THE RING OF PROJECTIVE INVARIANTS OF EIGHT POINTS ON THE LINE VIA REPRESENTATION THEORY

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Abstract. The ring of projective invariants of eight ordered points on the line is a quotient of the polynomial ring on V, where V is a fourteen-dimensional representation of S_8, by an ideal I_8, so the modular fivefold $(\mathbb{P}^1)^8 /\!/ \mathrm{GL}(2)$ is $\mathrm{Proj}(\mathrm{Sym}^\bullet(V)/I_8)$. We show that there is a unique cubic hypersurface S in $\mathbb{P}V$ whose equation s is skew-invariant, and that the singular locus of S is the modular fivefold. In particular, over $\mathbb{Z}[1/3]$, the modular fivefold is cut out by the 14 partial derivatives of s. Better: these equations generate I_8. In characteristic 3, the cubic s is needed to generate the ideal. The existence of such a cubic was predicted by Dolgachev. Over $\mathbb{Q}$, we recover the 14 quadrics found by computer calculation by Koike [Koi], and our approach yields a conceptual representation-theoretic description of the presentation. Additionally we find the graded Betti numbers of a minimal free resolution in any characteristic.

The proof over $\mathbb{Q}$ is by pure thought, using Lie theory and commutative algebra. Over $\mathbb{Z}$, the assistance of a computer was necessary. This result will be used as the base case describing the equations of the moduli space of an arbitrary number of points on $\mathbb{P}^1$, with arbitrary weighting, in [HMSV3], completing the program of [HMSV1]. The modular fivefold, and corresponding ring, are known to have a number of special incarnations, due to Deligne-Mostow, Kondo, and Freitag-Salvati Manni, for example as ball quotients or ring of modular forms respectively.

Contents

1. Introduction

Let n be an even integer, let $M_n = (\mathbb{P}^1)^n /\!/ \mathrm{GL}(2)$ be the GIT quotient of n ordered points on the projective line, and let R_n be the projective coordinate ring of M_n.

The first general results on the ring R_n were found by Kempe [Ke] in 1894. He showed that elements of R_n can naturally be interpreted as (formal linear combinations of) regular graphs on n vertices. We review this theory in §2. Kempe used this insight to prove that R_n is generated in degree one. Thus R_n has a natural set of generators: the *matchings* (regular degree one graphs) on n vertices.

Let I_n be the *ideal of relations*, the kernel of $\mathrm{Sym}(R_n^{(1)}) \to R_n$. The problem of giving a presentation for R_n is thus reduced to determining generators for the ideal I_n. For $n = 4$, $I_n = 0$. For $n = 6$, the ideal is generated by a single, beautiful cubic equation: the Segre cubic relation (see for example [DO, p. 17]). In [HMSV3], we will show that for $n \geq 8$, away from small characteristic, I_n is generated by an explicit simple

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class of quadrics. The argument will rely on the base case $n = 8$, the subject of this paper. It turns out that this case (like the cases $n = 4$ and $n = 6$) has some special extrinsic geometry.

Our main theorems are the following. Note that $R_8^{(1)}$ is 14 dimensional.

Theorem 1.1. *Over $\mathbb{Q}$, there is a unique (up to scaling) non-zero skew-invariant cubic polynomial s in $\mathrm{Sym}^3(R_8^{(1)})$. It vanishes on M_8, and M_8 is the singular locus of $s = 0$. Better: the 14 partial derivatives of s generate I_8 — the singular scheme of the affine cone of $s = 0$ is precisely the affine cone over M_8. These 14 partial derivatives have no syzygies of degree zero or one. In terms of graphs, s may be taken to be the skew-average of the cube of any matching.*

This result was predicted to us by Igor Dolgachev. It will be proved by pure thought, that is, without the use of a computer or long, explicit formulas. One consequence is a natural duality between the degree 1 piece of the ring (with representation corresponding to the partition $4 + 4$) and the degree 2 piece of the ideal (with sign-dual representation $2 + 2 + 2 + 2$) into the sign representation given by the cubic.

With the aid of a computer, we have a stronger integrality result:

Theorem 1.2. *Over $\mathbb{Z}$, there is a non-zero cubic polynomial s' in $\mathrm{Sym}^3(R_8^{(1)})$ (a rational multiple of the s of Theorem 1.1) such that M_8 is the singular locus of $s' = 0$. Better: the ideal I_8 is generated over $\mathbb{Z}$ by s' and its 14 partial derivatives. In particular, I_8 is generated over $\mathbb{Z}[1/3]$ by the 14 partial derivatives of s'. Over $\mathbb{Z}$, the cubic s' is not generated by its partial derivatives.*

This result is proved and discussed further in §9.

