. There are several other components which interpolate between ϕ_T and ψ_d, namely when three of the *points* v_{00}, v_x, v_y, v_{11} belong to a common projective line. To remedy the problem we consider the *Mumford quotient* $R(T, \mathbb{P}^2(\mathbb{C}))//PGL(3, \mathbb{C})$ where we assign weights as in the previous section. Let $R_{ss}(T, \mathbb{P}^2(\mathbb{C}))$ be the set of semi-stable points with respect to these weights. Our definition of weights prevents images of any three of the *points* v_{00}, v_x, v_y, v_{11} from belonging to a common projective line in $\mathbb{P}^2(\mathbb{C})$. It is clear that $R_{ss}(T, \mathbb{P}^2(\mathbb{C})) = PGL(3, \mathbb{C})\{\phi_T\}$. Thus we get

Lemma 8.19 *The weighted quotient $\mathcal{M}(T)(\mathbb{C}) = R(T, \mathbb{P}^2(\mathbb{C}))//PGL(3, \mathbb{C})$ consists of a single point which we can identify with the cross-section $\{\phi_T\}$ for the action of $PGL(3, \mathbb{C})$ on $R_{ss}(T, \mathbb{P}^2(\mathbb{C}))$.*

Suppose now that A is a general based arrangement, we assign weights as above. Then $BR(A, \mathbb{P}^2(\mathbb{C})) \subset R_s(A, \mathbb{P}^2(\mathbb{C})) = R_{ss}(A, \mathbb{P}^2(\mathbb{C}))$. By Lemma 8.18, $BR(A, \mathbb{P}^2(\mathbb{C}))$ is a cross-section to the action of $PGL(3, \mathbb{C})$ on $R_s(A, \mathbb{P}^2(\mathbb{C}))$. We obtain

Theorem 8.20 *The inclusion $BR(A, \mathbb{P}^2) \hookrightarrow R(A)$ induces an isomorphism $BR(A, \mathbb{P}^2(\mathbb{C})) \cong \mathcal{M}(A, \mathbb{P}^2(\mathbb{C}))$ between projective varieties.*

8.7 Functional arrangements

Suppose that A is a based marked arrangement with the *marking* $\nu = \{v_1, ..., v_n\}$ (see §8.4). We call the points in ν the *input* points and we shall assume that $\nu \cap T = \emptyset$. We also suppose that A has the second marking $\mu = \{w_1, ..., w_s\}$, which consists of distinct *output* points $\{w_1, ..., w_s\}$. (The sets μ and ν can intersect and we allow $\mu \cap T \neq \emptyset$.) Recall that $\iota(v_i, l_x), \iota(w_j, l_x)$ for all i, j.

Define the *projection maps* from the spaces of finite realizations of $\Pi : BR_0(A_\nu) \to \mathbb{A}^n$, $\Delta : BR_0(A_\nu) \to \mathbb{P}^s$ by

$$\Pi : \phi \mapsto (\phi(v_1), ..., \phi(v_n)) = (z_1, ..., z_n) \in \mathbb{A}^n$$

$$\Delta : \phi \mapsto (\phi(w_1), ..., \phi(w_s)) = (y_1, ..., y_s) \in \mathbb{P}^s$$

(here we identify $L_x - \{(\infty, 0)\}$ with the affine line $\mathbb{A}$).

Definition 8.21 *Suppose that arrangement A above satisfies the following axioms:*

- **(A2)** $BR_0(A_\nu) \subset BR_0(A_\mu)$, *i.e.*

$$\psi(v_j) \notin L_\infty, 1 \le j \le n \quad \Rightarrow \quad \psi(w_i) \notin L_\infty, 1 \le i \le s \ .$$

- **(A3)** *The projection Π is a biregular isomorphism of the variety $BR_0(A_\nu)$ onto $\mathbb{A}^n$.*

Such arrangement A is called a **functional** *arrangement on n variables.*

Each functional arrangement *defines* a vector-function $f : \mathbb{A}^n \to \mathbb{A}^s$ by

$$f(z_1, ..., z_n) := \Delta(\phi) = (\phi(w_1), ..., \phi(w_s))$$

where $\phi \in BR_0(A_\nu)$ corresponds to $(z_1, ..., z_n)$ under the map Π. We shall record this by writing $A = A^f$. It's easy to see that the vector-function f must be polynomial. Later on we will give some examples of functional arrangements and we will prove that **any** m-tuple of polynomials in $\mathbb{Z}[x_1, ..., x_n]$ can be defined by a functional arrangement.

Lemma 8.22 *The space $BR_0(A^f)$ is biregular isomorphic to the graph Γ_f of the function $f : \mathbb{A}^n \to \mathbb{A}^s$.*

Proof: Indeed, the natural projection $\pi : \Gamma_f \to \mathbb{A}^n$ is an isomorphism. Compose it with the isomorphism Π^{-1}. The result is the required isomorphism $\Gamma_f \to BR_0(A^f)$. $\square$

Now suppose that we are given two functional arrangements A^f, A^g which define the functions $f : \mathbb{A}^n \to \mathbb{A}^s$ and $g : \mathbb{A}^t \to \mathbb{A}$. We denote the variables for f by $(z_1, ..., z_n)$ and the variables for g by $(x_1, ..., x_t)$. We assume that they correspond to the *input points* $p_1, ..., p_n$ and $q_1, ..., q_t$ respectively. Denote by $w_0 \in A^g$ the *output point*. We would like to find an arrangement which *defines* the function $h = f(g(x_1, ..., x_t), z_2, ..., z_n)$. To do this we let A^h be the join $A^f *_{p_1 \equiv w_0} A^g$.

Lemma 8.23 *The arrangement A^h is functional and it* **defines** *the polynomial h which is the above composition of the functions f, g.*

Proof: The Axiom (A3) for A_h follows from the Axiom (A2) for the arrangements A_f, A_g. The Axiom (A4) follows from the fact that $BR_0(A^h)$ is the fiber product of the varieties $BR_0(A^f)$, $BR_0(A^g)$, see Lemma 8.3. $\square$

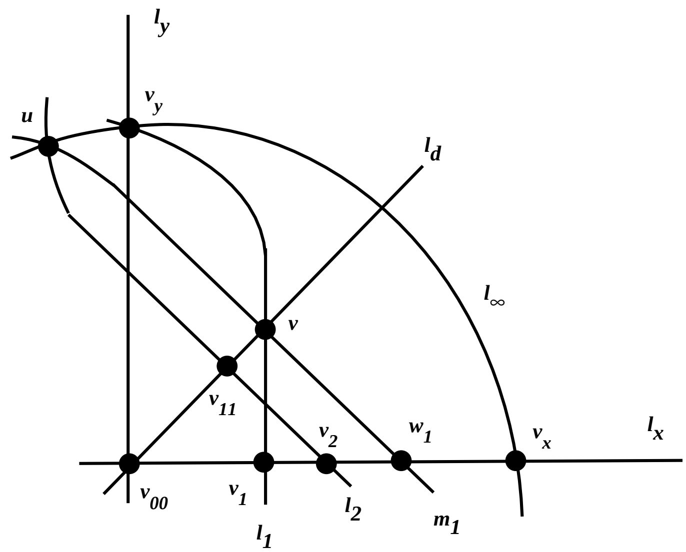

Figure 10: *The abstract arrangement C_M for the multiplication.*

9 Algebraic operations via arrangements

The following theorem goes back to the work of von Staudt [St] in the middle of the last century:

Theorem 9.1 *There are admissible functional arrangements C_A, C_M which* define *the functions*

$$A(z_1, z_2) = z_1 + z_2\ , \quad M(z_1, z_2) = z_1 \cdot z_2$$

Proof: Consider the functional arrangement C_M described on Figure 10. We omitted from the figure several (inessential) *lines* and *points* of the standard triangle T, however we still assume that C_M is a *based arrangement*. A generic projective realization ψ of this arrangement is described on Figure 11. Then the point of intersection of the line $M_1 := \psi(m_1)$ and the x-axis L_x is equal to ab. (See [H, page 45].) The addition is *defined* via the abstract arrangement on the Figure 12. Generic projective realization of C_A is described on the Figure 13. (See [H, page 44].) We will prove that C_M is a functional arrangement, leaving similar case of C_A to the reader. The Axioms **(A1)**, **(A2)** are clearly satisfied by the arrangement C_M , it is also easy to see that the morphism $\Pi : BR_0(C_M) \to \mathbb{A}^2$ is a bijection of complex points.

The problem is to prove that Π is invertible as a morphism. The example that the reader should keep in mind is the following. Consider the identity map

$$id : \{x = 0 : x \in \mathbb{C}\} \longrightarrow \{x^2 = 0 : x \in \mathbb{C}\}$$

Then id is a morphism which is bijective on complex points but not invertible as a morphism.

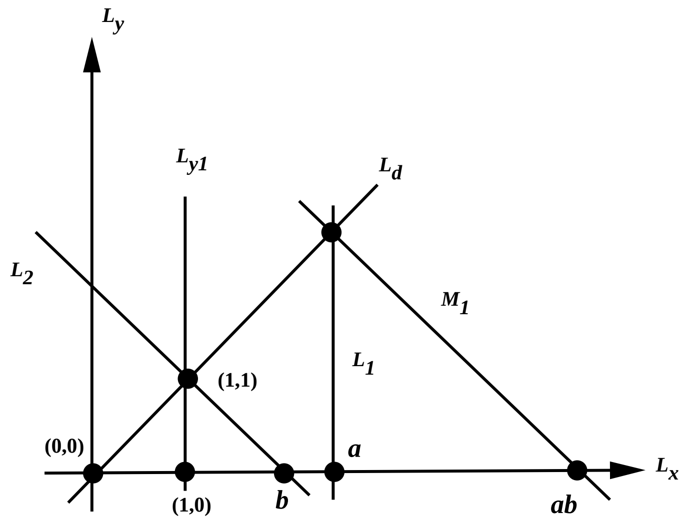

Figure 11: *Projective arrangement for multiplication.*

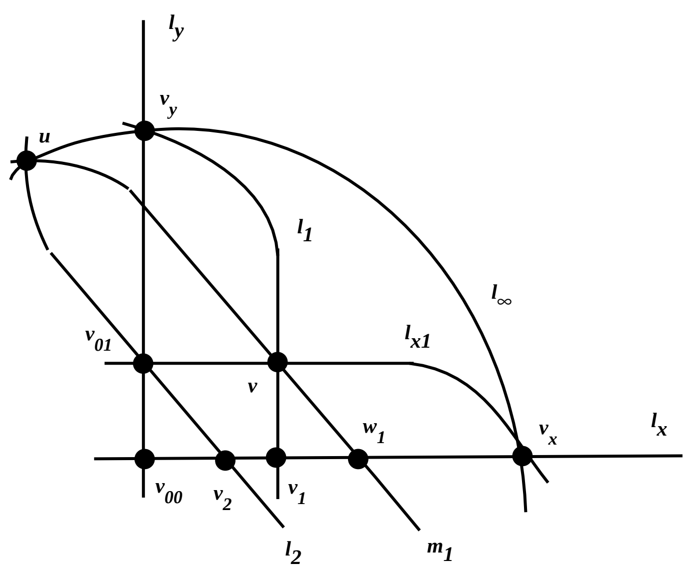

Figure 12: *The abstract arrangement C_A for addition.*

We will prove that $\Pi : BR_0(C_M) \to \mathbb{A}^2$ induces a bijection Π_R of $\mathcal{R}$-points for any commutative ring $\mathcal{R}$. This will imply that Π is an isomorphism by [EH, Proposition IV-2].

Let A_ν be a based marked arrangement. We first interpret the finiteness condition $\psi(v_j) \notin L_\infty, v_j \in \nu$, scheme-theoretically for any $\mathcal{R}$-valued point $\psi \in BR_0(A)(\mathcal{R})$. We will construct a subfunctor $U \subset \mathbb{P}^2$ corresponding to the affine plane $\mathbb{P}^2 - L_\infty$. Let $\mathcal{R}$ be a commutative ring, $\{e_1, e_2, e_3\}$ be the standard basis of $\mathcal{R}^3$ and let $M \subset \mathcal{R}^3$ be a submodule defined by $M = \mathcal{R}e_1 + \mathcal{R}e_2$. We define $U(\mathcal{R}) \subset \mathbb{P}^2(\mathcal{R})$ by

$$U(\mathcal{R}) = \{V \in \mathbb{P}^2(\mathcal{R}) : V \ \text{ is a complement to } \ M\}$$

It is then immediate, see [DG, §1.3.9], that $V \in U(\mathcal{R})$ implies that V is free and contains a unique vector (necessarily a basis) u of the form $u = xe_1 + ye_2 + e_3$. The map $\omega : U(\mathcal{R}) \to \mathcal{R}^2 + e_3$ given by $\omega(V) = (x, y, 1)$ is a natural (with respect to $\mathcal{R}$) bijection. Consequently U is represented by the polynomial algebra $\mathbb{Z}[X, Y]$. By [DG], loc. cit., U corresponds to an open subvariety, again denoted U, of $\mathbb{P}^2$ isomorphic to $\mathbb{A}^2$ (by the previous sentence).

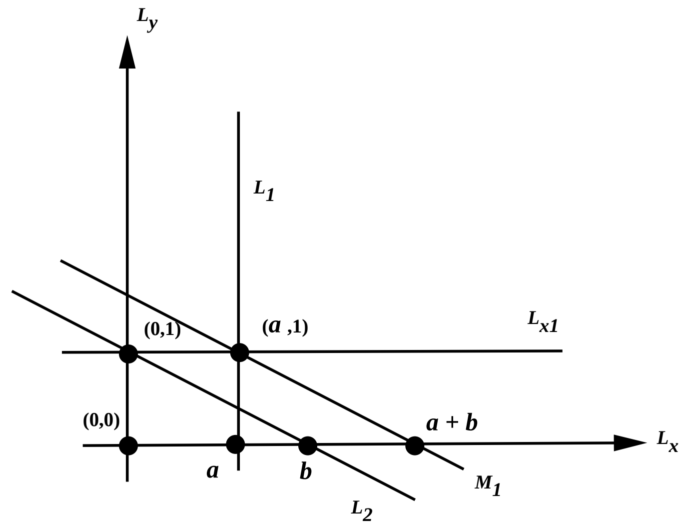

Figure 13: *Projective arrangement for addition.*

We observe that $U(\mathbb{C}) = P^2(\mathbb{C}) - L_\infty$. Thus U gives us the scheme-theoretic definition of $\mathbb{P}^2 - L_\infty$.

We now give the scheme-theoretic definition of the space of *finite* realizations $BR_0(A_\nu)$ by requiring that $\psi(v_j) \in U$, $v_j \in \nu$. In the case $\psi \in BR_0(A_\nu)(\mathcal{R})$ we have

$$\omega(\psi(v_j)) = (x_j, y_j, 1), v_j \in \nu$$

Moreover, since $\iota(v_j, l_x)$ we have: $y_j = 0, v_j \in \nu$. Thus we have associated a point $(x_1, ..., x_n) \in \mathcal{R}^n$ to each $\psi \in BR_0(A_\nu)(\mathcal{R})$. We will see in what follows that the co-ordinates $(x_1, ..., x_n)$ will play the same role as complex or real variables in the classical arguments.

We are now ready to prove

Proposition 9.2 $\Pi_{\mathcal{R}} : BR_0(C_M)(\mathcal{R}) \longrightarrow \mathcal{R}^2$ *is a bijection.*

Proof: Let $\psi \in BR_0(C_M)(\mathcal{R})$. We will prove that ψ is determined by $\psi(v_1), \psi(v_2)$ and the incidence relations in C_M. If $u = (x, y, z)$ we will use $[x : y : z]$ below to denote both the element in $\mathbb{P}^2(\mathcal{R})$ with the basis u and the element in $\mathbb{P}^2(\mathcal{R})^\vee$ with the basis $u^\vee$; the meaning will be clear from the context.

We now prove the proposition. Let $\omega(\psi(v_1)) = (a, 0, 1)$ and $\omega(\psi(v_2)) = (b, 0, 1)$. Now $\omega(\psi(v_1)) \times (0, 1, 0) = (1, 0, -a)$. Hence by the cross-product test $[1 : 0 : -a]$ is the unique line joining $\psi(v_1)$ and $[0, 1, 0]$. But $\psi(l_1) = \psi(v_1) \bullet L_y$, whence $\psi(l_1) = [1 : 0 : -a]$. We continue in this way, at each stage the cross-product test applies and we find in order

$$\psi(v) = \psi(l_1) \bullet \psi(l_d) = [a : a : 1]$$

$$\psi(l_2) = \psi(v_{11}) \bullet \psi(v_2) = [1 : b - 1 : -b]$$

$$\psi(u) = \psi(l_2) \bullet \psi(l_\infty) = [b - 1 : -1 : 0]$$

$$\psi(m_1) = \psi(u) \bullet \psi(v) = [-1 : 1 - b : ab]$$

$$\psi(w_1) = \psi(m_1) \bullet \psi(l_x) = [ab : 0 : 1]$$

This concludes the proof of Theorem 9.1 in the case of the arrangement C_M. The argument in the case of C_A is similar and is left to the reader. $\square$

Lemma 9.3 *For the function $D(x,y) = x - y$ there is a functional arrangement C_D which defines the function $D(x,y)$.*

Proof: Take the arrangement C_A corresponding to the function $A(x,y) = x + y$ and reverse the roles of input-output *points* v_2, w_1. $\square$

Remark 9.4 *Suppose that A is one of the arrangements C_A, C_M, C_D and $\psi \in BR(A, \mathbb{P}_2(\mathbb{C}))$ is a realization such that $\Pi(\psi) = (0,0)$, hence $\psi(w_1) = 0$, $\psi(m_1) = L_y$. Then $\psi(A) \subset \phi_T(T) \cup \{(\infty,\infty)\}$. (Recall that ϕ_T is the standard realization of the standard triangle.) Let's verify this for C_M. If $\psi(v_1) = 0, \psi(v_2) = 0$ then $\psi(l_1) = \psi(l_y), \psi(l_2) = \psi(l_d)$. Hence $\psi(u) = (\infty,\infty)$ and $\psi(v) = (0,0)$.*

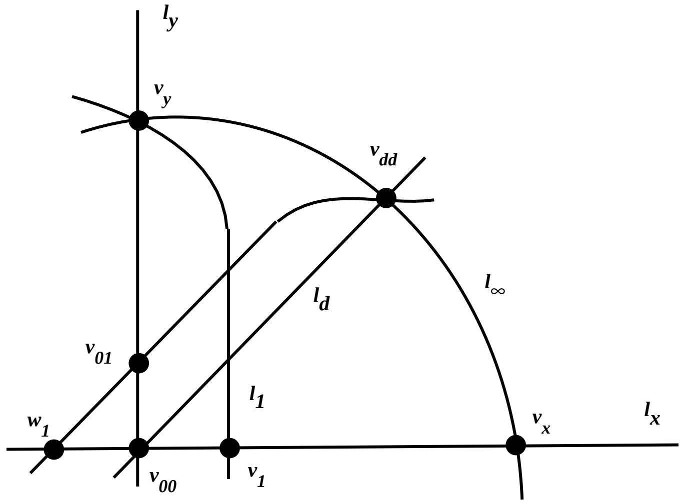

Figure 14: *Projective arrangement for the constant function $f_-(z) = -1$.*

Lemma 9.5 *There exist admissible functional arrangements C^+, C^- which define the constant functions*

$$f_+(z) = 1, f_-(z) = -1$$

We describe configuration only for the function f_-, the other case is similar. Consider the arrangement on the Figure 14, as usual we omit inessential lines and points form the standard triangle. Let v_1 be the input and w_1 be the output points. Under each realization ϕ of C^- the image $\phi(w_1)$ is the point $(-1,0)$. The image of l_1 is any vertical line in $\mathbb{A}^2$. We leave the proof of the fact that the arrangement C^- is functional to the reader. $\square$

Corollary 9.6 *For any polynomial f of n variables with integer coefficients there exists a functional arrangement A^f which* defines *f.*

Proof: Any such polynomial is a composition of the constant functions $f_\pm$, addition and multiplication. Thus the assertion is a straightforward application of Lemmas 8.23, 9.1, 9.3, 9.5 and Corollary 8.17. $\square$

Now we construct arrangements corresponding to polynomial vector-functions defined over $\mathbb{Z}$.