Remark 1.3. It would be ideal, of course, to have a pure thought proof of Theorem 1.2. Here is where we stand with respect to this. Theorem 1.1 automatically holds over $\mathbb{Z}[1/N]$ for some integer N. This integer cannot be determined from our proofs. However, using analogues of our proofs in positive characteristic, one can obtain a precise value of N so that the theorem remains true over $\mathbb{Z}[1/N]$. Unfortunately, we cannot get down to $N = 3$ by these methods. Since the positive characteristic arguments are more complicated, use less well-known facts and do not yield an optimal result, we decided not to include them.

1.1. **Other manifestations of this space, and this graded ring.** The extrinsic and intrinsic geometry of M_n for small n has special meaning often related to the representation theory of S_n. For example, M_4 relates to the cross ratio, M_5 is the quintic del Pezzo surface, and the geometry of the Segre cubic M_6 is well known (see for example [HMSV2] for the representation theory). The space M_8 might be the last of the M_n with such individual personality. For example, over $\mathbb{C}$, the space may be interpreted as a ball quotient in two ways:

(1) Deligne and Mostow [DM] showed that M_8 is isomorphic to the Satake-Baily-Borel compactification of an arithmetic quotient of the 5-dimensional complex ball, using the theory of periods of a family of curves that are fourfold cyclic covers of $\mathbb{P}^1$ branched at the 8 points.
(2) Kondo [Kon] showed that M_8 may also be interpreted in terms of moduli of certain K3 surfaces, and thus M_8 is isomorphic to the Satake-Baily-Borel compactification of a quotient of the complex 5-ball by $\Gamma(1 - i)$, an arithmetic subgroup of a unitary group of a hermitian form of signature $(1, 5)$ defined over the Gaussian integers. See also [FS2, p. 12] for further clarification and discussion.

Both interpretations are S_8-equivariant (see [Kon, p. 8] for the second).

Similarly, the graded ring R_8 we study has a number of manifestations:

(1) It is the ring of genus 3 hyperelliptic modular forms of level 2.
(2) Freitag and Salvati Manni showed that R_8 is isomorphic to the full ring of modular forms of $\Gamma(1-i)$ [FS2, p. 2], via the Borcherds additive lifting.
(3) The space of sections of multiples of a certain line bundle on $\overline{\mathcal{M}}_{0,8}$ (as there is a morphism $\overline{\mathcal{M}}_{0,8} \to M_8$, [Ka], see also [AL]).
(4) Igusa [I] showed that there is a natural map $A(\Gamma_3[2])/\mathcal{I}_3[2]^0 \to R_8$, where $A(\Gamma_3[2])$ is the ring of Siegel modular forms of weight 2 and genus 3. (See [FS2, §3] for more discussion.)
(5) It is a quotient of the third in a sequence of algebras related to the orthogonal group $\mathrm{O}(2m, \mathbb{F}_2)$ defined by Freitag and Salvati Manni (see [FS1], [FS2, §2]). (The cases $m = 5$ and $m = 6$ are related to Enriques surfaces.)

The Hilbert function $f(k) = \dim R_8^{(k)}$ was found by Howe [Ho, p. 155, §5.4.2.3]:

$$f(k) = \tfrac{1}{3}k^5 + \tfrac{5}{3}k^4 + \tfrac{11}{3}k^3 + \tfrac{13}{3}k^2 + 3k + 1, \qquad \text{for } k \geq 0.$$

The Hilbert series $H(t) = \sum_{k=0}^{\infty} f(k)t^k$ is

$$H(t) = \frac{1 + 8t + 22t^2 + 8t^3 + t^4}{(1-t)^6}.$$

Both of these formulas are given in [FS2, p. 7].

One reason for M_8 to be special is the coincidence $S_8 \cong \mathrm{O}(6, \mathbb{F}_2)$. A geometric description of this isomorphism in this context is given in [FS2, §4]. Another reason is Deligne and Mostow's table [DM, p. 86].

1.2. **Other manifestations of the cubic.** Let n be an even integer. There are natural generators of $R_n^{(1)}$, one for each directed matching on n labeled vertices (see §2.2). The group S_n acts on these coordinates in the obvious way. The signed sum of the cubes of these matchings s_n, regarded as an element of $\mathrm{Sym}^3(R_n^{(1)})$, is skew-invariant. By skew-invariance, it must vanish on those points of M_n where two of the n points come together. Hence s_n must be divisible by the discriminant, which has degree $\frac{1}{2}\binom{n}{2}$. Thus for $n \geq 6$, s_n vanishes on M_n. (This cubic appeared in the e-print [HMSV1e, §2.10], but was removed in the published version because its centrality was not yet understood.) For $n = 4$, s_4 vanishes precisely on the boundary of M_4. For $n = 6$, s_6 is the Segre cubic. For $n = 8$, s_8 is the s of Theorem 1.1 (although it must be scaled to give the s' of Theorem 1.2). And for $n > 8$, it may be shown that $s_n = 0$, so $n = 8$ is indeed the last interesting case.

1.3. **Outline of the proof of Theorem 1.1.** We now describe the main steps in the proof of Theorem 1.1:

(1) We first prove the existence and uniqueness of the cubic s, using just linear algebra.
(2) Next we prove that the partial derivatives of s generate $I_8^{(2)}$, using the structure of the relevant spaces as S_8-modules.
(3) We then prove that the partial derivates of s have no linear syzygies. This is where most of the work occurs.
(4) Finally we use the fact that R_8 is Gorenstein and step (3) to fill in the Betti diagram of R_8. From this we see that I_8 is generated by quadrics.

Steps (3) and (4) can of course be replaced by Koike's computer calculation [Koi], at the expense of the conceptual argument. As a simple corollary of the step (3) we find that $I_8^{(2)}$ generates $I_8^{(3)}$. We could then replace step (4) by an appeal to a result in [HMSV3] which states that I_n is generated by $I_n^{(2)}$ and $I_n^{(3)}$ for any n. However, we prefer to avoid referring to a later paper.

Step (3) may be further broken down as follows:

(3a) Let $\Psi : \mathrm{End}(R_8^{(1)}) \to I_8^{(3)}$ be the map given by $A \mapsto As$, where As is defined via the natural action of the Lie algebra $\mathrm{End}(R_8^{(1)}) \cong \mathfrak{gl}(14)$ on $\mathrm{Sym}^3(R_8^{(1)})$. We first observe that the space of linear syzygies between the partial derivaties of s is exactly $\mathfrak{g} = \ker \Psi$. We note that $\mathfrak{g}$ is a Lie subalgebra of $\mathrm{End}(R_8^{(1)})$ and is stable under the action of S_8.
(3b) Next, using general theory developed in §6 concerning G-stable Lie subalgebras of $\mathrm{End}(V)$, where V is a representation of G, and the classification of simple Lie algebras, we show that the only S_8-stable Lie subalgebras of $\mathrm{End}(R_8^{(1)})$ are 0, $\mathfrak{so}(14)$ and $\mathfrak{sl}(14)$ (ignoring the center). Thus $\mathfrak{g}$ must be one of these three Lie algebras.
(3c) Finally, we show that $\mathfrak{so}(14)$ does not annihilate any non-zero cubic. As $\mathfrak{g}$ is the annihilator of s we conclude $\mathfrak{g} = 0$.

1.4. **Relationship with** [HMSV3]**.** We now discuss the relationship between this paper and [HMSV3]. The two papers prove similar results but are logically and methodologically independent. (Except for a few peripheral remarks in this paper that rely on [HMSV3].) In [HMSV3] we prove that I_n is generated by quadrics for $n \geq 8$. The argument is inductive and uses the $n = 8$ case for its base. This base case is already known by the work of Koike and so, strictly speaking, [HMSV3] does not logically rely on the present paper. However, Koike's proof of the $n = 8$ case was by a computer calculation. Thus the present paper can be viewed as filling this conceptual gap. Together, this paper and [HMSV3] give a complete conceptual proof that I_n is generated by quadrics for $n \geq 8$.

As the main results of this paper and [HMSV3] are both concerned with quadric generation of I_n one might think that it would make more sense to combine the two papers. We feel that this is not the case for two reasons. First, as we have highlighted above, the $n = 8$ case has a number of special properties not shared by the general case. Had we combined the two papers, we feel that these beautiful features would have been obscured in the resulting, much larger paper. And second, although the results of the two papers are similar, the methods of proof are completely different. This paper uses Lie theory and commutative algebra while the main tools of [HMSV3] are toric degenerations and combinatorics.