Lemma 9.7 *Suppose that $f_1, ..., f_m \in \mathbb{Z}[x_1, ..., x_n]$. Then there exists an admissible abstract functional arrangement A^F which* defines *the vector-function $F = (f_1, ..., f_m)$.*

Proof: By Corollary 9.6 there exist functional arrangements $A^{f_1}, ..., A^{f_m}$ which *define* the polynomials f_j. Let v_{ij} be input point of A^{f_i} corresponding to the variable x_j, $1 \le i \le m$, $1 \le j \le n$. Let $C = \{v_1, ..., v_n\}$ be the arrangement which consists only of *points* v_j and has no *lines*, $B = C \cup T$. For each *point* v_j define $\psi_i : v_j \mapsto v_i$, this gives us embeddings $\psi_i : C \hookrightarrow A^{f_i}$. Using these embeddings define $A = (A^{f_1} * ... * A^{f_m}) \times_C B$. In the arrangement A we have $v_{ij} \equiv v_{kj} \equiv v_j$, $1 \le j, k \le n$, these are the *input points* of A. The output points $w_1, ..., w_m$ correspond to the *output points* of the functional arrangements $A^{f_1}, ..., A^{f_m}$. Then the fact that $A = A^F$ is the functional arrangement *defining* the vector-function F follows by iterated application of Corollary 8.5 and Theorem 8.17 similarly to the proof of Lemma 8.23. $\square$

10 Systems of polynomial equations

Suppose that we have a system of polynomial equations defined over $\mathbb{Q}$:

$$\Sigma = \begin{cases} f_1(x) = 0 \\ f_2(x) = 0 \\ \quad\vdots \\ f_m(x) = 0 \end{cases}$$

where $x = (x_1, ..., x_n)$, $x_j \in \mathbb{C}$. These equations determine an affine variety $S \subset \mathbb{A}^n$ defined over $\mathbb{Q}$. In the previous section we had constructed a functional arrangement $A = A^F$ which *defines* the vector-function $F = (f_1, ..., f_m)$. Recall that we have two projection morphisms

$$\Pi : BR_0(A, \mathbb{P}^2) \to \mathbb{A}^n \ , \quad \Delta : BR_0(A, \mathbb{P}^2) \to \mathbb{A}^m$$

so that the diagram of morphisms

$$\begin{array}{ccc} BR_0(A, \mathbb{P}^2) & \stackrel{\Pi}{\longleftrightarrow} & \mathbb{C}^n \\ \Delta\downarrow & & F\downarrow \\ \mathbb{C}^m & = & \mathbb{C}^m \end{array}$$

is commutative. By the Axiom (A3) the projection Π is an isomorphism. Let $T \subset A$ be the standard triangle, $w = v_{00}$ be its vertex. Define the new abstract arrangement A^Σ as the join

$$A^\Sigma = (\ldots((A *_{w_1 \equiv w} T) *_{w_2 \equiv w} T) \ldots *_{w_m \equiv w} T)$$

Then $BR_0(A^\Sigma)$ is the fiber product $\{\psi \in BR_0(A, \mathbb{P}^2) : \Delta(\psi) = 0\}$ and $BR_0(A^\Sigma) \cong \{x \in \mathbb{A}^n : F(x) = 0\}$, where the isomorphism is given by the restriction τ of Π to $BR_0(A)$. Thus we get the following

Theorem 10.1 *For any system of polynomial equations with integer coefficients*

$$\Sigma = \begin{cases} f_1(x) = 0 \\ f_2(x) = 0 \\ \quad\vdots \\ f_m(x) = 0 \end{cases}$$

there is an admissible based arrangement $A = A^\Sigma$ such that $\tau : BR_0(A, \mathbb{P}^2) \cong S$ is an isomorphism of quasi-projective varieties over $\mathbb{Q}$.

Definition 10.2 *We call the morphism $geo = \tau^{-1}$* **geometrization***: it allows us to do algebra (i.e. solve the system Σ of algebraic equations) via geometry (i.e. by studying the space of projective arrangements).*

Note that the arrangement $A = A^{\Sigma}$ is not uniquely determined by the affine variety S but also by its affine embedding (the system of polynomial equations) and particular formulae used to describe these equations. For instance, the equation $x^5 = 0$ can be described as $(x^2 \cdot x^2) \cdot x$ as well as $x \cdot (x^2 \cdot x^2)$ and $(x \cdot (x^2)) \cdot x^2$, etc.

Suppose that the system of equations Σ is defined over $\mathbb{Z}$ and has no constant terms. Then we can rewrite the system Σ so that it does not involve multiplicative constants, for instance the equation $2x^2y + (-1)z = 0$ is equivalent to

$$x^2y + x^2y - z = 0$$

Then the only subarrangements involved in construction of the arrangement A^{Σ} are arrangements for the multiplication, addition and subtraction (described in Lemmas 9.1, 9.3) and we do not need arrangements for the constant functions ± 1. Let $\psi_0 \in BR(A, \mathbb{P}^2)$ be the realization corresponding to the point $0 \in S$ under the isomorphism τ. Take any *line* $l \in A - T$ and a *point* $v \in A - T$. Then by using Remark 9.4 and the fact that arrangement for the composition of functions is a join of their arrangements (so it has no new points or lines) we conclude that the following holds:

Lemma 10.3 *$\psi_0(l)$ is one of the* **lines** *L_x, L_y, L_d and $\psi_0(v)$ is one of the* **points**

$$(0,0)\ ,(0,\infty)\ ,(\infty,0)\ ,(\infty,\infty)$$

for each **line** *$l \in A - T$ and each* **point** *$v \in A - T$.*

Suppose now that Q is an affine variety defined over $\mathbb{Q}$ and $q \in Q$ is a rational point. Then we can realize Q as an affine subvariety $S \subset \mathbb{A}^N$ defined over $\mathbb{Z}$ so that q goes to zero. Hence we get the following

Corollary 10.4 *For any affine variety Q defined over $\mathbb{Q}$ and a rational point $q \in Q$ there exists an abstract admissible arrangement A and an isomorphism[6] $geo : Q \to BR_0(A)$ so that the point q corresponds to a realization ψ so that $\psi(l)$ is one of the* **lines** *L_x, L_y, L_d and $\psi(v)$ is one of the* **points**

$$(0,0)\ ,(0,\infty)\ ,(\infty,0)\ ,(\infty,\infty)$$

for each **line** *$l \in A - T$ and each* **point** *$v \in A - T$.*

11 Groups corresponding to abstract arrangements

We will define several classes of groups corresponding to abstract arrangements. Let $\Gamma = \Gamma_A$ be the bipartite graph corresponding to an abstract arrangement A. We first construct the Coxeter group G^c_A without assuming that A is a based arrangement: we assign the label 2 to all edges of Γ and let $G^c_A := G^c_\Gamma$.

From now on we suppose that A is a based arrangement. We start by identifying the *point* v_{00} with the *line* l_∞, the *point* v_x with the *line* l_y and the *point* v_y with the *line* l_x in the standard triangle T. We also introduce the new edges

$$[v_{10}, v_{00}], \quad [v_{01}, v_{00}], \quad [v_{11}, v_{00}]$$

[6]of varieties defined over $\mathbb{Q}$

(Where $v_{10}, v_{00}, v_{11}, v_{01}$ are again *points* in the standard triangle T.) We will use the notation $\Lambda = \Lambda_A$ for the resulting graph. Put the following labels on the edges of Λ:

1) We assign the label 4 to the edges $[v_{10}, v_{00}], [v_{01}, v_{00}]$ and all the edges which contain v_{11} as a vertex (with the exception of $[v_{11}, v_{00}]$). We put the label 6 on the edge $[v_{11}, v_{00}]$.

2) We assign the label 2 to the rest of the edges.

Let $\Delta := \Lambda - [v_{11}, v_{00}]$. Now we have labelled graphs and we use the procedure from the Section 4 to construct:

(a) The *Artin group* $G_A^a := G_\Lambda^a$.

(b) We assign the label 3 to the vertex v_{11} and labels 2 to the rest of the vertices. Then we get the *Shephard* group $G_A^s := G_\Lambda^s$.

We will denote generators of the above groups g_v, g_l, where v, l are elements of A (corresponding to vertices of Λ).

Remark 11.1 *Suppose that A is a based arrangement. Then the group G_A^s admit an epimorphism onto a free product of at least 3 copies of $\mathbb{Z}/2$ and the group G_A^a has an epimorphism onto a free group of rank $r \geq 3$, where $r+3$ is the number of* **lines** *in A. Let's check this for $G = G_A^a$. Construct a new arrangement B by removing all the* **points** *in A (and the* **lines** *l_x, l_y, l_∞). Then $G_B^s \cong \mathbb{Z} * ... * \mathbb{Z}$ is the r-fold free product, where $r \geq 3$ is the number of lines in $A - \{l_x, l_y, l_\infty\}$. It is clear that we have an epimorphism $G_A^a \to G_B^a$. Hence all the groups G_A^a, G_A^s are not virtually solvable.*

As an illustration we describe an example of a labelled graph corresponding to the based functional arrangement *defining* the function $x \mapsto x^2$, see the Figure 15.

12 Representations associated with projective arrangements

This section is in a sense the heart of the paper. We start with an outline of the main idea behind it[7]. A projective arrangement ψ is *anisotropic* of $\psi(v) \in \mathbb{P}_0^2, \psi(l) \in (\mathbb{P}_0^2)^\vee$, all points and lines $v, l \in A$. The anisotropic condition defines Zariski open subsets of the previous arrangement varieties to be denoted $R(A, \mathbb{P}_0^2), BR(A, \mathbb{P}_0^2)$ and $BR_0(A, \mathbb{P}_0^2)$ respectively.

We now describe the morphism

$$alg : BR(A, \mathbb{P}_0^2) \to Hom(G_A^s, SO(3))$$

As we already saw in the Section 6, the correspondence between involutions in $\mathbb{P}^2$ and their isolated fixed points is a biregular isomorphism between PQ_0 and $\mathbb{P}_0^2$: any point x in $\mathbb{P}_0^2$ uniquely determines the "Cartan involution" around this point, i.e. the involution such that x is the isolated fixed point. The point $x = (1,1)$ in $\mathbb{A}^2 \subset \mathbb{P}^2$ also determines (uniquely up to inversion) the rotation of the order 3 around x, so that x is the neutral fixed point. (We will choose one of these rotations once and for all.) Similarly any line $L \in (\mathbb{P}_0^2)^\vee$ uniquely determines the reflection which keeps L pointwise fixed. Finally we encode the incidence relation between points and lines in $\mathbb{P}^2$ using algebra: two involutions generate the subgroup $\mathbb{Z}/2 \times \mathbb{Z}/2$ in $PO(3)$ iff the isolated fixed point of one belongs to the fixed line of another, rotations of the orders 2 and 3 anticommute iff the neutral fixed point of the rotation of the order 3 belongs to the fixed line of the involution, etc. Thus, given a geometric object (an anisotropic projective arrangement) we can constructs an algebraic object (a representation of the associated Shephard group). We call the mapping

$$alg : \quad \text{anisotropic projective arrangements} \quad \longrightarrow \quad \text{representations}$$

[7]Certain versions of this idea were used previously in our papers [KM1], [KM2], [KM3].

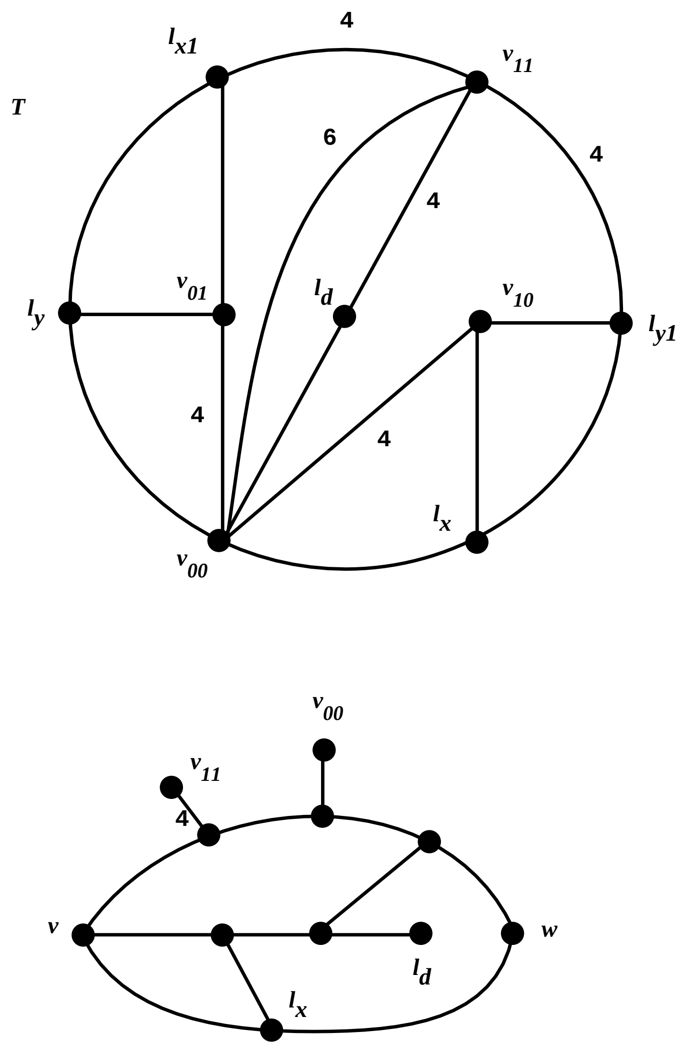

Figure 15: *Labelled graph of the functional arrangement for the function x^2. Identify vertices with the same labels. The* point v *is the "input", the* point w *is the "output".*

algebraization. This mapping is the key in passing from realization spaces of projective arrangements to representation varieties. The fact that this correspondence is a homeomorphism between sets of $\mathbb{C}$-points of appropriate subvarieties will be more or less obvious, however we will prove more: the *algebraization* is a biregular isomorphism of certain (quasi-) projective varieties, the latter requires more work.

12.1 Representations of Shephard groups

Let A be an abstract based arrangement with the graph Γ_A and G_A^s be the corresponding Shephard group with the graph Λ_A. For all edges e in the graph Λ_A the edge subgroups $G_e \subset G_A^s$ are finite. Recall that in Section 7.8 we had defined the space $\mathrm{Hom}_f(G, PO(3, \mathbb{C}))$ of nondegenerate representations of Shephard groups G. Thus as a direct corollary of Proposition 7.19 we get

Corollary 12.1 $\mathrm{Hom}_f(G^s, PO(3, \mathbb{C}))$ *is Zariski open and closed in* $\mathrm{Hom}(G^s, PO(3))$*.*

The significance of nondegenerate representations of Shephard groups is that they correspond to projective arrangements under the "algebraization" morphism alg.

Definition 12.2 *Suppose that $\rho \in \mathrm{Hom}_f(G^s, PO(3,\mathbb{C}))$ is a representation, then we associate a projective arrangement $\psi = \alpha(\rho) \in R(A, \mathbb{P}^2_0(\mathbb{C}))$ as follows:*

(a) If $v \in A$ is a point then we let $\psi(v)$ be the neutral fixed point of $\rho(g_v)$.

(b) If $l \in A$ is a line then we let $\psi(l)$ be the fixed line of the involution $\rho(g_v)$.

Clearly $\psi(v) \in \mathbb{P}^2_0$ and $\psi(l) \in (\mathbb{P}^2_0)^\vee$. Now we verify that ψ respects the incidence relation. Consider edges e in Λ_A connecting *points* to *lines*. Suppose that $e = [v, w]$ is an edge in Λ_A where v, w, e have the label 2. Then, $\rho \in \mathrm{Hom}_f(G_e, PO(3,\mathbb{C}))$ implies that $\rho(g_v), \rho(g_w)$ *anticommute* and hence $\psi(v) \cdot \psi(l) = 0$ (see §6.3). All other edges $e = [v, l]$ are labelled as: $\delta(v) = 3, \delta(l) = 2, \epsilon(e) = 4$, thus $\rho \in \mathrm{Hom}_f(G_e, PO(3,\mathbb{C}))$ again implies that $\rho(g_v), \rho(g_l)$ *anticommute*:

$$(\rho(g_v)\rho(g_l))^2 = \mathbf{1}$$

and hence $\psi(v) \cdot \psi(l) = 0$ (see §6.3). Note that the mapping

$$\alpha : \mathrm{Hom}_f(G^s, PO(3,\mathbb{C})) \to R(A, \mathbb{P}^2_0(\mathbb{C})), \quad \rho \mapsto \psi = \alpha(\rho)$$

is 2-1. Namely, we can modify any representation $\rho \in \mathrm{Hom}_f(G^s, PO(3,\mathbb{C}))$ by taking $\rho_-(g_{v_{11}}) := \rho(g_{v_{11}})^{-1}$ and $\rho_-(g_w) := \rho(g_w)$ for all $w \in V(\Gamma) - \{v_{11}\}$, then $\alpha(\rho) = \alpha(\rho^-)$. We denote the involution $\rho \mapsto \rho_-$ by ν. The mapping α is far from being onto $R(A, \mathbb{P}^2_0(\mathbb{C}))$ because of extra identifications and edges in the graph Λ (comparing to the Γ_A). The mapping *alg* will be the right-inverse to α if we restrict the target of α to *based realizations*:

Definition 12.3 *Suppose that A is an abstract based arrangement, $\psi \in BR(A, \mathbb{P}^2_0)$ is a realization.*[8] *We construct a homomorphism*

$$alg(\psi) = \rho_\psi : G^s_A \to PO(3,\mathbb{C})$$

as follows. If v is a point *in $A - \{v_{11}\}$, let $\rho(g_v)$ be the rotation of order 2 in $\mathbb{P}^2$ with the neutral fixed point $\psi(v)$. (Such rotation exists since $\psi(v)$ is anisotropic.) If $v = l$ is a* **line** *in A we let $\rho(g_v)$ be the reflection in the line $\psi(l)$ (equivalently this is the rotation of order 2 with the isolated fixed point $\psi(l)^\vee \in \mathbb{P}^2_0$. For the vertex $v = v_{11}$ we take the rotation of the order 3 around the point $(1,1)$. There are two such rotations, so we shall choose $\rho(g_v) : \psi(v_{00}) \mapsto \psi(v_x)$.*

Below we verify that ρ respects relations in G^s and determines a nondegenerate representation. For each $v \in \mathcal{P} - \{v_{11}\} \subset A$ and each *line* $l \in \mathcal{L} \subset A$ we have

$$\psi(v) \in \psi(l) \Rightarrow [\rho(g_l), \rho(g_v)] = \mathbf{1}, \quad \rho(g_l) \neq \rho(g_v)$$

Hence all the relations in G^s corresponding to the edges labelled by 2 are preserved by ρ and ρ is *nondegenerate* on such edges. Since the spherical distances between the points $(0,0)$ and $(1,0)$, and points $(0,0)$ and $(0,1)$ in $\mathbb{RP}^2$ equal $\pi/4$ we conclude that

$$(g_{v_{00}} g_{v_{10}})^4 = \mathbf{1} \quad , \quad (g_{v_{00}} g_{v_{01}})^4 = \mathbf{1}, (g_{v_{00}} g_{v_{10}})^2 \neq \mathbf{1}, (g_{v_{00}} \neq g_{v_{01}})^2 \neq \mathbf{1}$$

This implies that the Artin relations for the edges $[v_{00}, v_{10}], [v_{00}, v_{01}]$ are preserved by ρ and ρ is nondegenerate on these edges.

For each *line* l incident to the *point* v_{11} the rotations $\rho(g_l)$ and $\rho(g_{v_{11}})$ anticommute by Lemma 6.7 and we get:

$$\rho(g_l)\rho(g_{v_{11}})\rho(g_l)\rho(g_{v_{11}}) = \mathbf{1} \ , \quad \rho(g_{v_{11}})\rho(g_l)\rho(g_{v_{11}})\rho(g_l) = \mathbf{1}$$

[8] In particular $\psi(v_{00}) = \psi(l_\infty)^\vee, \psi(v_x) = \psi(l_y)^\vee, \psi(v_y) = \psi(l_x)^\vee$.

Hence the Artin relations for the edges $[l, v_{11}] \subset \Lambda$ are preserved by ρ. It is easy to check that the order of the element $\rho(g_{v_{00}} g_{v_{11}})$ equals 3. Recall that the edge $e = [v_{00}, v_{11}]$ in Λ has the label 6. Thus the relation

$$(g_{v_{00}} g_{v_{11}})^3 = (g_{v_{11}} g_{v_{00}})^3$$

associated with the edge e is preserved by ρ. We have proved

Proposition 12.4 *The mapping $alg : BR(A, \mathbb{P}^2_0(\mathbb{C})) \to \mathrm{Hom}_f(G^s_A, PO(3,\mathbb{C}))$ is such that $\alpha \circ alg = id$. The space $BR(A, \mathbb{P}^2_0(\mathbb{C}))$ lies in the image of the mapping α.*

Let $Res_T : \mathrm{Hom}_f(G^s_A, PO(3,\mathbb{C})) \to \mathrm{Hom}_f(G^s_T, PO(3,\mathbb{C}))$ be the restriction homomorphism. Define the varieties

$$\mathrm{BHom}_f(G^s_A, PO(3,\mathbb{C})) := Res_T^{-1}\{\rho_{\phi_T}, \nu(\rho_{\phi_T})\}$$

$$\mathrm{BHom}^+_f(G^s_A, PO(3,\mathbb{C})) := Res_T^{-1}\{\rho_{\phi_T}\}$$

of *based representations*. Clearly $\mathrm{BHom}^+_f(G^s_A, PO(3,\mathbb{C}))$ is the image $alg(BR(A, \mathbb{P}^2_0(\mathbb{C})))$ (as a set) and the mapping

$$alg : BR(A, \mathbb{P}^2_0(\mathbb{C})) \to \mathrm{BHom}^+_f(G^s_A, PO(3,\mathbb{C}))$$

is a bijection.

Lemma 12.5 *The representation $\rho = \rho_{\phi_T} : G^s_T \to PO(3,\mathbb{R}) \subset PO(3,\mathbb{C})$ corresponding to the canonical realization $\phi = \phi_T$ of the standard triangle T has finite image. The centralizer of the group $\rho(G^s_T)$ in $PO(3,\mathbb{C})$ is trivial.*

Proof: It is clear that the group $\rho(G^s_T)$ has invariant finite set

$$\Sigma = \{(1,1), (-1,1), (-1,-1), (1,-1)\} \subset \mathbb{P}^2(\mathbb{R}) \subset \mathbb{P}^2(\mathbb{C}) \quad .$$

Any three distinct points of Σ do not lie on a projective line in $\mathbb{P}^2(\mathbb{C})$, thus if $g \in PO(3,\mathbb{C})$ fixes Σ pointwise then $g = id$. This implies finiteness of $\rho(G^s_T)$. It is easy to check that $\rho(G^s_T)$ equals A_4, the alternating group of the 4-element set Σ. Any element of $g \in PO(3,\mathbb{C})$ centralizing $\rho(G^s_T)$ must fix Σ pointwise, hence $g = \mathbf{1}$. $\square$

Proposition 12.6 *The group $PO(3,\mathbb{C})$ acts simply-transitively by conjugations on the set $\mathrm{Hom}_f(G^s_T, PO(3,\mathbb{C}))$.*

Proof: Suppose that $\rho \in \mathrm{Hom}_f(G^s_T, PO(3,\mathbb{C}))$. We note that $G^s = G^s_T$ contains the abelian subgroup $\mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2$ generated by the involutions $g_{v_{00}}, g_{v_x}, g_{v_y}$. Since ρ is nondegenerate we conclude that the restriction of ρ to this subgroup is injective. Thus, by the classification of finite subgroups of $PO(3,\mathbb{C})$, we conclude that ρ can be conjugate to a representation (which we again denote by ρ) so that the projective arrangement $\psi = \alpha(\rho)$ has the property:

$$\psi(v_{00}) = (0,0), \psi(v_x) = (0,\infty), \psi(v_y) = (\infty, 0)$$

Thus necessarily: $\psi(l_x) = L_x, \psi(l_y) = L_y, \psi(l_\infty) = L_\infty$. Let G_{10} denote the subgroup of G^s generated by $g_{v_{00}}, g_{v_y} = g_{l_x}, g_{v_{10}}$. The restriction of ρ to G_{10} factors through the finite Coxeter group $G^c_{10} = G_{10}/\langle\langle (g_{v_{00}} g_{v_{10}})^2 \rangle\rangle$. There are only two homomorphisms $\rho^c : G^c_{10} \to PO(3,\mathbb{C})$ such that

$$\rho^c(g_{v_{00}}) = \rho(g_{v_{00}}), \rho^c(g_{v_y}) = \rho(g_{v_y}), \rho^c(g_{v_{10}}) \neq \rho^c(g_{v_{00}}), \rho^c(g_{v_{10}}) \neq \rho^c(g_{v_y})$$

Namely, for one of them the isolated fixed point of $\rho^c(g_{v_{10}})$ has the affine coordinates $(1,0)$, for the second it has the affine coordinates $(-1,0)$. The reflection in the line L_y conjugates one representation to the other and commutes with the elements $\rho(g_{v_{00}}), \rho(g_{v_x}), \rho(g_{v_y})$. Thus, after adjusting ρ by this conjugation (if necessary), we conclude that
$\alpha(\rho) = \psi : v_{10} \mapsto (1,0)$. Similar argument works for the vertex v_{01}. It remains to determine $\psi(l_{x1}), \psi(l_{y1}), \psi(v_{11})$ and $\psi(l_d)$. Since $\rho(g_{l_{x1}})$ commutes with the elements $\rho(g_{v_{01}})$ and $\rho(g_{v_{\infty 0}})$ and does not coincide with either, the line $\psi(l_{x1})$ contains the points $(0,1)$ and $(\infty, 0)$. Similarly the line $\psi(l_{y1})$ contains $(1,0)$ and $(0,\infty)$.

We next determine $\psi(v_{11})$. Since the edge $[v_{11}, l_{y1}]$ has label 4, the elements $\rho(g_{v_{11}})$ and $\rho(g_{l_{y1}})$ anticommute, so $\psi(v_{11}) \in L_{y1}$ (the line in $\mathbb{P}^2$ joining $(1,0)$ and $(0,\infty)$). Similarly $\psi(v_{11}) \in L_{x1}$. Finally we determine $\psi(l_d)$. Since $\rho(g_{l_d})$ commutes with $\rho(g_{v_{00}})$ we have $(0,0) \in \psi(l_d)$. Also $\rho(g_{l_d})$ anticommutes with $\rho(g_{v_{11}})$, so $(1,1) \in \psi(l_d)$. We conclude that

$$\psi = \alpha(\rho) = \phi_T$$

and either $\rho = \rho_{\phi_T}$ or $\rho = \nu(\rho_{\phi_T})$. It now follows from Lemma 12.5 that the action of the group $PO(3,\mathbb{C})$ on $\mathrm{Hom}_f(G_T^s, PO(3,\mathbb{C}))$ is free. $\square$

Corollary 12.7 $\mathrm{Hom}_f(G_T^s, PO(3,\mathbb{C}))$ *equals the* $AdPO(3,\mathbb{C})$*-orbit of the set* $\{\rho_{\phi_T}, \nu(\rho_{\phi_T})\}$. *In particular, both* ρ_{ϕ_T} *and* $\nu(\rho_{\phi_T})$ *are locally rigid.*

Note however that this corollary doesn't apriori imply that the variety $\mathrm{Hom}_f(G_T^s, PO(3,\mathbb{C}))$ is smooth since it could be nonreduced. To prove smoothness we need the following:

Proposition 12.8 *The representations* $\rho_{\phi_T}, \nu(\rho_{\phi_T}) : G_T^s \to PO(3,\mathbb{C})$ *are infinitesimally rigid.*

We will prove the proposition only for ρ_{ϕ_T}, the second case easily follows. Proposition 12.8 will immediately follow from the more general

Proposition 12.9 *Let* $G^a = G_T^a$, *define* $\tilde{\rho} : G^a \to PO(3,\mathbb{C})$ *by composing* ρ_{ϕ_T} *with the canonical projection* $G_T^a \to G_T^s$. *Then the representation* $\tilde{\rho}$ *is infinitesimally rigid.*

Proof: Our proof is based on the results of Section 7. The reader will notice that the proof follows the lines of the proof of Proposition 12.6. We first consider the subgroup F in G^a generated by $g_{v_{00}}, g_{l_x}, g_{l_y}$, these generators mutually commute, hence the subgroup is abelian. Let $\sigma \in Z^1(G^a, ad \circ \tilde{\rho})$. By Lemma 7.5 the restriction of σ to each cyclic subgroup $\langle g_{v_{00}} \rangle, \langle g_{l_x} \rangle, \langle g_{l_y} \rangle$ is cohomologically trivial, thus $\sigma|_F$ comes from a cocycle on the finite Coxeter group

$$F / \langle\langle\ g_{v_{00}}^2\ ,\ g_{l_x}^2\ ,\ g_{l_y}^2\ \rangle\rangle$$

which implies that $\sigma|_F$ is a coboundary. By adjusting the cocycle σ by a coboundary we may assume that $\sigma|_F = 0$. Now we consider the subgroup F_{10} generated by $g_{l_x}, g_{v_{10}}$. Because of the Artin relations in G_T^a this is again an abelian subgroup whose image under $\tilde{\rho}$ is dihedral. Hence $\sigma|_{F_{10}}$ is also a coboundary. Let H_{10} denote the subgroup generated by elements of F_{10} and $g_{v_{00}}$, we recall that $(g_{v_{00}} g_{v_{10}})^2 = (g_{v_{10}} g_{v_{00}})^2$.

Since the restriction of σ to each generator of H_{10} is exact, the cocycle $\sigma|_{H_{10}}$ comes from the finite Coxeter group $H_{10}/\langle\langle g_{v_{00}}^2, g_{l_x}^2, g_{v_{10}}^2 \rangle\rangle$. This implies that $\sigma|_{H_{10}}$ is a coboundary. Since $\sigma(g_{l_x}) = 0, \sigma(g_{v_{00}}) = 0$ and the fixed points of $\tilde{\rho}(g_{l_x}), \tilde{\rho}(g_{v_{00}})$ are distinct we conclude that $\sigma|_{H_{10}} = 0$. The same argument implies that $\sigma(g_{v_{01}}) = 0$. We repeat our argument for the two abelian subgroups generated by $g_{l_x}, g_{v_{10}}, g_{l_{y1}}$ and by $g_{l_x}, g_{v_{01}}, g_{l_{x1}}$, it follows that $\sigma(g_{l_{y1}}) = 0, \sigma(g_{l_{x1}}) = 0$.

Then we use Lemma 7.17, where $a_1 = g_{l_{x1}}, a_2 = g_{l_{y1}}, a_3 = g_{v_{00}}, b = g_{v_{11}}$, to conclude that $\sigma(g_{v_{11}}) = 0$. Finally $\sigma(g_{l_d}) = 0$ since the Shephard subgroup of G^s generated by $g_{v_{11}}, g_{l_d}, g_{v_{00}}$ is again finite. $\square$

Corollary 12.10 *The variety* $\mathrm{Hom}_f(G^s_T, PO(3,\mathbb{C}))$ *is smooth. The variety* $\mathrm{BHom}_f(G^s_T, PO(3,\mathbb{C}))$ *is a (scheme-theoretic) cross-section for the action of* $PO(3,\mathbb{C})$ *by conjugation on the variety* $\mathrm{Hom}_f(G^s_T, PO(3,\mathbb{C}))$.

Proof: Smoothness of $\mathrm{Hom}_f(G^s_T, PO(3,\mathbb{C}))$ follows from infinitesimal rigidity of the representations $\rho_{\phi_T}, \nu\rho_{\phi_T}$, see Theorem 2.4. In Proposition 12.6 we proved that the morphism

$$\lambda : PO(3,\mathbb{C}) \times \mathrm{BHom}_f(G^s_T, PO(3,\mathbb{C})) \hookrightarrow \mathrm{Hom}_f(G^s_T, PO(3,\mathbb{C}))$$

given by the action of $PO(3,\mathbb{C})$ by conjugation, is a bijection. Thus $\mathrm{Hom}_f(G^s_T, PO(3,\mathbb{C}))$ is also smooth, which implies that the morphism λ is actually an isomorphism of varieties. $\square$

Corollary 12.11 *The variety* $\mathrm{BHom}_f(G^s_A, PO(3,\mathbb{C}))$ *is a (scheme-theoretic) cross-section for the action of* $PO(3,\mathbb{C})$ *by conjugation on the variety* $\mathrm{Hom}_f(G^s_A, PO(3,\mathbb{C}))$.

Proof: Consider the $Ad(PO(3,\mathbb{C}))$-equivariant restriction morphism

$$Res_T : \mathrm{Hom}_f(G^s_A, PO(3,\mathbb{C})) \longrightarrow \mathrm{Hom}_f(G^s_T, PO(3,\mathbb{C}))$$

It was proven in Corollary 12.10 that the subvariety $\mathrm{BHom}_f(G^s_T, PO(3,\mathbb{C}))$ is a cross-section for the action of $PO(3,\mathbb{C})$ by conjugation on $\mathrm{Hom}_f(G^s_T, PO(3,\mathbb{C}))$. Thus the pull-back variety

$$Res_T^{-1}\mathrm{Hom}_f(G^s_T, PO(3,\mathbb{C})) \equiv \mathrm{BHom}_f(G^s_A, PO(3,\mathbb{C}))$$

is a cross-section as well. $\square$

Our goal is to show that the mapping $alg : BR(A, \mathbb{P}^2_0(\mathbb{C})) \to \mathrm{BHom}^+_f(G^s_A, PO(3,\mathbb{C}))$ is an isomorphism of varieties over $\mathbb{Q}$, this will be proven in the next section.

12.2 alg is an isomorphism of varieties

We first establish that $alg : BR(A, \mathbb{P}^2_0(\mathbb{C})) \to \mathrm{BHom}^+_f(G^s_A, PO(3,\mathbb{C}))$ is an isomorphism of varieties in two elementary cases. Let C be an arrangement whose graph Γ_C has only one edge $e = [v, l]$, m is the number of isolated vertices v_j in Γ_C. Let A be a based arrangement which is the disjoint union of the standard triangle and C.

Lemma 12.12 $alg : BR(A, \mathbb{P}^2_0(\mathbb{C})) \to \mathrm{BHom}^+_f(G^s_A, PO(3,\mathbb{C}))$ *is an isomorphism of varieties over* $\mathbb{Q}$.

Proof: We already know that alg is a bijection. It is clear that the restriction morphisms

$$Res : \mathrm{BHom}^+_f(G^s_A, PO(3,\mathbb{C})) \to \mathrm{Hom}_f(G^s_C, PO(3,\mathbb{C}))$$

$$res : BR(A, \mathbb{P}^2(\mathbb{C})) \to BR(C, \mathbb{P}^2(\mathbb{C}))$$

are isomorphisms of varieties. Let $alg : BR(C, \mathbb{P}^2(\mathbb{C})) \to \mathrm{Hom}_f(G^s_C, PO(3,\mathbb{C}))$ denote the mapping induced by the restriction. The group G^s_C is the free product

$$G_e^s * \underbrace{\mathbb{Z}/2 * \ldots * \mathbb{Z}/2}_{m \text{ times}}$$

Hence

$$\mathrm{Hom}_f(G_C^s, PO(3)) \cong \mathrm{Hom}_f(G_e^s, PO(3)) \times (\mathrm{Hom}_f(\mathbb{Z}/2, PO(3)))^m$$

Since the edge group G_e^s and the vertex groups $\mathbb{Z}/2$ are finite, the variety $\mathrm{Hom}_f(G_C^s, PO(3))$ is smooth (Proposition 7.8). The quasi-projective variety $R(C, \mathbb{P}_0^2)$ again splits as the direct product

$$\mathcal{I}_0 \times \mathbb{P}_0^2 \times \ldots \times \mathbb{P}_0^2 \ ,$$

where $\mathcal{I}_0$ is the anisotropic incidence variety, see Section 8.4. The anisotropic incidence variety is smooth by Proposition 8.11, hence the product smooth as well.

Let B denote the arrangement obtained by removing the incidence relation between v and l in C and G_B^s be the corresponding Coxeter group. Then

$$alg : R(B, \mathbb{P}_0^2) \longrightarrow \mathrm{Hom}_f(G_B, PO(3))$$

is an isomorphism of smooth varieties (the left-hand side is $(\mathbb{P}_0^2)^{m+2}$ and the right-hand side is $p(R)^{m+2}$, see Lemma 6.3). Thus $alg : BR(A, \mathbb{P}_0^2) \longrightarrow \mathrm{BHom}_f(G, PO(3))$ is an isomorphism of smooth varieties. $\square$

We now consider a relative version of the above lemma. Suppose that a based abstract arrangement D is the fiber sum $T \times_{w \equiv v} C$ where C is the arrangement above, w, v are elements of T and C respectively.

Lemma 12.13 *$alg : BR(D, \mathbb{P}_0^2(\mathbb{C})) \to \mathrm{BHom}_f^+(G_D^s, PO(3, \mathbb{C}))$ is an isomorphism of varieties.*

Proof: Let $\alpha = \rho_{\phi_T}(g_v)$, $G_e \subset G_D^s$ is the edge subgroup corresponding to $e = [v, l]$. It is clear that the restriction morphism

$$\mathrm{BHom}_f^+(G_A^s, PO(3, \mathbb{C})) \xrightarrow{Res} \{\rho \in \mathrm{Hom}_f(G_C^s, PO(3, \mathbb{C})) : \rho(g_v) = \alpha\}$$
$$=: F_{g_v, \alpha}(G_C^s, PO(3, \mathbb{C}))$$

is an isomorphism of varieties. The group G_e is finite, hence by Proposition 7.8 the variety $F_{g_v,\alpha}(G_C^s, PO(3, \mathbb{C}))$ is smooth. Similarly let $q := \phi_T(v)$, the restriction morphism

$$res : BR(D, \mathbb{P}_0^2(\mathbb{C})) \to \{\psi \in R(C, \mathbb{P}_0^2(\mathbb{C})) : \psi(v) = q\} =: F_{v,q}(C, \mathbb{P}_0^2(\mathbb{C}))$$

is an isomorphism of varieties. Then $F_{v,q}(C, \mathbb{P}_0^2(\mathbb{C}))$ is isomorphic the product of the *relative anisotropic incidence variety*

$$\mathcal{I}_0(q) = \{l \in (\mathbb{P}_0^2)^\vee : q \cdot l = 0\}$$

(see §8.4) with m copies of $\mathbb{P}_0^2$. It is clear that the mapping alg induces a bijection of the sets of complex points

$$alg : \mathcal{I}_0(q)(\mathbb{C}) \longrightarrow F_{g_v,\alpha}(G_e, PO(3, \mathbb{C}))$$

According to Lemma 8.11 the variety $\mathcal{I}(q)$ is smooth and we repeat the arguments from the proof of Lemma 12.12. $\square$

Now we consider the case when A is a general based arrangement. Let $X := BR(A, \mathbb{P}_0^2)$, $Y := \mathrm{BHom}_f^+(G_A^s, PO(3))$.