1.5. **Other results.** In the course of our study we have found some miscellaneous results which do not fit into the rest of the paper.

First, Miles Reid pointed out that the secant variety of M_8 necessarily lies in the cubic $s = 0$, by Bezout's theorem. The secant variety is 11-dimensional, as expected (this is a computer calculation), and is thus a hypersurface in the cubic.

The second result concerns the ring R_8. As stated above, elements of R_n may be represented as formal sums of regular graphs on n vertices. In particular, the matchings on n points span $R_n^{(1)}$. By embedding the n vertices into the unit circle we obtain a notion of planarity. A theorem of Kempe states that the planar graphs give a basis for R_n. We observed (using a computer) the following result, which is particular to the case of 8 points:

Proposition 1.4. *The squares of the non-planar matchings form a basis for $R_8^{(2)}$. This holds over $\mathbb{Z}[1/2]$.*

1.6. **Acknowledgments.** Foremost we thank Igor Dolgachev, who predicted to us that Theorem 1.1 is true. Without him this paper would not have been written. We also thank Shrawan Kumar and Riccardo Salvati Manni for helpful comments.

2. Review of the ring R_L

In this section we give a precise definition of the ring R_n and recall some facts about how S_n acts on R_n. A more thorough treatment of these topics is given in [HMSV3].

Before we begin, we remark that we prefer to work as functorially as possible. This results in greater clarity and does not cost much. Thus, rather than working with an integer n we work with a set L of cardinality n. We will therefore have a ring R_L in place of R_n. Also, rather than working with $\mathbb{P}^1$ we work with $\mathbb{P}U$ where U is a two-dimensional vector space. For this section we work over an arbitrary commutative base ring k. In most of the remainder of the paper we will take k to be a field of characteristic 0.

2.1. **The ring R_L.** Let k be a commutative ring, let L be a finite set of even cardinality n, let U be a free rank two k-module and let ω be a non-degenerate symplectic form on U. We are interested in the GIT quotient $M_L = (\mathbb{P}U)^L /\!/ \mathrm{GL}(U)$. By definition, M_L is $\mathrm{Proj}(R_L)$ where

$$R_L = \bigoplus_{d=0}^{\infty} \Gamma((\mathbb{P}U)^L, \mathscr{O}(d)^{\boxtimes L})^{\mathrm{GL}(U)}.$$

Here the action of $\mathrm{GL}(U)$ on the dth-graded piece is the usual action twisted by the $(-d/2)$th power of the determinant. Thus taking $\mathrm{GL}(U)$-invariants is the same as taking $\mathrm{SL}(U)$-invariants. We can therefore rewrite the above formula as

$$R_L = \left[\bigoplus_{d=0}^{\infty} (\mathrm{Sym}^d\, U^*)^{\otimes L} \right]^{\mathrm{SL}(U)}. \tag{1}$$

Here $U^* = \mathrm{Hom}(U, k)$. We take (1) as the *definition* of R_L for the purposes of this paper. Note that R_L and M_L depend upon k but this is absent from the notation. We will write $(R_L)_k$ when we want to emphasize the dependence on k. Of course, $(R_L)_k = (R_L)_{\mathbb{Z}} \otimes k$. We note that the symmetric group $S_L = \mathrm{Aut}(L)$ acts on R_L by permuting the tensor factors.

We now take a moment to comment about tensor powers, such as the one appearing in (1). For a k-module V and a finite set L we define $V^{\otimes L}$ in the obvious manner: it is the universal k-module with a multilinear map from $\mathrm{Hom}(L, V)$. We think of pure tensors in $V^{\otimes L}$ as *functions* from L to V. For an integer n, we write $V^{\otimes n}$ for $V^{\otimes \{1,\dots,n\}}$. The construction $V^{\otimes L}$ is functorial in both V and L.

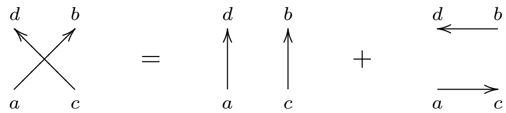

FIGURE 1. The Plücker relation.