Theorem 12.14 *The mapping $alg : X \to Y$ is a biregular isomorphism of quasi-projective varieties.*

Proof: Let F (resp. R) be the functor of points of X (resp. Y). Then alg is an isomorphism of the varieties X and Y if and only if F and R are naturally isomorphic (see [EH, Proposition IV-2]). Let B be the based arrangement obtained by removing all edges from the graph of A that are not in the graph of T (and retaining all vertices). Let G^s_B be the corresponding Shephard group, clearly

$$G^s_B \cong G^s_T * \underbrace{\mathbb{Z}/2 * \ldots * \mathbb{Z}/2}_{m \ \text{times}}$$

where m is the cardinality of the vertex set $\mathcal{V}(A-T)$. Thus

$$BR(B, \mathbb{P}^2_0(\mathbb{C})) \cong \mathbb{P}^2_0(\mathbb{C})^m, \mathrm{BHom}^+_f(G^s_B, PO(3)) \cong \mathrm{Hom}_f(\mathbb{Z}/2, PO(3,\mathbb{C}))^m$$

are smooth varieties and

$$\widetilde{alg} : \tilde{X} := BR(B) \longrightarrow \tilde{Y} := \mathrm{BHom}^+_f(G^s_B, PO(3))$$

is an isomorphism of smooth varieties (see the proof of Lemma 12.12). We let $\tilde{F}$ and $\tilde{R}$ be the functors of points of $\tilde{X}$ and $\tilde{Y}$. The isomorphism $\widetilde{alg} : \tilde{X} \to \tilde{Y}$ induces a natural isomorphism of functors $\eta : \tilde{F} \to \tilde{R}$. The functors F and R are subfunctors of $\tilde{F}$ and $\tilde{R}$. We now make explicit the inclusions $F \subset \tilde{F}, R \subset \tilde{R}$.

Let $\mathcal{E}_0$ be the collection of edges of Γ_A that are not edges of Γ_T. Suppose $e_{ij} = [v_i, \ell_j] \in \mathcal{E}_0$ is such an edge. Let X_{ij} be the subvariety of $BR_0(B)$ defined by

$$x_i\alpha_j + y_i\beta_j + z_i\gamma_j = 0$$

where $[x_i : y_i : z_i]$ and $[\alpha_j : \beta_j : \gamma_j]$ are homogeneous coordinates on $BR_0(B)$ corresponding to v_i, l_j respectively. We let F_{ij} be the subfunctor of $\tilde{F}$ which as the functor of points associated to the subvariety X_{ij}. Then we have

$$F = \bigcap_{[v_i l_j] \in \mathcal{E}_0} F_{ij} \subset \tilde{F} \qquad (2)$$

Similarly if the edge e_{ij} has the label ϵ_{ij} then we let R_{ij} be the subfunctor of $\tilde{R}$ corresponding to the subvariety Y_{ij} defined by the Artin relation

$$(g_{v_i} g_{\ell_j})^{\epsilon_{ij}} = (g_{\ell_j} g_{v_i})^{\epsilon_{ij}} \ .$$

We have

$$R = \bigcap_{(ij) \in \mathcal{E}} R_{ij} \subset \tilde{R} \qquad (3)$$

Lemma 12.15 *$\eta : \tilde{F} \to \tilde{R}$ induces an isomorphism from F_{ij} to R_{ij}.*

Proof: Let X_{ij} and Y_{ij} be the subvarieties of $BR(B, \mathbb{P}^2_0), \mathrm{BHom}^+_f(G^s_B, PO(3))$ corresponding to the subfunctors F_{ij}, R_{ij}. Then Lemmas 12.12 and 12.13 imply that alg induces an isomorphism of smooth varieties $X_{ij} \to Y_{ij}$. Hence η induces an isomorphism of the corresponding subfunctors. $\square$

The above lemma and equations (2), (3) immediately imply that η induces a natural isomorphism from F to R. Theorem 12.14 follows. $\square$

Now we can prove one of the two main results of this paper. Let $S \subset \mathbb{C}^n$ be an affine variety defined over $\mathbb{Q}$. We will consider S as a quasi-projective variety in $\mathbb{P}^n(\mathbb{C})$.

Theorem 12.16 *For any variety S as above there exists a Zariski open subset $U \subset S(\mathbb{C})$ containing all real points and a based arrangement A so that the corresponding Shephard group G_A^s has the property:*

There is a Zariski open subset W in $\mathrm{Hom}(G_A^s, PO(3,\mathbb{C}))//PO(3,\mathbb{C})$ which is biregular isomorphic to U.

Remark 12.17 *W is never Zariski dense in the character variety $X(G_A^s, PO(3,\mathbb{C}))$.*

Proof: Given the variety S we construct an abstract based arrangement A such that $BR_0(A,\mathbb{P}^2)$ is biregular isomorphic to S via the isomorphism *geo* (Theorem 10.1 and Corollary 10.4). Let $U := \tau(BR_0(A,\mathbb{P}^2_0(\mathbb{C})))$, it is Zariski open in $S(\mathbb{C})$ and contains all real points since the subvariety $BR_0(A,\mathbb{P}^2_0)$ is Zariski open in $BR_0(A,\mathbb{P}^2)$ and contains all real points. Theorem 12.14 implies that we have an isomorphic embedding with Zariski open image $\mathrm{BHom}_f^+(G_A^s, PO(3,\mathbb{C}))_0$

$$alg : BR_0(A,\mathbb{P}^2_0) \hookrightarrow \mathrm{BHom}_f^+(G_A^s, PO(3,\mathbb{C}))$$

Corollary 12.11 implies that $\mathrm{BHom}_f^+(G_A^s, PO(3))$ is a cross-section for the action of $PO(3,\mathbb{C})$ by conjugation on the Zariski component $\mathrm{Hom}_f^+(G_A^s, PO(3))$ of $\mathrm{Hom}(G_A^s, PO(3))$. Thus we get an open monomorphism of varieties

$$\mathrm{BHom}_f^+(G_A^s, PO(3,\mathbb{C}))_0 \hookrightarrow X(G_A^s, PO(3,\mathbb{C})) = \mathrm{Hom}(G_A^s, PO(3,\mathbb{C}))//PO(3,\mathbb{C})$$

so that the image is a Zariski open subvariety W in the character variety. Therefore the composition

$$\theta : U \xrightarrow{geo} BR_0(A,\mathbb{P}^2(\mathbb{C})) \xrightarrow{alg} \mathrm{BHom}_f^+(G_A^s, PO(3,\mathbb{C}))_0 \longrightarrow W \subset X(G_A^s, PO(3,\mathbb{C}))$$

is the required isomorphism onto a Zariski open subvariety W of the character variety. $\square$

Proposition 12.18 *Suppose that S is an affine variety over $\mathbb{Q}$ and $q \in \mathbb{Q}$ is a rational point. Then there is an abstract arrangement A (as in Theorem 12.16) so that the representation $\rho = alg \circ geo(q)$ has finite image. The centralizer of the subgroup $\rho(G_A^s) \subset PO(3,\mathbb{C})$ is trivial.*

Proof: Follows from Corollary 10.4 and Lemma 12.5. $\square$

As an example we consider the configuration space of the arrangement[9]

$$A = \{v; l_1, l_2 : \iota(v,l_1), \iota(v,l_2)\}$$

Take the corresponding Coxeter group $G = G_A^c$. Then $R(A,\mathbb{P}^2_0) \cong \mathbb{P}^2_0 \times \mathbb{P}^1_0 \times \mathbb{P}^1_0$ corresponds to the representations of G which are *nondegenerate*, i.e. the elements $\rho(g_v), \rho(g_{l_1}), \rho(g_{l_2})$, $\rho(g_v g_{l_1})$, $\rho(g_v g_{l_2})$ have order 2. However there are some other components of $\mathrm{Hom}(G, PO(3,\mathbb{C}))$ which are described by assigning which of the elements $\rho(g_v), \rho(g_{l_1}), \rho(g_{l_2}), \rho(g_v g_{l_1}),\ \rho(g_v g_{l_2})$ are equal to $\mathbf{1}$. If $\rho(g_v) = \mathbf{1}$ then any representation factors through the free product $\mathbb{Z}/2 * \mathbb{Z}/2$ and the corresponding component of $\mathrm{Hom}(G, PO(3,\mathbb{C}))$ is isomorphic to $\mathbb{P}^2_0 \times \mathbb{P}^2_0$. The reader will verify that besides

$$\mathrm{Hom}_f(G, PO(3,\mathbb{C})) \cong \mathbb{P}^2_0 \times \mathbb{P}^1_0 \times \mathbb{P}^1_0$$

and the above component isomorphic to $\mathbb{P}^2_0 \times \mathbb{P}^2_0$ there are 4 components isomorphic to $\mathbb{P}^2_0 \times \mathbb{P}^1_0$, 4 components isomorphic to $\mathbb{P}^2_0$ and one component which consists of a single reduced point (the last corresponds to the trivial representation).

[9] This is not a based arrangement.

12.3 Representations of Artin groups

Our next goal is to prove a theorem analogous to Theorem 12.16 for Artin groups. Take a based admissible arrangement A, consider the germ $(\mathrm{Hom}(G^a_A, PO(3,\mathbb{C})), \tilde{\rho})$, where $\tilde{\rho} = \omega(\rho)$ is the pull-back of $\rho \in \mathrm{Hom}_f(G^s_A, PO(3,\mathbb{C}))$.

Theorem 12.19 *The morphism* $\omega : (\mathrm{Hom}(G^s_A, PO(3,\mathbb{C})), \rho) \to (\mathrm{Hom}(G^a_A, PO(3)), \tilde{\rho})$ *is an analytical isomorphism of germs.*

Proof: Let $m \geq 1$ be an integer, consider the m-th order Zariski tangent spaces T^m_ρ of the variety $\mathrm{Hom}(G^s_A, PO(3,\mathbb{C}))$ at ρ and $T^m_{\tilde{\rho}}$ of the variety $\mathrm{Hom}(G^a_A, PO(3,\mathbb{C}))$ at $\tilde{\rho}$, there is a well-defined mapping $\omega_m : T^m_\rho \to T^m_{\tilde{\rho}}$ induced by ω. We will think of elements of m-th order Zariski tangent spaces as *formal curves* or homomorphisms from G^s, G^a to $PO(3)(\mathcal{A})$ which is the set of $\mathcal{A}$-points of $PO(3)$ and $\mathcal{A}$ is a certain Artin local $\mathbb{C}$-algebra, see §5.

Our goal is to prove that ω_m is an isomorphism for all m.

Lemma 12.20 ω_m *is injective.*

Suppose that $\alpha \neq \beta \in T^m_{\tilde{\rho}}$ are formal curves of the order m. We think of them as representations of G^s into $PO(3)(\mathcal{A})$. Then $\omega_m(\alpha), \omega_m(\beta)$ are homomorphisms of G^a into $PO(3)(\mathcal{A})$ defined by composing with the projection $G^a \to G^s$. Clearly $\omega_m(\alpha) \neq \omega_m(\beta)$. □

Lemma 12.21 ω_m *is surjective.*

Proof: Let $\xi \in T^m_{\tilde{\rho}}(G^a, PO(3,\mathbb{C}))$ and v is a *point* in $A - \{v_{11}\}$. To prove that ξ belongs to the image of ω_m it is enough to check that the restriction of ω_m to each vertex-subgroup of G^a is a *trivial deformation* (see Lemma 5.1). We first consider the *points* $v \in A$ distinct from v_{11}. Then v is incident to at least one *line* $l \in A$ and hence the image of the edge-subgroup $\rho(G^s_{vl}) = \tilde{\rho}(G^a_{vl})$ is generated by two distinct commuting involutions (since ρ is "nondegenerate"). Thus by Lemma 7.5 the restriction $\rho|G^a_{vl}$ is infinitesimally rigid, in particular $\xi|G^a_v$ is a *trivial deformation*. In the case $v = v_{11}$ we use Proposition 12.9 to conclude that the restriction of ξ to the vertex-subgroup G^a_v is a trivial deformation as well. Suppose that $l \in A$ is a *line*. Then admissibility of the arrangement A implies that l is incident to at least one *point* w in $A - \{v_{11}\}$. Hence we repeat the same argument as in the case of *points* in A. □

Thus, we established that the morphism ω induces an isomorphism of Zariski tangent spaces of all orders. Hence Theorem 12.19 follows from Lemma 2.1. □

Corollary 12.22 *For any admissible based arrangement A the morphism*

$$\omega : \mathrm{Hom}_f(G^s_A, PO(3)) \to \mathrm{Hom}(G^a_A, PO(3))$$

is open (in the classical topology) and is an analytic isomorphism onto its image.

Corollary 12.23 *For any admissible based arrangement A the character variety of G^a_A inherits all singularities of the representation variety of the group G^s_A corresponding to points of $BR(A, \mathbb{P}^2_0(\mathbb{C}))$.*

The image $\mathrm{Hom}_f(G^a_A, PO(3,\mathbb{C})) := \omega(\mathrm{Hom}_f(G^s_A, PO(3,\mathbb{C})))$ is a constructible set.

Proposition 12.24 $\mathrm{Hom}_f(G^a_A, PO(3,\mathbb{C}))$ *is closed in* $\mathrm{Hom}(G^a_A, PO(3,\mathbb{C}))$

Proof: The image Z of the monomorphism

$$\mathrm{Hom}(G_A^s, PO(3,\mathbb{C})) \longrightarrow \mathrm{Hom}(G_A^a, PO(3,\mathbb{C}))$$

is described by the equations

$$\rho(g_v)^{\delta(v)} = \mathbf{1}, \quad v \in \mathcal{V}(\Lambda_A)$$

where $\delta(v)$ is the label of the vertex v. Thus Z is closed. On the other hand, by Corollary 12.1, the space $\mathrm{Hom}_f(G_A^s, PO(3,\mathbb{C}))$ is closed in $\mathrm{Hom}(G_A^s, PO(3,\mathbb{C}))$, the proposition follows. $\square$

Corollary 12.25 *For any admissible based arrangement the space* $\mathrm{Hom}_f(G_A^a, PO(3,\mathbb{C}))$ *is a union of Zariski connected components of* $\mathrm{Hom}(G_A^a, PO(3,\mathbb{C}))$.

Proof: The set $\mathrm{Hom}_f(G_A^a, PO(3,\mathbb{C}))$ is Zariski open since it is constructible and open in the classical topology. $\square$

Theorem 12.26 *The morphism* $\omega : \mathrm{Hom}_f(G_A^s, PO(3)) \to \mathrm{Hom}_f(G_A^a, PO(3)))$ *is a biregular isomorphism of varieties.*

Proof: We will first prove that the reduced varieties corresponding to

$$X := \mathrm{Hom}_f(G_A^s, PO(3)) \quad \text{and} \quad Y := \mathrm{Hom}_f(G_A^a, PO(3))$$

are isomorphic. We construct nonsingular varieties U, W containing X, Y and an extension of ω to an isomorphism $U \to W$ which carries X to Y bijectively as sets. Let C be the arrangement obtained from A by removing the incidence relations everywhere outside of the standard triangle. Thus we get two nonsingular varieties $Q = \mathrm{Hom}_f(G_C^a, PO(3))$ and $U := \mathrm{Hom}_f(G_C^s, PO(3))$ with $X \subset U$ and $Y \subset Q$.

Clearly $\omega : U \to V$ is a biregular isomorphism and it bijectively carries X to Y. Thus the reduced varieties X^r, Y^r are isomorphic via ω. Hence the assertion of Theorem follows from combination of Corollary 12.22 and Theorem 2.9. $\square$

As a corollary we get the following generalization of Theorem 12.16:

Theorem 12.27 *Let S be an affine variety defined over $\mathbb{Q}$. Then there exists an admissible based arrangement A so that the corresponding Artin group G_A^a has the property:*

There is a Zariski open subset in S (containing all real points) which is biregular isomorphic to a Zariski open and closed subvariety in $\mathrm{Hom}(G_A^a, PO(3,\mathbb{C}))//PO(3,\mathbb{C})$. *Suppose that $q \in S$ is a rational point. Then the abstract arrangement A can be chosen so that the representation $\rho = \omega \circ alg \circ geo(q)$ has finite image. The centralizer of the subgroup* $\rho(G_A^a) \subset PO(3,\mathbb{C})$ *is trivial.*

13 Differential graded algebras and Lie algebras

In this section we discuss differential graded Lie algebras, differential algebras and their Sullivan's minimal models. These definitions will be used in the following two sections. Let $\mathbf{k}$ be the ground field. A *graded Lie algebra* over $\mathbf{k}$ is a $\mathbf{k}$-vector space

$$L^\bullet = \oplus_{i \geq 0} L^i$$

graded by (nonnegative) integers and a family of bilinear mappings

$$[\cdot,\cdot] : L^i \otimes L^j \longrightarrow L^{i+j}$$

satisfying graded skew-commutativity:

$$[\alpha,\beta] + (-1)^{ij}[\beta,\alpha] = 0$$

and the graded Jacobi identity:

$$(-1)^{ki}[\alpha,[\beta,\gamma]] + (-1)^{ij}[\beta,[\gamma,\alpha]] + (-1)^{jk}[\gamma,[\alpha,\beta]] = 0$$

where $\alpha \in L^i, \beta \in L^j, \gamma \in L^k$. We say that L is *bigraded* if L is graded as above and

$$L = \oplus_{q\geq 0} L_q = \oplus_{i,q\geq 0}\ L^i_q$$

where $L^i_q = L^i \cap L_q$. We also require:

$$[\cdot,\cdot] : L_q \otimes L_p \longrightarrow L_{p+q}$$

A *differential graded Lie algebra* is a pair (L, d) where $d : L \to L$ is a *derivation* of the degree ℓ, i.e.

$$d : L^i \longrightarrow L^{i+\ell}, \quad d \circ d = 0, \quad \text{and}$$
$$d[\alpha,\beta] = [d\alpha,\beta] + (-1)^{i\ell}[\alpha, d\beta]$$

Remark 13.1 *In this paper we will use only the degree $\ell = 1$.*

Suppose that $L = L^\bullet_\bullet$ is a bigraded Lie algebra, then the differential graded Lie algebra (L, d) is called a *differential bigraded Lie algebra* if there is a number s such that d has the bidegree (ℓ, s):

$$d : L^i_q \longrightarrow L^{i+\ell}_{q+s}, \quad \text{for all} \quad i, q$$

Remark 13.2 *In this paper we will use only the bidegree $(\ell, s) = (1, 0)$. Similarly one defines* trigraded *differential Lie algebras $L^\bullet_{\bullet,\bullet}$*

Let $L^\bullet$ be a differential graded Lie algebra, and suppose that $\mathcal{J} \subset L^\bullet$ is a vector subspace. Then $\mathcal{J}$ is an *ideal* in $L^\bullet$ if:

- $\mathcal{J}$ is *graded*, i.e. $\mathcal{J} = \oplus_{n=0}^{\infty}(\mathcal{J} \cap L^n)$;
- $d(\mathcal{J}) \subset \mathcal{J}$;
- For each $\gamma \in \mathcal{J}, \alpha \in L$ we have: $[\gamma, \alpha] \in \mathcal{J}$.

Remark 13.3 *If L is bigraded then we require ideals in L to be bigraded as well, i.e.*

$$\mathcal{J} = \oplus_{n,q\geq 0}\ (\mathcal{J} \cap L^n_q)$$

Lemma 13.4 *If $\mathcal{J} \subset L$ is an ideal in a differential graded (bigraded) Lie algebra then the quotient $L/\mathcal{J}$ has the natural structure of a differential graded (bigraded) Lie algebra so that the natural projection $L \to L/\mathcal{J}$ is a morphism.*

Proof: The proof is straightforward and is left to the reader. $\square$

Let $(L^\bullet, d)$ be a differential graded Lie algebra and $\mathfrak{g}$ be a Lie algebra. An *augmentation* is a homomorphism $\epsilon : L \to \mathfrak{g}$ such that $\oplus_{i\geq 1} L^i \subset \ker(\epsilon)$ and $\epsilon \neq 0$. An *augmentation ideal* of ϵ is the kernel of ϵ.

We recall the definition of the complete local $\mathbf{k}$-algebra $R_{L^\bullet}$ (see [Mi]) associated to a differential graded Lie algebra $L^\bullet$ over $\mathbf{k}$. We will give a definition which conveys the intuitive meaning of $R_{L^\bullet}$ but has the defect of being non-functorial. We have the sequence

$$0 \longrightarrow L^0 \xrightarrow{d} L^1 \xrightarrow{d} L^2 \xrightarrow{d} \ldots$$

Choose a complement $C^1 \subset L^1$ to the 1-coboundaries $B^1 \subset L^1$. Let $\mathcal{F} : C^1 \to L^2$ be the polynomial mapping given by $\mathcal{F}(\eta) = d\eta + \frac{1}{2}[\eta, \eta]$. Then $R_{L^\bullet}$ is the completion at 0 of the coordinate ring of the affine subvariety of C^1 defined by the equation $\mathcal{F}(\eta) = 0$.

Since L^1 and C^1 are infinite dimensional in many applications, further definitions of affine variety and completion are needed in infinite dimensional vector spaces. These are given in [BuM, §1]. Suppose next that (X, x) is an analytic germ. We will say that $L^\bullet$ *controls* (X, x) if $R_{L^\bullet}$ is isomorphic to the complete local ring $\widehat{O_{X,x}}$.

Definition 13.5 *Let $A^\bullet, B^\bullet$ be cochain complexes and $\rho : A^\bullet \to B^\bullet$ be a morphism. Then ρ is called a* **quasi-isomorphism** *if it induces an isomorphism of all cohomology groups. The morphism is called a* **weak equivalence** *if it induces an isomorphism of* $\mathrm{H}^0, \mathrm{H}^1$ *and a monomorphism of* H^2.

Weak equivalence induces an equivalence relation on the category of differential graded Lie algebras: algebras $A_1^\bullet, A_m^\bullet$ are (weakly) equivalent if there is a chain of weak equivalences:

$$A_1^\bullet \to A_2^\bullet \leftarrow A_2^\bullet \to \ldots \leftarrow A_m^\bullet$$

In the following two sections we will use the following theorem about controlling differential graded Lie algebras proven in [GM]:

Theorem 13.6 *Suppose that $L^\bullet$ and $N^\bullet$ are weakly equivalent differential graded Lie algebras which control analytical germs. Then the germs controlled by the algebras $L^\bullet$ and $N^\bullet$ are analytically isomorphic.*

A *differential graded algebra* $A^\bullet$ is defined similarly to a differential graded Lie algebra : just instead of the Lie bracket $[\cdot, \cdot]$ satisfying graded Jacobi identity we have an associative multiplication:

$$\wedge : A^i \otimes A^j \longrightarrow A^{i+j}$$

satisfying the properties:

- $d : A^i \to A^{i+1}$, $d \circ d = 0$;
- $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^i \alpha \wedge d\beta$, for all $\alpha \in A^i$;
- $\alpha \wedge \beta = (-1)^{ij} \beta \wedge \alpha$, for all $\alpha \in A^i$, $\beta \in A^j$;
- A has the unit $1 \in A^0$.

To get a differential graded Lie algebra from a differential graded algebra $A^\bullet$ take a Lie algebra $\mathfrak{g}$ and let $L^\bullet = A^\bullet \otimes \mathfrak{g}$ (see [GM] for details).

Suppose that V is a vector space over $\mathbf{k}$, $(A^\bullet, d)$ is a differential graded algebra and $f : V \to Z^2$ is a linear mapping, where Z^2 is the space of 2-cocycles of $A^\bullet$. Then the *Hirsch* *extension* $A^\bullet \otimes_f V$ is a differential graded algebra which (as an algebra) equals to $A^\bullet \otimes \Lambda(V)$ and the restriction of the differential on $A^\bullet \otimes_f V$ to $A^\bullet$ equals d and the restriction to V equals f.

A differential graded algebra $\mathcal{M}^\bullet$ is called *1-minimal* if:

(a) $\mathcal{M}^0 = \mathbf{k}$;

(b) $\mathcal{M}^\bullet$ is the increasing union of differential subalgebras:

$$\mathbf{k} = \mathcal{M}_{[0]} \subset \mathcal{M}_{[1]} \subset \mathcal{M}_{[2]} \subset ...$$

with each $\mathcal{M}_{[i]} \subset \mathcal{M}_{[i+1]}$ a Hirsch extension;

(c) The differential d on $\mathcal{M}$ is decomposable, i.e. for each $\alpha \in \mathcal{M}$ we have:

$$d(\alpha) = \sum_{j,i} \beta_j \wedge \gamma_i \ , \quad \beta_j \ , \gamma_i \in \oplus_{s \geq 1} \mathcal{M}^s$$

Definition 13.7 *Suppose that $A^\bullet$ is a differential graded algebra. Then a* **1-minimal model** *for $A^\bullet$ is a morphism $\rho : \mathcal{M}^\bullet \to A^\bullet$ such that:*

- *The differential graded algebra $\mathcal{M}^\bullet$ is 1-minimal;*
- *The morphism ρ is a weak equivalence.*

We refer the reader to [Sul], [GrM], [Mo2] for further discussion of the definition, properties and construction of 1-minimal models.

14 Hain's theorem and its applications

In this section we give an exposition of a work of R. Hain [Hai] which shows that the singularities in representation varieties of fundamental groups of smooth complex algebraic varieties are quasi-homogeneous. In fact some assumption on the representation ρ which is being deformed is also required. In [Hai] the analogue of our Theorem 14.6 is proven under the assumption that ρ was the monodromy representation of an admissible variation of mixed Hodge structure. One does not obtain a restriction on weights working in this generality.

Let M be a smooth connected manifold with the fundamental group Γ. Let $\mathbf{G}$ be the Lie group of real points of an algebraic group G with the Lie algebra $\mathfrak{g}$ defined over $\mathbb{R}$ and $\rho : \Gamma \to \mathbf{G}$ is a homomorphism. Let P be the flat bundle over M associated to ρ and adP the associated $\mathfrak{g}$-bundle. Then $\mathcal{A}^\bullet(M, adP)$, the complex of smooth adP-valued differential forms on M is a differential graded Lie algebra . We define an augmentation $\epsilon : \mathcal{A}^\bullet(M, adP) \to \mathfrak{g}$ by evaluating degree zero forms at a base-point $x \in M$ and sending the rest of forms to zero. Let $\mathcal{A}^\bullet(M, adP)_0$ be the kernel of ϵ. The following theorem follows immediately from [GM, Theorem 6.8].

Theorem 14.1 *$\mathcal{A}^\bullet(M, adP)_0$ controls the germ* $(\mathrm{Hom}(\Gamma, \mathbf{G}), \rho)$.

The point of this section is that if M is smooth connected complex algebraic variety and ρ has finite image then $\mathcal{A}^\bullet(M, adP)_0$ is quasi-isomorphic to a differential graded Lie algebra which has a structure of a mixed Hodge complex. By a theorem of Hain this implies that

$$R_{\mathcal{A}^\bullet(M,adP)_0 \otimes \mathbb{C}}$$

is a quasi-homogeneous ring. We now give details.

A *real mixed Hodge complex* (abbreviated MHC) is a pair of complexes $K^\bullet_{\mathbb{R}}$ and $K^\bullet_{\mathbb{C}}$ (real and complex respectively), together with a quasi-isomorphism $\alpha : K^\bullet_{\mathbb{R}} \otimes \mathbb{C} \to K^\bullet_{\mathbb{C}}$ such that $K^\bullet_{\mathbb{R}}$ is a complex of real vector spaces equipped with an increasing filtration $W_\bullet$ (called the *weight* filtration) and $K^\bullet_{\mathbb{C}}$ is equipped with an increasing (weight) filtration $W_\bullet$ and a decreasing filtration $F^\bullet$ (called the *Hodge* filtration). The data

$$K^\bullet_{\mathbb{R}}, K^\bullet_{\mathbb{C}}, \alpha, W_\bullet, F^\bullet$$

satisfies the axioms described in [D2, Scholie 8.1.5] (take $A = \mathbb{R}$).

By a theorem of Deligne ([D2, Scholie 8.1.9]) the cohomology of a MHC has a mixed Hodge structure, [D1, Scholie 2.3.1]. It is important in what follows that the filtrations on $\mathrm{H}^\bullet(K_{\mathbb{C}})$ induced by $W_\bullet$ and $F^\bullet$ can be canonically split, [D1, Section 1.2.8]. Thus the filtration $W_\bullet$ induces a canonical grading on $\mathrm{H}^\ell(K_{\mathbb{C}})$ and consequently on $\mathrm{H}^\ell(K_{\mathbb{C}})^*$, $\ell = 0, 1, 2$.

If V is a finite-dimensional vector space over $\mathbb{C}$ we will let $\mathbb{C}[[V]]$ denote the completion of the symmetric algebra $\mathbb{C}[V]$ at the maximal ideal corresponding to 0. Thus $\mathbb{C}[[\mathrm{H}^\ell(K_{\mathbb{C}})^*]]$ has a canonical decreasing filtration (as an algebra) induced by the grading of $\mathrm{H}^\ell(K_{\mathbb{C}})^*$, $\ell = 0, 1, 2$.

We will say that a MHC is a *mixed Hodge differential graded Lie algebra* if the complexes $K^\bullet_{\mathbb{R}}$ and $K^\bullet_{\mathbb{C}}$ are differential graded Lie algebras ([GM, §1.1]), such that α is bracket preserving and the filtrations satisfy

(i) $[W_p(K^\bullet_{\mathbb{R}}), W_q(K^\bullet_{\mathbb{R}})] \subset W_{p+q}(K^\bullet_{\mathbb{R}})$;

(ii) $[F^p(K^\bullet_{\mathbb{C}}), F^q(K^\bullet_{\mathbb{C}})] \subset F^{p+q}(K^\bullet_{\mathbb{C}})$.

In this case we will use $L^\bullet_{\mathbb{R}}$ and $L^\bullet_{\mathbb{C}}$ in place of $K^\bullet_{\mathbb{R}}$ and $K^\bullet_{\mathbb{C}}$. Let $\mathfrak{m}$ denote the maximal ideal of $\mathbb{C}[[\mathrm{H}^1(L^\bullet_{\mathbb{C}})^*]]$. We now have

Theorem 14.2 *(Hain's Theorem.) Suppose* $L^\bullet = (L^\bullet_{\mathbb{R}}, L^\bullet_{\mathbb{C}}, \alpha, W_\bullet, F^\bullet)$ *is a mixed Hodge differential graded Lie algebra with* $\mathrm{H}^0(L^\bullet_{\mathbb{R}}) = 0$. *Then* $R_{L^\bullet_{\mathbb{C}}}$ *is quasi-homogeneous (see Section 3). Moreover there exists a morphism of graded vector spaces*

$$\delta : \mathrm{H}^2(L^\bullet_{\mathbb{C}})^* \to Gr^W \mathbb{C}[[\mathrm{H}^1(L^\bullet_{\mathbb{C}})^*]]$$

with image of δ *contained in* $\mathfrak{m}^2$ *such that* $R_{L^\bullet_{\mathbb{C}}}$ *is the quotient of* $\mathbb{C}[[\mathrm{H}^1(L^\bullet_{\mathbb{C}})^*]]$ *by the graded ideal generated by the image of* δ.

Remark 14.3 *Hain further proves that* $\delta \equiv Q^* \pmod{\mathfrak{m}^3}$, Q^* *is dual to* Q *where*

$$Q : S^2\mathrm{H}^1(L^\bullet) \to \mathrm{H}^2(L^\bullet)$$

is given by the cup-product $Q(\eta) = [\eta, \eta]$.

We will say an element $\eta \in \mathrm{H}^i(L^\bullet_{\mathbb{C}})$ has *weight* n if $\eta \in W_n\mathrm{H}^i(L^\bullet_{\mathbb{C}})$ but $\eta \notin W_{n-1}\mathrm{H}^i(L^\bullet_{\mathbb{C}})$. We will combine Hain's theorem with the following theorem to obtain our desired result about the singularities in representation varieties of fundamental groups of smooth complex algebraic varieties. Suppose now that M is a smooth connected complex algebraic variety, a representation $\rho : \pi_1(M) \to \mathbf{G}$ with finite image, the bundle adP, etc., are as above.

Theorem 14.4 *Under the conditions above there is a filtration* $W_\bullet$ *on* $A^\bullet(M, adP)$ *and a filtration* $F^\bullet$ *on* $A^\bullet(M, adP_{\mathbb{C}})$ *such that for the canonical map* $\alpha : A^\bullet(M, adP) \otimes \mathbb{C} \to A^\bullet(M, adP_{\mathbb{C}})$ *the algebra*

$$L^\bullet = (A^\bullet(M, adP), A^\bullet(M, adP_{\mathbb{C}}), \alpha, W_\bullet, F^\bullet)$$

is a mixed Hodge differential graded Lie algebra . Moreover the weights of $\mathrm{H}^1(A^\bullet(M, adP_{\mathbb{C}}))$ *are* 1 *and* 2 *and the weights of* $\mathrm{H}^2(A^\bullet(M, adP_{\mathbb{C}}))$ *are* 2, 3 *and* 4.

Proof: Let $\tilde{M}$ be the finite cover of M corresponding to $\ker(\rho)$. Let $\Phi \cong \rho(\Gamma)$ be the group of covering transformations. By [Sum] there exists an equivariant completion $\tilde{N}$ of $\tilde{M}$. But according to [BiM] there is also a canonical resolution of singularities N of $\tilde{N}$ so that the complement $N - \tilde{M}$ is a divisor $D = D_1 \cup ... \cup D_m$ with normal crossings.

Hence the action of Φ extends to N, which is a smooth Φ-equivariant completion of $\tilde{M}$. Hence Φ acts on the log-complex of $\tilde{M}$ defined using the compactification N. It is a basic result of Deligne ([D1, Theorem 1.5]) that one may use the log-complex to define:

(a) A subcomplex $A^\bullet(\tilde{M}) \subset \mathcal{A}^\bullet(\tilde{M})$ so that the inclusion is a quasi-isomorphism.

(b) Filtrations $W_\bullet, F^\bullet$ on $A^\bullet(\tilde{M})$ and $A^\bullet(\tilde{M}) \otimes \mathbb{C}$ which satisfy the axioms of MHC.

By construction these filtrations are Φ-invariant. We tensor with $\mathfrak{g}$ (regarded as a mixed Hodge differential graded Lie algebra concentrated in degree zero). Since Φ acts on $\mathfrak{g}$ via $ad\rho$, it also acts on the tensor product. We obtain the required mixed Hodge differential graded Lie algebra $L^\bullet$ by taking Φ-invariants. To derive weight restrictions we use results of Morgan [Mo1], [Mo2], who proved it for $A^\bullet(\tilde{M}) \otimes \mathbb{C}$, etc. The operations of tensoring with $\mathfrak{g}$ and taking Φ-invariants will not change these restrictions on weights. □

Remark 14.5 *Choose a point $m \in X$ and a point $\tilde{m} \in \tilde{M}$ over m. We define augmentations (see [GM, §3.1]) $\epsilon : A^\bullet(M, adP) \to \mathfrak{g}$ and $\tilde{\epsilon} : A^\bullet(\tilde{M}) \otimes \mathfrak{g} \to \mathfrak{g}$ by evaluation at m and $\tilde{m}$ respectively. We let $A^\bullet(M, adP)_0$ and $A^\bullet(\tilde{M})_0 \otimes \mathfrak{g}$ be the augmentation ideals. Then all statements in Theorem 14.4 hold when $A^\bullet(M, adP)$ and $A^\bullet(M, adP_{\mathbb{C}})$ are replaced by $A^\bullet(M, adP)_0$ and $A^\bullet(M, adP_{\mathbb{C}})_0$. We abbreviate the corresponding mixed Hodge differential graded Lie algebras by $L_0^\bullet$.*

Let Γ, ρ, G be as above. Let Z be the representation variety $\mathrm{Hom}(\Gamma, G)$. By combining Theorems 14.2, 14.4 we obtain

Theorem 14.6 *The germ $(Z_{\mathbb{C}}, \rho)$ is analytically equivalent to a quasi-homogeneous cone with generators of weights 1 and 2 and relations of weights 2, 3 and 4. Suppose that G is reductive and there is a local cross-section S through ρ to G-orbits. Then the conclusion is valid not just for the germ $(Z_{\mathbb{C}}, \rho)$ but also for $(\mathrm{Hom}(\Gamma, G)//G, [\rho])$.*

Proof: We apply Theorems 14.2 and 14.4 to deduce that the complete local $\mathbb{C}$-algebra $(R_{L_0^\bullet})_{\mathbb{C}}$ has a presentation of the required type. But by Theorem 14.1, $(R_{L_0^\bullet})_{\mathbb{C}}$ is isomorphic to the complete local ring associated to the germ $(Z_{\mathbb{C}}, \rho)$. We obtain the corresponding result for $(\mathrm{Hom}(\Gamma, G_{\mathbb{C}})//G_{\mathbb{C}}, [\rho])$ by replacing $L_0^\bullet$ by $L^\bullet$ and applying [KM3, Theorem 2.4]. Note that if S exists then $\mathrm{H}^0(L^\bullet) = 0$. □

There are infinitely many germs $(Y, 0)$, where $Y \subset \mathbb{C}^n$ is an affine variety defined over $\mathbb{Z}$, so that $(Y, 0)$ is not quasi-homogeneous with the weights of relations between 2 and 4. We can even assume that 0 is an isolated singular point, see the Section 3. Thus as a consequence of Theorems 14.6, 12.16 we obtain the following

Theorem 14.7 *Among the Artin groups G_A^a there are infinitely many mutually nonisomorphic groups which are not isomorphic to fundamental groups of smooth complex algebraic varieties.*

Proof: Let Y be an affine variety defined over $\mathbb{Q}$ and $y \in Y$ be a rational point. Assume that the analytical germ (Y, x) is not quasi-homogeneous (with the weights of variables 1, 2 and weights of generators 2, 3, 4). Let A be an affine arrangement corresponding to the pair (Y, y) as in Proposition 12.18, so that the representation $\rho_s : G_A^s \to PO(3, \mathbb{C})$ corresponding to y has finite image and the group $\rho_s(G_A^s)$ has trivial centralizer in $PO(3, \mathbb{C})$.