2.2. **Graphical description of R_L.** Let $e = (i, j)$ be an element of $L \times L$. Since e is ordered, we have a natural isomorphism $(U^*)^{\otimes\{i,j\}} = (U^*)^{\otimes 2}$. We can thus transfer the symplectic form ω on U, thought of as an element of $(U^*)^{\otimes 2}$, to an element ω_e of $(U^*)^{\otimes\{i,j\}}$. Explicitly, if we pick a symplectic basis $\{x, y\}$ of U^* then $\omega_e = x_i y_j - x_j y_i$, where $x_i y_j$ is just shorthand for the function $\{i, j\} \to U^*$ which takes i to x and j to y. Note that ω_e is invariant under $\mathrm{SL}(U) = \mathrm{Sp}(U)$.

We now give a description of R_L in terms of graphs. We say that a directed graph with vertex set L is *regular* if each vertex has the same valence. This common valence is then called the *degree* of the graph. Let Γ be a regular directed graph of degree d. We define an element X_Γ of $R_L^{(d)}$ by $X_\Gamma = \prod \omega_e$, the product taken over the edges e of Γ. (These may be interpreted as Specht polynomials.) As each ω_e is invariant under $\mathrm{SL}(U)$, so is X_Γ. The fact that Γ is regular of degree d ensures that X_Γ belongs to $(\mathrm{Sym}^d\, U^*)^{\otimes L}$.

It is a fact from classical invariant theory that the X_Γ span R_L as a k-module. The next matter, of course, is to determine the relations between the various X_Γ. To begin with, we clearly have $X_\Gamma X_{\Gamma'} = X_{\Gamma\cdot\Gamma'}$, where $\Gamma \cdot \Gamma'$ denotes the graph on L whose edge set is the union of those of Γ and Γ'. We then have the following easily verified relations:

- (Sign relation.) $X_\Gamma = -X_{\Gamma'}$ if Γ' is obtained from Γ by reversing the direction of a single edge.
- (Loop relation.) $X_\Gamma = 0$ if Γ contains a loop.
- (Plücker relation, see Fig. 1.) Let Γ be a regular directed graph and let (a, b) and (c, d) be two edges of Γ. Let Γ' (resp. Γ'') be the graph obtained by replacing these two edges with the edges (a, d) and (c, b) (resp. (a, c) and (b, d)). Then
$$X_\Gamma = X_{\Gamma'} + X_{\Gamma''}.$$

It is now a second fact from classical invariant theory that these three types of relations generate all the relations amongst the X_Γ. (The loop relation is implied by the sign relation if 2 is invertible in k.)

To be a bit more precise, let $\widetilde{R}_L$ be the free graded k-module with basis $\{\widetilde{X}_\Gamma\}$ as Γ varies over directed regular graphs on L. The grade of $\widetilde{X}_\Gamma$ is the degree of Γ. We turn $\widetilde{R}_L$ into a ring by defining $\widetilde{X}_\Gamma \widetilde{X}_{\Gamma'} = \widetilde{X}_{\Gamma\cdot\Gamma'}$. We then have a map $\widetilde{R}_L \to R_L$ given by $\widetilde{X}_L \mapsto X_L$. The two theorems of classical invariant theory referred to above about amount to the assertion that this map is surjective and the kernel is generated by the sign, loop and Plücker relations.

2.3. **Facts needed about R_L.** We now recall some of its properties of R_L that will be relevant to us. The first and perhaps most important is the following (for a proof, see [HMSV3] or [HMSV1]):

Proposition 2.1 (Kempe)**.** *The ring R_L is generated as a k-algebra by its degree one piece.*

We emphasize that this holds for all k, or equivalently, for $k = \mathbb{Z}$. We remark that Kempe proved another theorem: if one fixes an embedding of L into the unit circle, so that one can make sense of what it means for a graph to be planar, then the X_Γ with Γ planar form an basis of R_L as a k-module. Thus, for instance, one can count planar graphs to determine the Hilbert function of R_L.

We write V_L for the first graded piece of R_L. It is spanned by regular graphs of degree one. We call such graphs *matchings*. Kempe's theorem says that the map $\mathrm{Sym}(V_L) \to R_L$ is surjective. We let I_L be the kernel of this map. We call I_L the *ideal of relations*. The present paper is concerned with finding generators for I_L when L has cardinality eight.