Let $\rho = \omega(\rho_s) : G^a_A \to PO(3, \mathbb{C})$, where G^a_A is the *Artin group* of the arrangement A. Recall that we have an open embedding

$$\omega \circ alg \circ geo : Y \cong BR_0(A) \hookrightarrow \mathrm{Hom}(G^a_A, PO(3))//PO(3) = X(G^a_A, PO(3))$$

Suppose that G^a_A is the fundamental group of a smooth complex algebraic variety. Then Theorem 14.6 can be applied to the germ $(X(G^a_A, PO(3)), [\rho])$ provided we can construct a local cross-section through ρ to the $PO(3, \mathbb{C})$-orbits. In the definition of local cross-section we take

$$U := \mathrm{Hom}_f(G^a_A, PO(3, \mathbb{C})) \quad \text{and} \quad S := \omega[\mathrm{BHom}_f(G^s_A, PO(3, \mathbb{C}))] \ .$$

Then U is open by Corollary 12.25 and S is a cross-section because $\mathrm{BHom}_f(G^s_A, PO(3, \mathbb{C}))$ is a cross-section for the action of $PO(3, \mathbb{C})$ on $\mathrm{Hom}_f(G^s_A, PO(3, \mathbb{C}))$ and the morphism $\omega : \mathrm{Hom}_f(G^s_A, PO(3, \mathbb{C})) \to U$ is an isomorphism. We get a contradiction. To see that there are infinitely many nonisomorphic examples we refer to the argument at the end of the introduction. $\square$

As the simplest example of $(Y, 0)$ we can take the germ $(\{x^5 = 0\}, 0)$. We describe the Coxeter graph of the Artin group corresponding to this singularity on the Figure 16. We let $x^5 = (x^2)^2 \cdot x$. To get the labelled graph[10] Λ of G^a from the diagram on the Figure 16 identify vertices marked by the same symbols.

15 Sullivan's minimal models and singularities of representation varieties

The goal of this section is to give a direct proof of Theorem 15.1 below (which is Theorem 1.12 of the Introduction).

Theorem 15.1 *Let M be a smooth connected complex algebraic variety with the fundamental group Γ. Let $\mathbf{G}$ be the Lie group of real points of an algebraic group G defined over $\mathbb{R}$; let $\mathfrak{g}$ be the Lie algebra of $\mathbf{G}$. Suppose that $\rho : \Gamma \to \mathbf{G}$ is a representation with finite image. Then the germ $(\mathrm{Hom}(\Gamma, G), \rho)$ is analytically isomorphic to a quasi-homogeneous cone with generators of weights 1 and 2 and relations of weights 2, 3 and 4. In the case there is a local cross-section through ρ to $Ad(G)$-orbits, then the same conclusion is valid for the quotient germ $(X(\Gamma, G), [\rho])$ of the character variety.*

Proof: We begin the proof by choosing a smooth Φ-equivariant compactification $\tilde{N} = \tilde{M} \cup D$ of $\tilde{M}$ as in the proof of Theorem 14.4.

In what follows we shall use the following simple lemma:

Lemma 15.2 *Suppose that Φ is a finite group and*

$$\begin{array}{ccc} & & H \\ & & {\scriptstyle g}\big\downarrow \\ E & \xrightarrow{f} & F \end{array}$$

is a diagram of morphisms of Φ-modules over $\mathbb{C}$ such that $f(E) \subset g(H)$. Then f admits a Φ-equivariant lifting $\tilde{f} : E \to H$.

[10] See §§4, 11.

Proof: Since $f(E) \subset g(H)$ there exists a linear mapping $h : E \to H$ which lifts f. Then we let

$$\tilde{f} = Av(h) := |\Phi|^{-1} \sum_{\phi \in \Phi} \phi \circ h \circ \phi^{-1} \qquad \square$$

Recall that Morgan in [Mo2] defines a *mixed Hodge diagram*

$$\mathcal{E}(logD) \xleftarrow{\varphi} E_{C^\infty}(\tilde{M}) \otimes \mathbb{C} \xrightarrow{\bar{\varphi}} \bar{\mathcal{E}}(logD)$$

associated with the pair $(\tilde{N}, D)$. The log-complex $\mathcal{E}(logD)$ is a subcomplex of $\mathbb{C}$-valued differential forms on $\tilde{M}$ and the complex $\bar{\mathcal{E}}(logD)$ is the log-complex with the *opposite* complex structure.

The mixed Hodge diagram must satisfy certain properties described in [Mo2]. In particular, it has a structure of a mixed Hodge complex, i.e. $E_{C^\infty}(\tilde{M})$ has an increasing filtration W and $\mathcal{E}(logD)$ has a pair of filtrations: an increasing weight filtration W and decreasing Hodge filtration F; $\varphi, \bar{\varphi}$ must preserve the weight filtrations and be quasi-isomorphisms.

Proposition 15.3 *There exists a Φ-invariant mixed Hodge diagram with Φ-invariant structure of a mixed Hodge complex. The identity embedding*

$$id : \mathcal{E}(logD) \hookrightarrow \mathcal{A}(\tilde{M}) \otimes \mathbb{C}$$

is a quasi-isomorphism.

Proof: First we describe the *log-complex* $\mathcal{E}(logD)$ on $\tilde{M}$ associated with the compactification N. Let $z \in D$ be a point of p-fold intersection

$$z \in D_{i_1} \cap ... \cap D_{i_p}$$

where each D_{i_j} is locally (near z) is given by the equation $z_{i_j} = 0$. Then elements σ of $\mathcal{E}$ are $\mathbb{C}$-valued differential forms on $\tilde{M}$ which can be (locally with respect to the z_j-coordinates) written as

$$\sum_J \eta_J \frac{dz_{i_1}}{z_{i_1}} \wedge ... \wedge \frac{dz_{i_p}}{z_{i_p}}$$

where each η_j extends to a C^∞-form in a neighborhood of z in N. Thus near any generic point $z \in D_j$ the form σ has at worst a simple pole. Since the group Φ acts holomorphically on N leaving D invariant we conclude that this group acts naturally on the log-complex $\mathcal{E}(logD)$. The complex $\mathcal{E}(logD)$ has the *weight* and *Hodge* filtrations which are defined in a canonical way. Recall that $W_\ell(\mathcal{E}(logD))$ consists of differential forms of the type

$$\omega = \sum_J \omega_J \wedge \frac{dz_{j_1}}{z_{j_1}} \wedge ... \wedge \frac{dz_{j_t}}{z_{j_t}} \ , \ t \leq \ell$$

and $F^p(\mathcal{E}(logD))$ consists of differential forms of the type

$$\omega = \sum_J \omega_J \wedge dz_{j_1} \wedge ... dz_{j_s} \wedge \frac{dz_{j_{s+1}}}{z_{j_{s+1}}} \wedge ... \wedge \frac{dz_{j_t}}{z_{j_t}} \ , \ t \geq p$$

where the forms ω_J extend smoothly over the divisor D. Thus both filtrations are Φ-invariant.

Now we describe the second complex $E_{C^\infty}(\tilde{M})$ associated to (N, D). For each component D_j of D choose a regular neighborhood N_j in Φ-invariant way, i.e. if $\phi \in \Phi$ and $\phi : D_i \to D_j$ then $\phi : N_i \to N_j$. Let $[\omega_j] \in \mathrm{H}^2(N_j, \partial N_j; \mathbb{R})$ be the Thom's class. By using

Lemma 15.2 we choose a 2-form ω_j representing the class $[\omega_j]$ so that for $\phi \in \Phi$ mapping D_i to D_j we have: $\phi^*\omega_j = \omega_i$. Then Morgan takes 1-forms $\gamma_j \in \mathcal{E}(logD)$ supported on N_j so that $d\gamma_j = \omega_j$. We can choose γ_j so that $\phi^*\gamma_j = \gamma_i$ for each $\phi \in \Phi$ such that $\phi : D_i \to D_j$ by using Lemma 15.2 again.

The differential graded algebra $E_{C^\infty}(\tilde{M})$ consists of global sections of a certain sheaf $\mathcal{S}$ of algebras that we will describe below. Let $U \subset \tilde{M}$ be an open subset missing all regular neighborhoods N_j. Then sections of $\mathcal{S}$ over U are real-valued infinitely differentiable differential forms on U.

Let N^p denote subset of $\tilde{N}$ consisting of p-fold intersections between the regular neighborhoods N_j. Take a connected open subset $U \subset N^p$ which is disjoint from N^{p+1}. We suppose that U is contained in the p-fold intersection $N_{i_1} \cap ... \cap N_{i_p}$. Then sections of $\mathcal{S}$ over U are elements of the Hirsch extension:

$$\mathcal{A}^\bullet(U) \otimes_d \Lambda(\tau_{i_1}, ..., \tau_{i_p})$$

where $\mathcal{A}^\bullet(U)$ is the complex of real-valued differential forms on U and

$$d\tau_{i_j} = \omega_{i_j}|_U$$

It is clear that the group Φ acts on the sheaf $\mathcal{S}$ and on the differential graded algebra $E_{C^\infty}(\tilde{M})$ of its sections as well. The complex $E_{C^\infty}(\tilde{M})$ has a canonically defined *weight* filtration $W_\bullet$ which is therefore invariant under the action of Φ. (This filtration is similar to the weight filtration on the log-complex, just use τ_j-s instead of the forms dz_j/z_j.) Finally Morgan defines a morphism

$$\alpha : E_{C^\infty}(\tilde{M}) \otimes \mathbb{C} \to \mathcal{E}(logD)$$

by mapping τ_j to γ_j and differential forms supported in $\tilde{M}$ to themselves. Morgan proves that this morphism and the identity embedding

$$\mathcal{E}(logD) \hookrightarrow \mathcal{A}^\bullet(\tilde{M}) \otimes \mathbb{C}$$

induce isomorphisms of cohomology groups. The mixed Hodge structure on the diagram is Φ-invariant by the construction. This finishes the proof of Proposition 15.3. $\square$

Remark 15.4 *In what follows we shall use the notation $A^\bullet$ to denote the differential graded algebra $E_{C^\infty}(\tilde{M})$ and $A^\bullet_{\mathbb{C}}$ its complexification.*

Given the weight filtration W on $A^\bullet$, Morgan defines an increasing *Dec-weight filtration* $Dec\, W_\bullet(A^\bullet)$ as

$$Dec\, W_\ell(A^k) = \{x : x \in W_{k-\ell}(A^k), dx \in W_{k-\ell+1}(A^{k-1})\}$$

Let $\nu : \mathcal{N} \to A^\bullet$ denote a 1-minimal model:

$$\mathcal{N} = (\ 0 \longrightarrow \mathcal{N}^0 = \mathbb{R} \xrightarrow{0} \mathcal{N}^1 \xrightarrow{d} \mathcal{N}^2 \xrightarrow{d} ...\ )$$

Recall the basic properties of $\mathcal{N}$ and ν proven in [Mo2, §6 and Lemma 7.2]:

(a) The *Dec*-weight filtration $Dec\, W_\bullet(A^\bullet_{\mathbb{C}})$ of $A^\bullet_{\mathbb{C}}$ pulls back to $\mathcal{N}_{\mathbb{C}}$ to a *weight* filtration $W_\bullet(\mathcal{N}_{\mathbb{C}})$ which *splits* so that $\mathcal{N}_{\mathbb{C}}$ becomes *bigraded* with the differential of the bidegree $(1,0)$:

$\mathcal{N}^j_{\mathbb{C}} = \oplus_{i\geq 0}\mathcal{N}^j_i$, $\mathcal{N}^j_i = \mathcal{N}^j_{\mathbb{C}} \cap \mathcal{N}_i$, where $(\mathcal{N}_i)_{\mathbb{C}}$ consists of elements of $\mathcal{N}_{\mathbb{C}}$ of the *weight* i,

$$d : \mathcal{N}_i \to \mathcal{N}_i, \quad \wedge : \mathcal{N}_i \otimes \mathcal{N}_j \to \mathcal{N}_{i+j} \quad ,$$

$\mathcal{N}_0 = \mathcal{N}^0 = \mathbb{C}$ and each $\mathcal{N}_j^i$ is finite-dimensional.

(b) The weight filtration on $\mathcal{N}$ induces a weight filtration on $\mathrm{H}^\bullet(\mathcal{N})$ so that the induced weights on $\mathrm{H}^1(\mathcal{N})$ are $1, 2$, the induced weights on $\mathrm{H}^2(\mathcal{N})$ are $2, 3, 4$.

(c) The homomorphism $\nu : \mathcal{N} \to A^\bullet$ is a weak equivalence.

(d) $\mathcal{N}$ is a 1-minimal differential algebra. Together with (a) it implies that the restriction of the differential d to $\mathcal{N}_1^1$ is identically zero.

Proposition 15.5 *Suppose that $A^\bullet$ is a differential graded commutative algebra (over the ground field $\mathbf{k} = \mathbb{C}$ or $\mathbb{R}$) and Φ is a finite group acting on $A^\bullet$. Then the action of Φ on $A^\bullet$ lifts to an action of Φ on a certain 1-minimal model $\mathcal{N}$ for $A^\bullet$.*

Proof: We will prove the proposition by constructing $\mathcal{N}$ in Φ-invariant way. Let $\mathcal{N}^0 := \mathbf{k}$. Take $\mathcal{N}_{[1]}^1 := \mathrm{H}^1(A^\bullet)$ and $\mathcal{N}_{[1]}$ be the differential graded algebra freely generated by $\mathcal{N}^0 = \mathcal{N}_{[1]}^0$ and $\mathcal{N}_{[1]}^1$. We need a homomorphism

$$\nu = \nu_{[1]} : \mathcal{N}_{[1]} \to A^\bullet$$

which induces an isomorphism of the 1-st cohomology groups. The group Φ naturally acts on $\mathrm{H}^1(A^\bullet)$. We have the epimorphism of Φ-modules

$$Z^1(A^\bullet) \longrightarrow \mathrm{H}^1(A^\bullet)$$

By Lemma 15.2 this epimorphism admits a Φ-invariant splitting $\nu^1 : \mathrm{H}^1(A^\bullet) \to Z^1(A^\bullet)$. Thus we let ν be the identity embedding of $\mathcal{N}^0 = \mathbf{k}$ to A^0 and $\nu|\mathcal{N}_{[1]}^1$ be ν^1. We continue the construction of $(\mathcal{N}, \nu)$ by induction. Suppose that $(\mathcal{N}_{[i]}, \nu_{[i]})$ are constructed and the homomorphism of Φ-modules $\nu_{[i]} : \mathcal{N}_{[i]} \to A^\bullet$ induces an isomorphism of H^1 but the induced mapping of H^2 has nonzero kernel. This kernel is canonically isomorphic to the relative cohomology group $\mathrm{H}^2(\mathcal{N}_{[i]}^\bullet, A^\bullet)$. Choose a Φ-equivariant section $\sigma_{[i]}$ to the projection from the space of relative cocycles $Z^2(\mathcal{N}_{[i]}^\bullet, A^\bullet)$ onto $\mathrm{H}^2(\mathcal{N}_{[i]}, A^\bullet)$. Let

$$p_1 : Z^2(\mathcal{N}_{[i]}^\bullet, A^\bullet) \to Z^2(\mathcal{N}_{[i]}^\bullet)$$

and

$$p_2 : Z^2(\mathcal{N}_{[i]}^\bullet, A^\bullet) \to A^1$$

be the projections. Both are Φ-equivariant. Define

$$d : \mathrm{H}^2(\mathcal{N}_{[i]}^\bullet, A^\bullet) \to Z^2(\mathcal{N}_{[i]})$$

by $d = p_1 \circ \sigma_{[i]}$ and let

$$\nu_{i+1} : \mathrm{H}^2(\mathcal{N}_{[i]}^\bullet, A^\bullet) \to A^1 \ , \quad \nu = p_2 \circ \sigma_{[i]}$$

Define

$$\mathcal{N}_{[i+1]} := \mathcal{N}_{[i]} \otimes_d \mathrm{H}^2(\mathcal{N}_{[i]}, A^\bullet)$$

and extend ν_{i+1} to $\nu_{[i+1]} : \mathcal{N}_{[i+1]} \to A^\bullet$ multiplicatively. The group Φ acts on $\mathcal{N}_{[i+1]}$ in the natural way and the homomorphism $\nu_{[i+1]}$ is Φ-equivariant. $\square$

We now assume that we have a mixed Hodge diagram:

$$\mathcal{E} \xleftarrow{\varphi} A_{\mathbb{C}} \xrightarrow{\overline{\varphi}} \overline{\mathcal{E}}$$

and a finite group Φ acting on the diagram compatibly with the differential graded algebra structures so that φ and $\overline{\varphi}$ are Φ-equivariant. In [Mo2, §6], Morgan constructs a trigraded 1-minimal model

$$\begin{tikzcd} \mathcal{E} & A_{\mathbb{C}} \arrow[l, "\varphi"'] \arrow[r, "\overline{\varphi}"] & \overline{\mathcal{E}} \\ & \mathcal{N}^{\bullet}_{\bullet,\bullet} \arrow[ul] \arrow[u] \arrow[ur] & \end{tikzcd}$$

for the above mixed Hodge diagram (see [Mo2, Page 270] for definition). We will say that $\mathcal{N}^{\bullet}_{\bullet,\bullet}$ is a Φ-equivariant trigraded 1-minimal model for the above mixed Hodge diagram if all three morphisms with the source $\mathcal{N}$ are Φ-equivariant and the trigrading of $\mathcal{N}$ is Φ-invariant.

Remark 15.6 *Morgan calls $\mathcal{N}^{\bullet}_{\bullet,\bullet}$ a* **bigraded** *minimal model.*

Proposition 15.7 *There exists a Φ-equivariant trigraded 1-minimal model for the mixed Hodge diagram*

$$\mathcal{E} \xleftarrow{\varphi} A_{\mathbb{C}} \xrightarrow{\overline{\varphi}} \overline{\mathcal{E}}$$

The filtration $Dec\, W_{\bullet}(A_{\mathbb{C}})$ pulls back to a filtration $Dec\, W_{\bullet}(\mathcal{N}_{\mathbb{C}})$ given by

$$Dec\, W_q(\mathcal{N}_{\mathbb{C}}) := \oplus_{r+s\leq q}\ \mathcal{N}_{r,s}$$

Consequently the filtration $Dec\, W_{\bullet}(\mathcal{N}_{\mathbb{C}})$ is Φ-equivariantly split.

Proof: We will check that Morgan's construction can be made Φ-equivariant. To do this we examine Morgan's induction step when he passes from a trigrading on $\mathcal{N}_{[i]}$ to one on $\mathcal{N}_{[i+1]}$. This step is carried out on the page 176 of Morgan's paper and involves a study of the diagram

$$\begin{tikzcd} Z^2(\mathcal{N}_{[i]},\mathcal{E}) \arrow[dr, shift left] & Z^2(\mathcal{N}_{[i]},A_{\mathbb{C}}) \arrow[l, "j"'] \arrow[r, "\overline{j}"] \arrow[d, shift left] & Z^2(\mathcal{N}_{[i]},\overline{\mathcal{E}}) \arrow[dl, shift left] \\ & \mathrm{H}^2(\mathcal{N}_{[i]},A_{\mathbb{C}}) \arrow[ul, shift left, "s"] \arrow[u, shift left, "p"] \arrow[ur, shift left, "s'"] & \end{tikzcd}$$

By induction $\mathrm{H}^2(\mathcal{N}^{\bullet}_{[i]})$ has a Φ-equivariant mixed Hodge structure. Since $\mathrm{H}^2(\mathcal{N}^{\bullet}_{[i]}, A^{\bullet}_{\mathbb{C}})$ is the kernel of the canonical (thus Φ-equivariant) morphism

$$\mathrm{H}^2(\mathcal{N}^{\bullet}_{[i]}) \to \mathrm{H}^2(A^{\bullet}_{\mathbb{C}})$$

it inherits a Φ-equivariant mixed Hodge structure. Consequently by Deligne's Theorem (see [Mo2, Proposition 1.9]) $\mathrm{H}^2(\mathcal{N}^{\bullet}_{[i]}, A^{\bullet}_{\mathbb{C}})$ has a canonical (hence Φ-invariant) bigrading. Thus it suffices to check that the cross-sections s, p and s' as well as the maps h, h' in [Mo2, Page 176] can be chosen to be Φ-equivariant. The cross-sections s, p, s' are required to satisfy the linear conditions (1)–(3) of [Mo2, Page 176]. It is immediate that our averages $Av(s), Av(p), Av(s')$ (defined as in Lemma 15.2 with respect to the action of Φ) again satisfy (1)–(3). Finally the maps

$$h : \mathrm{H}^2(\mathcal{N}^{\bullet}_{[i]}, A^{\bullet}_{\mathbb{C}}) \to DecW_{r+s}(\mathcal{E})$$

and

$$h' : \mathrm{H}^2(\mathcal{N}^{\bullet}_{[i]}, A^{\bullet}_{\mathbb{C}}) \to DecW_{r+s}(\overline{\mathcal{E}})$$

must satisfy certain lifting conditions. By Lemma 15.2 we can take h, h' to be Φ-equivariant.

By definition (see Proposition 15.5) we have

$$\mathcal{N}_{[i+1]} = \mathcal{N}_{[i]} \otimes_d \mathrm{H}^2(\mathcal{N}^\bullet_{[i]}, A^\bullet_{\mathbb{C}})$$

we extend the trigrading from $\mathcal{N}_{[i]}$ and $\mathrm{H}^2(\mathcal{N}^\bullet_{[i]}, A^\bullet_{\mathbb{C}})$ to $\mathcal{N}_{[i+1]}$ multiplicatively. We obtain a Φ-equivariant trigrading on $\mathcal{N}_{[i+1]}$ and a new diagram

$$\begin{array}{ccccc} \mathcal{E} & \longleftarrow & A_{\mathbb{C}} & \longrightarrow & \overline{\mathcal{E}} \\ & \nwarrow & \uparrow & \nearrow & \\ & & \mathcal{N}_{[i+1]} & & \end{array}$$

of equivariant maps satisfying Morgan's axioms. This completes the proof of the proposition. $\square$

We will no longer need the trigrading on $\mathcal{N}$ and instead will consider the bigrading:

$$\mathcal{N}^k_q := \oplus_{r+s=q} \mathcal{N}^k_{r,s}$$

which defines a splitting of the Dec-filtration $Dec\, W_\bullet(\mathcal{N}_{\mathbb{C}})$.

Let P be the flat bundle over M associated to ρ and adP the associated $\mathfrak{g}$-bundle; similarly $ad\tilde{P} = \tilde{M} \times \mathfrak{g}$. We let $adP_{\mathbb{C}}$, $ad\tilde{P}_{\mathbb{C}}$ denote the complexifications of these vector bundles. Let $\mathcal{A}^\bullet(M, adP)$, $\mathcal{A}^\bullet(\tilde{M}, ad\tilde{P})$ denote the differential graded Lie algebras of adP, $ad\tilde{P}$–valued differential forms. According to [KM3, Theorem 2.4] we have:

Theorem 15.8 *If there is a local cross-section through ρ to the $Ad(G)$-orbits then the differential graded Lie algebra $\mathcal{A}^\bullet(M, adP)$* **controls** *the germ $(X(\Gamma, G), [\rho])$.*

We tensor $\mathcal{A}^\bullet(\tilde{M})$ with the Lie algebra $\mathfrak{g}$ (regarded as a differential graded Lie algebra concentrated in the degree zero). Since Φ acts on $\mathfrak{g}$ via $ad\rho$, it also acts on the tensor product. Since $ad\tilde{P}$ is trivial we have isomorphisms

$$\mathcal{A}^\bullet(M, adP) \cong \mathcal{A}^\bullet(\tilde{M}, ad\tilde{P})^\Phi \cong (\mathcal{A}^\bullet(\tilde{M}) \otimes \mathfrak{g})^\Phi$$

We conclude that under the conditions of Theorem 15.8 the differential graded Lie algebra $(\mathcal{A}^\bullet(\tilde{M}) \otimes \mathfrak{g})^\Phi$ controls the germ of the character variety $(X(\Gamma, G), [\rho])$ (see Theorem 13.6).

We will need similar results for the representation variety $\mathrm{Hom}(\Gamma, G)$ itself. Pick a point $m \in \tilde{M}$. We define an augmentation $\epsilon : \mathcal{A}^\bullet(M, adP) \to \mathfrak{g}$ by evaluating degree zero forms at a base-point $m \in M$ and sending the rest of forms to zero. Let $\mathcal{A}^\bullet(M, adP)_0$ be the kernel of ϵ. Recall that by Theorem 14.1 $\mathcal{A}^\bullet(M, adP)_0$ controls the germ $(\mathrm{Hom}(\Gamma, G), \rho)$.

We lift the augmentation ϵ to $\mathcal{A}^\bullet(\tilde{M}, ad\tilde{P}_{\mathbb{C}})$ as follows. Let $\tilde{m}$ be a point in $\tilde{M}$ which projects to m. Then for each $\omega \in \mathcal{A}^0(\tilde{M}, ad\tilde{P}_{\mathbb{C}})$ let

$$\tilde{\epsilon}(\omega) := |\Phi|^{-1} \sum_{\gamma \in \Phi} \gamma \cdot \omega(\tilde{m})$$

where $|\Phi|$ is the order of the group Φ. We extend $\tilde{\epsilon}$ to the rest of $\mathcal{A}^\bullet(M, ad\tilde{P})$ by zero. It is clear that the restriction of $\tilde{\epsilon}$ to

$$\mathcal{A}^\bullet(\tilde{M}, ad\tilde{P})^\Phi \cong \mathcal{A}^\bullet(M, adP)$$

is the same as ϵ. We let $\mathcal{A}^\bullet(M, adP)_0 := \ker(\tilde{\epsilon})$. It is immediate that

$$\mathcal{A}^\bullet(M, adP)_0 \cong \mathcal{A}^\bullet(\tilde{M}, ad\tilde{P})^\Phi_0$$

We let $\beta : \mathcal{A}^\bullet(M) \to \mathbb{R}$ be the evaluation at m and $\tilde{\beta}$ be the lift of β to $\mathcal{A}^\bullet(\tilde{M})$ as above. We set $\mathcal{A}^\bullet(\tilde{M})_0 := \ker \tilde{\beta}$. It is immediate that the isomorphism above carries $\tilde{\epsilon}$ to $\tilde{\beta} \otimes id$ and we obtain induced isomorphisms

$$\mathcal{A}^\bullet(M, adP)_0 \cong \mathcal{A}^\bullet(\tilde{M}, ad\tilde{P})_0^\Phi \cong (\mathcal{A}^\bullet(\tilde{M})_0 \otimes \mathfrak{g})^\Phi$$

By [GM, Theorem 6.8] we conclude that $(\mathcal{A}^\bullet(\tilde{M})_0 \otimes \mathfrak{g})^\Phi$ controls the germ $(\mathrm{Hom}(\Gamma, G), \rho)$. By modifying the above argument in an obvious way we find that $(\mathcal{A}^\bullet(\tilde{M})_0 \otimes \mathfrak{g} \otimes \mathbb{C})^\Phi$ controls the germ $(\mathrm{Hom}(\Gamma, G_{\mathbb{C}}), \rho)$.

Finally we note that $\tilde{\beta}$ induces an augmentation of $A^\bullet = E^\bullet(\tilde{M}) = E^\bullet(\tilde{M})$. We let $A_0^\bullet$ denote $\ker \tilde{\beta}|A^\bullet$. Repeating the above arguments we find that $(A_0^\bullet \otimes \mathfrak{g})^\Phi$ controls the germ $(\mathrm{Hom}(\Gamma, G), \rho)$, $(A_0^\bullet \otimes \mathfrak{g} \otimes \mathbb{C})^\Phi$ controls the germ $(\mathrm{Hom}(\Gamma, G_{\mathbb{C}}), \rho)$ and (under the conditions of Theorem 15.8) $(A^\bullet \otimes \mathfrak{g})^\Phi$ and $(A^\bullet \otimes \mathfrak{g} \otimes \mathbb{C})^\Phi$ control the germs

$$(X(\Gamma, G), [\rho]) \quad \text{and} \quad (X(\Gamma, G_{\mathbb{C}}), [\rho])$$

respectively.

The action of the finite group Φ lifts from $A^\bullet \otimes \mathfrak{g}$ to the tensor product $\mathcal{N} \otimes \mathfrak{g}$ (recall that the action on the Lie algebra $\mathfrak{g}$ is induced by the adjoint representation $ad\rho$ of the group $\pi_1(M)$). Let $\mathcal{M} \subset \mathcal{N}_{\mathbb{C}} \otimes \mathfrak{g}$ denote the subalgebra defined as:

$$\mathcal{M} = (\mathcal{N}_{\mathbb{C}})^\Phi \;\; \text{is the space of} \;\; \Phi-\text{invariants.}$$

Let μ denote the restriction of ν to $\mathcal{M}$, the image of μ lies in the algebra of Φ-invariants $(A^\bullet \otimes \mathfrak{g})^\Phi$. Similarly we let $\mathcal{L} \subset \mathcal{M}$ denote the kernel of $\tilde{\epsilon} \circ \mu$.

Lemma 15.9 *The homomorphisms* $\mu : \mathcal{M} \to (A^\bullet \otimes \mathfrak{g} \otimes \mathbb{C})^\Phi, \mu : \mathcal{L} \to (A^\bullet \otimes \mathfrak{g} \otimes \mathbb{C})_0^\Phi$ *induce isomorphisms of* $\mathrm{H}^0, \mathrm{H}^1$ *and monomorphisms of* H^2.

Proof: Standard. □

Remark 15.10 *Note that* $\mathrm{H}^0(\mathcal{A}^\bullet(M, adP_{\mathbb{C}})_0) \cong \mathrm{H}^0((A^\bullet \otimes \mathfrak{g} \otimes \mathbb{C})_0^\Phi) \cong \mathrm{H}^0(\mathcal{L}) \cong 0$ *and, under the assumption of Theorem 15.8,* $\mathrm{H}^0(\mathcal{A}^\bullet(M, adP_{\mathbb{C}})) \cong \mathrm{H}^0((A^\bullet \otimes \mathfrak{g} \otimes \mathbb{C})^\Phi) \cong \mathrm{H}^0(\mathcal{M}) \cong 0$.

Corollary 15.11 *The differential graded Lie algebra $\mathcal{L}$ controls the germ* $(\mathrm{Hom}(\Gamma, G_{\mathbb{C}}), \rho)$ *and (under the conditions of Theorem 15.8) the differential graded Lie algebra $\mathcal{M}$ controls the germ* $(X(\Gamma, G_{\mathbb{C}}), [\rho])$.

The split weight filtration on $\mathcal{N}_{\mathbb{C}}$ defines a split weight filtration on $\mathcal{N} \otimes \mathfrak{g} \otimes \mathbb{C}$ (by taking tensor products of the components $\mathcal{N}_i$ with $\mathfrak{g}$). Using Proposition 15.7 restrict the split weight filtration from $\mathcal{N}_{\mathbb{C}}$ to $\mathcal{M}$ and $\mathcal{L}$. Clearly these split filtrations of $\mathcal{M}$ and $\mathcal{L}$ satisfy the properties (a)-(c) (of $\mathcal{N}$), where in (a) instead of $\wedge$ we take the Lie bracket. The property (d) fails, however we will need only its weak version:

(d') The restriction of the differential to $\mathcal{M}_1^1$ and $\mathcal{L}_1^1$ is identically zero.

Our further arguments are the same in the cases of $\mathcal{M}$ and $\mathcal{L}$ so we will discuss only $\mathcal{M}$.

Note that $\mathcal{J}_5 := \oplus_{i=5}^\infty \mathcal{M}_i$ is an *ideal* in $\mathcal{M}$. We let $\mathcal{J}$ be the ideal generated by $\mathcal{J}_5$ and $\mathcal{M}_4^1$:

$$\mathcal{J} = \mathcal{J}_5 \oplus \mathcal{M}_4^1 \oplus d(\mathcal{M}_4^1)$$

The quotient $\mathcal{M}/\mathcal{J}$ is again a differential graded Lie algebra. The property (b) of $\mathcal{M}$ (weight restrictions on the cohomology groups) implies that the projection morphism $\mathcal{M} \to \mathcal{Q} := \mathcal{M}/\mathcal{J}$ induces isomorphisms of the 1-st and 2-nd cohomology groups. Hence by Theorem 13.6 the differential graded Lie algebras $\mathcal{Q}$ and $\mathcal{M}$ control germs which are analytically isomorphic. So it is enough to prove that $\mathcal{Q}$ controls a quasi-homogeneous germ with the correct weights and we shall consider the differential graded Lie algebra $\mathcal{Q}$ from now on.

Remark 15.12 *We shall use the notation $\mathcal{Q}_p$ to denote the projection of $\mathcal{M}_p$ ($p \le 4$), elements of $\mathcal{Q}_p$ will be denoted η_p. Lemma 13.4 implies that $\mathcal{Q} = \oplus_p \mathcal{Q}_p$ so that:*

- $d : \mathcal{Q}_p \to \mathcal{Q}_p$ *for each* p*;*
- $[\cdot,\cdot] : \mathcal{Q}_p \otimes \mathcal{Q}_q \to \mathcal{Q}_{p+q}$ *;*
- *The induced weights on* $\mathrm{H}^1(\mathcal{Q}^\bullet)$ *are* $1, 2$ *and the induced weights on* $\mathrm{H}^2(\mathcal{Q}^\bullet)$ *are* $2, 3, 4$*;*
- $d(\mathcal{Q}_1^1) = 0$.
- $\mathcal{Q}^0 = \mathcal{Q}_0 = 0$ *(see Remark 15.10).*

We split each vector space $\mathcal{Q}_p^k$ into the direct sum $\mathcal{H}_p^k \oplus B_p^k \oplus C_p^k$, where

- The space of coboundaries B_p^k is the image of $d_p : \mathcal{Q}_p^{k-1} \to \mathcal{Q}_p^k$;
- The space $\mathcal{H}_p^k$ of "harmonic forms" is a complement to B_p^k in $Z_p^k := \ker(d_p : \mathcal{Q}_p^k \to \mathcal{Q}_p^{k+1})$;
- C_p^k is a complement to Z_p^k in $\mathcal{Q}_p^k$.

We let $\beta_p : \mathcal{Q}_p^k \to B_p^k$ denote the projection with the kernel $\mathcal{H}_p^k \oplus C_p^k$ ("coclosed k-forms"). We let $I_p : B_p^k \to C_p^{k-1}$ denote the inverse to the differential d_p. This allows us to define the "co-differential"

$$\delta_p : \mathcal{Q}_p^k \to C_p^{k-1} \subset \mathcal{Q}_p^{k-1}, \quad \delta_p = I_p \circ \beta_p$$

(whose kernel is $\mathcal{H}_p^k \oplus C_p^k$). Let $\Pi_p : \mathcal{Q}_p \to \mathcal{H}_p$ denote the projection with the kernel $B_p \oplus C_p$. Clearly the projection

$$\mathcal{H}_p^k \to \mathrm{H}^k(\mathcal{Q}_p^\bullet)$$

is an isomorphism of vector spaces. Notice that $\mathcal{Q}^1 = \mathcal{Q}_1^1 \oplus \mathcal{Q}_2^1 \oplus \mathcal{Q}_3^1$. Consider the variety $V \subset \mathcal{Q}^1$ given by the equation

$$d\eta + [\eta, \eta]/2 = 0, \quad \eta \in \mathcal{Q}^1 \tag{4}$$

The algebra $\mathcal{Q}$ controls the germ $(V, 0)$ (see §13) and our goal is to show that this germ is quasi-homogeneous with correct weights. Since $\eta = \eta_1 + \eta_2 + \eta_3$ the equation (4) is equivalent to the system:

$$d\eta_1 = 0, \quad d\eta_2 + [\eta_1, \eta_1]/2 = 0, \quad d\eta_3 + [\eta_1, \eta_2] = 0, \quad [\eta_1, \eta_3] + [\eta_2, \eta_2]/2 = 0$$

Recall that the differential d is identically zero on $\mathcal{Q}_1^1$ and the restriction

$$d : \mathcal{Q}_3^1 \to \mathcal{Q}_3^2$$

has zero kernel (since $\mathrm{H}^1(\mathcal{Q}^\bullet)$ has no weight 3 elements). Therefore the equation $d\eta_3 + [\eta_1, \eta_2] = 0$ is equivalent to the system of three equations:

$$d[\eta_1, \eta_2] = 0, \quad \Pi_3[\eta_1, \eta_2] = 0 (\text{i.e. } [\eta_1, \eta_2] \text{ is exact},$$

$$\eta_3 + \delta_3[\eta_1, \eta_2] = 0 \quad ,$$

equivalently (since $d\eta_1 = 0$ for all η_1)

$$[\eta_1, d\eta_2] = 0, \quad \Pi_3[\eta_1, \eta_2] = 0, \quad \eta_3 + \delta_3[\eta_1, \eta_2] = 0$$

Note however that the equation $d\eta_2 + [\eta_1, \eta_1] = 0$ together with the graded Lie identity imply that $[\eta_1, d\eta_2] = 0$. Thus we eliminate the variable η_3 and the system of equations (4) is equivalent to:

$$d\eta_2 + [\eta_1, \eta_1]/2 = 0, \quad \Pi_3[\eta_1, \eta_2] = 0, \quad -[\eta_1, \delta_3[\eta_1, \eta_2]] + [\eta_2, \eta_2]/2 = 0 \tag{5}$$

The mappings d, δ_3, Π_3 are linear and the bracket $[\cdot, \cdot]$ is quadratic. We conclude that the system of equations (5) is quasi-homogeneous where the weights of the generators (i.e. the components of) η_j are $j = 1, 2$ and the weights of the relations are $2, 3$ and 4. The only problem is that the first polynomial equation has nonzero linear term. To resolve this problem we let $\eta_2 = \eta_2' + \eta_2''$, where $\eta_2' \in Z_2^1, \eta_2'' \in C_2^1$. Thus (similarly to the case of η_3) we get:

the equation $d\eta_2 + [\eta_1, \eta_1]/2 = 0$ is equivalent to the system:

$$\Pi_2[\eta_1, \eta_1] = 0, \quad \eta_2'' + \delta_2[\eta_1, \eta_1]/2 = 0$$

Thus we have $\eta_2 = \eta_2' - \delta_2[\eta_1, \eta_1]/2$ and instead of the system of equations (5) we get the system

$$\Pi_2[\eta_1, \eta_1] = 0, \quad \Pi_3[\eta_1, \eta_2' - \delta_2[\eta_1, \eta_1]] = 0,$$

$$-[\eta_1, \delta_3[\eta_1, \eta_2' - \delta_2[\eta_1, \eta_1]/2]] + [\eta_2' - \delta_2[\eta_1, \eta_1]/2, \eta_2' - \delta_2[\eta_1, \eta_1]/2]/2 = 0$$

which is quasi-homogeneous with the required weights. Theorem 15.1 follows. $\square$

16 Malcev Lie algebras of Artin groups

Out discussion of the material below follows [ABC]. Let $\mathbf{k}$ be a field of zero characteristic and Γ be a group. We define the $\mathbf{k}$-*unipotent completion* (or *Malcev completion*) $\Gamma \otimes \mathbf{k}$ of Γ by the following universal property:

- There is a homomorphism $\eta \otimes \mathbf{k} : \Gamma \to \Gamma \otimes \mathbf{k}$.
- For every $\mathbf{k}$-unipotent Lie group U and any homomorphism $\rho : \Gamma \to U$ there is a lift $\tilde{\rho} : \Gamma \otimes \mathbf{k} \to U$ so that $\tilde{\rho} \circ (\eta \otimes \mathbf{k}) = \rho$.
- $\Gamma \otimes \mathbf{k}$ and $\eta \otimes \mathbf{k}$ are unique up to an isomorphism.
- $\Gamma \otimes \mathbf{k}$ is $\mathbf{k}$-pro-unipotent.