We need to recall some facts about how V_L and some related spaces decompose under the symmetric group $S_L = \mathrm{Aut}(L)$. For simplicity, we now take k to be a field of characteristic zero, although analogues of the statements remain true so long as $n!$ is invertible in k. Recall that the irreducible representations of S_L over k correspond to Young diagrams, or partitions. The representation V_L of S_L is irreducible and corresponds to the Young diagram with two rows and $n/2$ columns. The result we need is the following:

Proposition 2.2. *Assume k is a field of characteristic zero. In the following table, each S_L-module is multiplicity free. The set of irreducibles it contains corresponds to the given set of partitions.*

S_L-module	Set of partitions of n
$\mathrm{Sym}^2(V_L)$	at most four parts, all even
$\bigwedge^2 V_L$	exactly four parts, all odd
$V_L^{\otimes 2}$	union of previous two sets
$I_L^{(2)}$	exactly four parts, all even

This proposition is proved in [HMSV3]. However, we will only need this result in the case $n = 8$, where it can easily be checked by computer or even by hand.

3. The skew-invariant cubic

Until the final section of the paper, we take k to be a field of characteristic zero.

In this section we prove the existence and uniqueness of the skew-invariant cubic s of Theorem 1.1, as well as establishing the formula for it in terms of graphs. We fix once and for all a set L of cardinality eight and write V, *etc.*, in place of V_L, *etc.* We often write G in place of $S_L = \mathrm{Aut}(L)$.

Proposition 3.1. *The space of skew-invariants in $\mathrm{Sym}^3(V)$ is one-dimensional. It is spanned by the skew-average of the cube of any matching.*

The proof of Proposition 3.1 is elementary linear algebra. However, to state it correctly we need some preparation. For now we allow L to be any finite set of even cardinality n. We write $\mathcal{M}_L$ for the set of directed matchings on L. The symmetric group S_L acts transitively on $\mathcal{M}_L$. The alternating group A_L clearly does not act transitively, and thus has exactly two orbits. We fix a bijection

$$\mathrm{sgn} : \mathcal{M}_L/A_L \to \{\pm 1\}.$$

Thus for each matching Γ we have a sign $\mathrm{sgn}\,\Gamma$, and for $\sigma \in S_L$ we have $\mathrm{sgn}(\sigma\Gamma) = \mathrm{sgn}(\sigma)\,\mathrm{sgn}(\Gamma)$.

Now let W be a k-vector space of dimension n and let η be a non-degenerate symplectic form on W. For an ordered pair $e = (i, j)$ in $L \times L$ we define $\eta_e \in (W^*)^{\otimes\{i,j\}}$ as we defined ω_e in §2.2. For a matching Γ on L we define η_Γ as the product $\prod \eta_e$, taken over the edges of Γ. Finally, we define $\eta^{L/2} \in (W^*)^{\otimes L}$ by the formula

$$\eta^{L/2} = \sum_{\Gamma \in \mathcal{M}_L} \mathrm{sgn}(\Gamma)\eta_\Gamma.$$

Clearly $\eta^{L/2}$ is skew-invariant for the action of S_L.

Lemma 3.2. *The space of S_L-skew-invariants in $(W^*)^{\otimes L}$ is one-dimensional and spanned by $\eta^{L/2}$. The group $\mathrm{SL}(W)$ leaves $\eta^{L/2}$ invariant.*

Proof. This is the usual method of building a volume form out of a symplectic form. □

We now prove the proposition. We return to our original notation.

Proof of Proposition 3.1. Recall that $V = ((U^*)^{\otimes L})^{\mathrm{SL}(U)}$. We thus have

$$\mathrm{Sym}^3 V = (V^{\otimes 3})_{S_3} = ((((U^*)^{\otimes L})^{\mathrm{SL}(U)})^{\otimes 3})_{S_3} = ((((U^*)^{\otimes L})^{\otimes 3})^{\mathrm{SL}(U)^3})_{S_3} = ((((U^*)^{\otimes 3})^{\otimes L})^{\mathrm{SL}(U)^3})_{S_3}.$$

Here the subscript S_3 denotes co-invariants by the the symmetric group S_3. (Symmetric powers are most naturally defined by taking co-invariants, not invariants.) Putting $W = U^{\otimes 3}$, we may write this formula as

$$\mathrm{Sym}^3 V = (((W^*)^{\otimes L})^{\mathrm{SL}(U)^3})_{S_3}.$$

Note that $\mathrm{SL}(U)^3 \rtimes S_3$ acts on W and the action used in the above formula is the natural one. Furthermore, the action of $G = S_L$ given by permuting factors commutes with $\mathrm{SL}(U)^3 \rtimes S_3$.