Remark 16.1 *Recall that any group U above is torsion-free and nilpotent.*

Definition 16.2 *The group $\Gamma \otimes \mathbf{k}$ has a* $\mathbf{k}$-pro-nilpotent *Lie algebra $\mathcal{L}(\Gamma, \mathbf{k})$. This algebra is called the $\mathbf{k}$-Malcev Lie algebra of Γ.*

We will take $\mathbf{k} = \mathbb{R}$ in what follows. Thus we shall denote $\mathcal{L}(\Gamma) := \mathcal{L}(\Gamma, \mathbb{R})$, $\eta := \rho \otimes \mathbb{R}$, etc.

Example 16.3 *Suppose that the group Γ has a generating set consisting of elements of finite order. Then $\mathcal{L}(\Gamma) = 0$ because $\Gamma \otimes \mathbb{R} = \{1\}$. In particular, if Γ is a Shephard group where all vertices have nonzero labels then $\mathcal{L}(\Gamma) = 0$.*

Let H be a finite-dimensional real vector space, $L(H)$ is the free Lie algebra spanned by H. It can be described as follows. Consider the tensor algebra $T(H)$ of tensors of all possible degrees on H, define the Lie bracket of $T(H)$ by $[u,v] = u \otimes v - v \otimes u$. Then $L(H)$ is the Lie subalgebra in $T(H)$ generated by elements of H.

Let F_r be a free group of rank r and H be the r-dimensional real vector space, then

$$L(H) \cong \mathcal{L}(F_r) \quad .$$

An element $u \in L(H)$ is said to have the degree $\leq d$ if $u \in \oplus_{i \leq n} H^{\otimes i}$. The *degree* of u equals d if $deg(u) \leq d$ but $deg(u)$ is not $\leq d-1$. I.e. the degree of u is the highest degree of monomial in the expansion of u as a linear combination of tensor products of elements of H. For instance, *quadratic elements* of $L(H)$ are elements of the degree 2, i.e. they have the form of nonzero linear combinations

$$\sum_j [u_j, v_j], \quad u_j \ , v_j \ \in H$$

A *quadratically presented* Lie algebra is the quotient $L(H)/J$ where J is an ideal generated by a (possibly empty) set of quadratic elements.

Theorem 16.4 *(P. Deligne, P. Griffith, J. Morgan, D. Sullivan, [DGMS].) Suppose that M is a compact connected Kähler manifold, then the Malcev Lie algebra $\mathcal{L}(\pi_1(M))$ is quadratically presented.*

Theorem 16.5 *(J. Morgan, [Mo1], [Mo2].)* **"Morgan's test."** *Suppose that M is a smooth connected complex algebraic variety. Then the Malcev Lie algebra $\mathcal{L}(\pi_1(M))$ is the quotient $L(H)/J$, where $L(H)$ is a free Lie algebra and the ideal J is generated by elements of degrees $2 \leq d \leq 4$.*

Remark 16.6 *Until now Morgan's theorem was the only known restriction on the fundamental groups of smooth complex algebraic varieties, besides finite presentability. Much more restrictions are known in the case of smooth* **complete** *varieties and compact Kähler manifolds, see [ABC].*

Below we compute Malcev algebras of Artin groups. Suppose that G^a is an Artin group. Let n be the number of generators of G^a. Define a Lie algebra over $\mathbb{R}$

$$\mathcal{L} := \langle X_1, ..., X_n | [X_i, X_j] = 0 \text{ if } \epsilon(i,j) \neq \infty \text{ is even }, X_i = X_j \text{ if } \epsilon(i,j) \neq \infty \text{ is odd } \rangle$$

where $[X, Y]$ denotes the Lie algebra commutator. Clearly this Lie algebra is quadratically presentable. Let x_i denote the generator of G^a corresponding to the vertex v_i.

To compute Malcev completions we will need the following two lemmas

Lemma 16.7 *Suppose that $\rho : G^a \to N$ is a homomorphism to a torsion-free nilpotent group. Then for all x_i, x_j such that $2q_{ij} = \epsilon(i,j) \neq \infty$ we have $[\rho(x_i), \rho(x_j)] = \mathbf{1}$. For all x_i, x_j such that $2q_{ij} + 1 = \epsilon(i,j) \neq \infty$ we have $\rho(x_i) = \rho(x_j)$.*

Proof: The assertion is obvious if N is Abelian. So we assume that the assertion is valid for all $(s-1)$-step nilpotent torsion-free groups $\bar{N}$. Let N be s-step nilpotent. Let $Z(N)$ denote the center of N, let $\bar{N} := N/Z(N)$ and $p : N \to \bar{N}$ be the projection. Then by the induction hypothesis:

- $[p(\rho(x_i)), p(\rho(x_j))] = \mathbf{1}$, provided that $2q_{ij} = \epsilon(i,j) \neq \infty$.

- $p(\rho(x_i)) = p(\rho(x_j))$, provided that $2q_{ij}+1 = \epsilon(i,j) \neq \infty$.

(1) Consider the case $2q_{ij} = \epsilon(i,j)$. Then $\rho(x_i)\rho(x_j) = \rho(x_j)\rho(x_i)z$, for some $z \in Z(N)$. Thus the relation

$$(x_i x_j)^{q_{ij}} = (x_j x_i)^{q_{ij}}$$

implies that

$$z^{q_{ij}}\rho[(x_j x_i)^{q_{ij}}] = \rho[(x_j x_i)^{q_{ij}}]$$

Since N is torsion-free we conclude that $z = 1$ and hence $\rho([x_i, x_j]) = \mathbf{1}$.

(2) Another case is when $2q_{ij}+1 = \epsilon(i,j)$. Then $\rho(x_i) = z\rho(x_j)$, for some $z \in Z(N)$. The relation

$$(x_i x_j)^{q_{ij}} x_i = (x_j x_i)^{q_{ij}} x_j$$

implies that $z = 1$. $\square$

Remark 16.8 *In our paper we use only Artin groups with even labels.*

Lemma 16.9 *Suppose that U is an $\mathbb{R}$-unipotent group, $a, b \in U$ are commuting elements. Then $[\log(a), \log(b)] = 0$ in the Lie algebra of U.*

Proof: Since U is unipotent we can think of U as the subgroup of the group of upper-triangular matrices with 1-s on the diagonal. Then for any $g \in U$ we have: $\log(g) = \log(\mathbf{1} - (\mathbf{1} - g))$, $h = \mathbf{1} - g$ is a nilpotent matrix, thus $\log(\mathbf{1} - h)$ is a polynomial of h. Since matrices a, b commute, any polynomial functions of them commute as well. Thus $[\log(a), \log(b)] = 0$. $\square$

Theorem 16.10 *Under the above conditions $\mathcal{L} \cong \mathcal{L}(G^a)$ is the Malcev Lie algebra of G^a.*

Proof: Let F denote the free group on $x_1, ..., x_n$ and $\pi : F \to G^a$ be the quotient map. Let $F \otimes \mathbb{R}$ be the $\mathbb{R}$-unipotent completion of F and $\eta : F \to F \otimes \mathbb{R}$ be the canonical homomorphism. Let $\mathcal{L}(F, \mathbb{R})$ be the Lie algebra of $F \otimes \mathbb{R}$. Put $g_i = \eta(x_i)$, $1 \leq i \leq n$ and $X_i := \log(g_i)$. Let $\mathcal{I}$ be the ideal in $\mathcal{L}(F, \mathbb{R})$ generated by the commutators $[X_i, X_j]$, for even labels $\epsilon(i,j) \neq \infty$ and the elements $X_i - X_j$ if $\epsilon(i,j) \neq \infty$ is odd. Let $\mathcal{Q} := \mathcal{L}(F, \mathbb{R})/\mathcal{I}$ be the quotient Lie algebra and Q be the corresponding pro-unipotent Lie group over $\mathbb{R}$. Let $\widehat{\pi} : F \otimes \mathbb{R} \to Q$ be the quotient map. Put $\bar{g}_i := \widehat{\pi}(g_i)$ and $\bar{X}_i := d\widehat{\pi}(X_i)$. Then $\bar{g}_i = \exp(\bar{X}_i)$, $1 \leq i \leq n$. Consequently $[\bar{g}_i, \bar{g}_j] = \mathbf{1}$ for all vertices i, j connected by an edge with even label and $\bar{g}_i = \bar{g}_j$ for all vertices i, j connected by an edge with odd label. Hence we have a commutative diagram

$$\begin{array}{ccccc}
F & \xrightarrow{\eta} & F \otimes \mathbb{R} & \longrightarrow & \mathcal{L}(F, \mathbb{R}) \\
{\scriptstyle\pi}\downarrow & & {\scriptstyle\widehat{\pi}}\downarrow & & \downarrow \\
G^a & \xrightarrow{\tau} & Q & \longrightarrow & \mathcal{Q}
\end{array}$$

where $\tau(\pi(x_i)) = \bar{g}_i$, $1 \leq i \leq n$. We claim that Q is the Malcev completion of G^a. It is clear that any homomorphism $\rho : Q \to U$ from Q to a unipotent group U is determined by its pull-back to G^a (because its pull-back to $F \otimes \mathbb{R}$ is determined by its further pull-back to F). So let $\rho : G^a \to U$ be a homomorphism with U a unipotent group over $\mathbb{R}$. The homomorphism $\pi^*\rho$ extends to a morphism $\widehat{\rho} : F \otimes \mathbb{R} \to U$. Note that U is necessarily nilpotent and torsion-free.

According to Lemma 16.7 for each pair of (i, j) such that $\epsilon(i, j)$ is even we have:

$$\widehat{\rho}([g_i, g_j]) = \pi^* \rho([x_i, x_j]) = \rho([x_i, x_j]) = \mathbf{1}$$

Hence $\log(\widehat{\rho}(g_i))$ and $\log(\widehat{\rho}(g_j))$ commute in the Lie algebra u of U (see Lemma 16.9). Therefore $d\widehat{\rho}(X_i)$ and $d\widehat{\rho}(X_j)$ commute in u. The case of odd labels $\epsilon(i, j)$ is similar. This implies that $d\widehat{\rho}$ descends to $\mathcal{Q}$ and consequently $\widehat{\rho}$ descends to Q. $\square$

Corollary 16.11 *If G^a is any Artin group then $\mathcal{L}(G^a)$ is quadratically presented.*

Thus the Artin groups constructed in Theorem 14.7 satisfy *Morgan's test* of being fundamental groups of smooth complex algebraic varieties.

17 Representation varieties near the trivial representation

The second author would like to thank Carlos Simpson for explaining Theorem 17.1 in this section.

Let Γ be a finitely-generated group, $\Gamma \otimes \mathbb{R}$ is its Malcev completion, $\eta \otimes \mathbb{R} : \Gamma \to \Gamma \otimes \mathbb{R}$ is the canonical homomorphism. Let $\mathcal{L}(\Gamma)$ denote the Malcev Lie algebra of Γ. Let $\mathbf{G}$ be the set of real points of an algebraic group G defined over $\mathbb{R}$, $\mathfrak{g}$ be the Lie algebra of $\mathbf{G}$. The homomorphism $\eta \otimes \mathbb{R}$ induces the pull-back morphism $\eta^* : \mathrm{Hom}(\Gamma \otimes \mathbb{R}, \mathbf{G}) \to \mathrm{Hom}(\Gamma, \mathbf{G})$. Let $\rho_0 : \Gamma \otimes \mathbb{R} \to \mathbf{G}$ be the trivial representation .

Theorem 17.1 1. *If the Lie algebra $\mathcal{L}(\Gamma)$ is quadratically presentable then the variety $\mathrm{Hom}(\mathcal{L}(\Gamma), \mathfrak{g})$ is given by homogeneous quadratic equations.*

2. *The varieties $\mathrm{Hom}(\Gamma \otimes \mathbb{R}, \mathbf{G})$ and $\mathrm{Hom}(\mathcal{L}(\Gamma), \mathfrak{g})$ are naturally isomorphic.*

3. *The morphism η^* induces an isomorphism of germs*

$$(\mathrm{Hom}(\Gamma \otimes \mathbb{R}, \mathbf{G}), \rho_0) \longrightarrow (\mathrm{Hom}(\Gamma, \mathbf{G}), \eta^*(\rho_0))$$

Proof: The property (1) is obvious. To prove (2) note that the group $\Gamma \otimes \mathbb{R}$ is $\mathbb{R}$-pro-unipotent, thus we have a natural isomorphism between the representation variety of $\Gamma \otimes \mathbb{R}$ and of its Lie algebra.

Now consider (3). Let $\mathcal{A}$ be an Artin local $\mathbb{R}$-algebra. We recall that $G(\mathcal{A})$ is the set of $\mathcal{A}$-points of G, algebraically the group $G(\mathcal{A})$ is the semidirect product $N_{\mathcal{A}} \rtimes \mathbf{G}$, where $N_{\mathcal{A}}$ is a certain $\mathbb{R}$-unipotent group (kernel of the natural projection $p_0 : G(\mathcal{A}) \to \mathbf{G}$). Consider the space

$$\mathrm{Hom}_0(\Gamma, G(\mathcal{A})) := \{\rho : \Gamma \to G(\mathcal{A}) | p_0(\rho) = \rho_0\} \cong \mathrm{Hom}(\Gamma, N_{\mathcal{A}})$$

Thus, by the definition of $\Gamma \otimes \mathbb{R}$, for each Artin local $\mathbb{R}$-algebra $\mathcal{A}$ the morphism η^* induces a natural bijection between the $\mathbb{R}$-points of the varieties $\mathrm{Hom}(\Gamma, N_{\mathcal{A}})$ and $\mathrm{Hom}(\Gamma \otimes \mathbb{R}, N_{\mathcal{A}})$. Then we have an induced isomorphism between functors

$$\mathrm{Hom}(\Gamma, G) : \mathcal{A} \mapsto set \ , \quad \mathrm{Hom}(\Gamma \otimes \mathbb{R}, G) : \mathcal{A} \mapsto set$$

of $\mathcal{A}$-points. Therefore (by [GM]) η^* induces an isomorphism of the germs

$$(\mathrm{Hom}(\Gamma \otimes \mathbb{R}, \mathbf{G}), \rho_0 \ ) \longrightarrow (\mathrm{Hom}(\Gamma, \mathbf{G}), \eta^*(\rho_0))$$

$\square$

Proposition 17.2 *Let Γ be a Coxeter group or a Shephard group (where all vertices have nonzero labels). Then the trivial representation $\rho_0 : \Gamma \to \mathbf{G}$ is infinitesimally rigid (and hence is an isolated reduced point) in* $\mathrm{Hom}(\Gamma, \mathbf{G})$.

Proof: The group Γ is generated by elements of finite order. Let ζ be a cocycle in $Z^1(\Gamma, \mathfrak{g})$. Then $\zeta|_{\langle g_j \rangle}$ is a coboundary for each generator g_j of Γ (since g_j has finite order). However $\rho_0(g_j) = \mathbf{1}$, hence $\zeta(g_j) = 0$ for all j. We conclude that $\zeta = 0$. $\square$

Theorem 17.3 *Let Γ be any Artin group. Then the representation variety* $\mathrm{Hom}(\Gamma, \mathbf{G})$ *has at worst quadratic singularity at the trivial representation.*

Proof: Combine Corollary 16.11 and Theorem 17.1. $\square$

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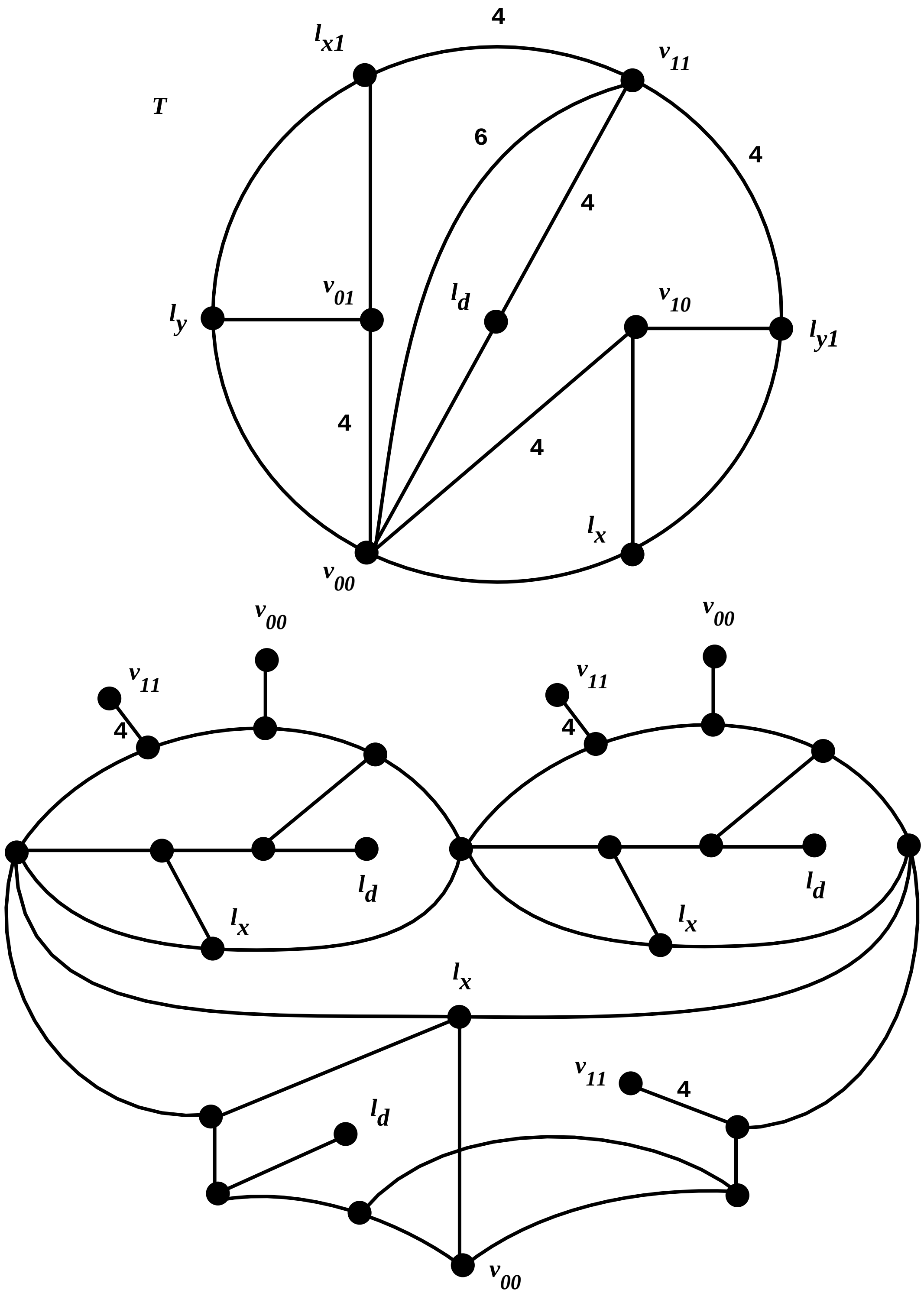

Figure 16: *Labelled graph of an Artin group.*