The space W has a natural symplectic form, namely $\eta = \omega^{\otimes 3}$. The group $\mathrm{SL}(U)^3 \rtimes S_3$ clearly preserves this form, and thus the map $\mathrm{SL}(U)^3 \rtimes S_3 \to \mathrm{GL}(W)$ lands in $\mathrm{Sp}(W) \subset \mathrm{SL}(W)$. We now appeal to the lemma (note W is eight-dimensional and L has cardinality eight). We see that $\eta^{L/2}$ is non-zero and spans the space of skew-invariants in $(W^*)^{\otimes L}$. Furthermore, the group $\mathrm{SL}(U)^3 \rtimes S_3$ leaves $\eta^{L/2}$ invariant. We have thus shown that the space of skew-invariants in

$$(((W^*)^{\otimes L})^{\mathrm{SL}(U)^3})^{S_3}$$

is one-dimensional and spanned by $\eta^{L/2}$. Since we are not in characteristic 2 or 3, the map

$$(((W^*)^{\otimes L})^{\mathrm{SL}(U)^3})^{S_3} \to (((W^*)^{\otimes L})^{\mathrm{SL}(U)^3})_{S_3} = \mathrm{Sym}^3 V$$

is an isomorphism. Thus the space of skew-invariants in $\mathrm{Sym}^3 V$ is one-dimensional and spanned by the image s of $\eta^{L/2}$.

We now wish to express the skew-invariant s in terms of graphs. We have, by definition,

$$\eta^{L/2} = \sum_{\Gamma \in \mathcal{M}_L} \mathrm{sgn}(\Gamma)\eta_\Gamma.$$

Now, η_Γ is equal to $\prod \omega_e^{\otimes 3}$. The image of this in $\mathrm{Sym}^3 V$ is just $\prod \omega_e^3$, which is, by definition, X_Γ^3. We thus have

$$s = \sum_{\Gamma \in \mathcal{M}_L} \mathrm{sgn}(\Gamma) X_\Gamma^3.$$

(The skew-average of a cube of a matching is equal to αs for some $\alpha \mid 8!$.) This completes the proof of the proposition. □

4. The partial derivatives of s span $I_8^{(2)}$

In this section we prove that the 14 partial derivatives of s span $I_8^{(2)}$ and are linearly independent, thus establishing part of Theorem 1.1. We keep the notation from the previous section.

Proposition 4.1. *Let s be a non-zero skew-invariant element of $\mathrm{Sym}^3(V)$. Then the 14 partial derivatives of s are linearly independent and span $I^{(2)}$.*

Proof. For an element v^* of the dual space V^* define a derivation ∂_{v^*} of $\mathrm{Sym}(V)$ by the formula

$$\partial_{v^*}(v_1 \cdots v_n) = \sum_{i=1}^{n} \langle v^*, v_i \rangle \cdot (v_1 \cdots \hat{v}_i \cdots v_n)$$

where the hat indicates that that factor is to be omitted. We have a map

$$\Phi : V^* \otimes ks \to \mathrm{Sym}^2(V), \qquad v^* \otimes as \mapsto a\partial_{v^*} s.$$

The proposition states that Φ is injective with image $I^{(2)}$.

The crucial fact is that Φ is a map of G-modules. Now, as a G-module, V is irreducible and corresponds to the Young diagram with 2 rows and 4 columns. As with any representation of the symmetric group, V is self-dual. Thus $V^* \otimes ks$ is the irreducible representation with 4 rows and 2 columns (since G acts on ks by the sign representation). Now, by Proposition 2.2, $\mathrm{Sym}^2(V)$ is multiplicity free. Furthermore, that proposition shows that $I^{(2)}$ is irreducible and corresponds to the Young diagram with 4 rows and 2 columns. It thus follows that Φ must have image contained in $I^{(2)}$. Since the domain of Φ is irreducible, it follows that Φ is either zero or injective. But Φ cannot be zero since the non-zero polynomial s must have some non-zero partial derivative. This proves the proposition. □

5. The partial derivatives of s have no linear syzygies — set-up

The goal of the next few sections is to establish the following proposition:

Proposition 5.1. *The partial derivatives of s have no linear syzygies.*

This proposition means that if $\sum_{i=1}^{14} x_i \partial_i s = 0$ with x_i in $\mathrm{Sym}^1(V)$ then $x_i = 0$ for all i. We will not prove Proposition 5.1 in this section but we will reduce the proof to a problem that we will soon solve.

Consider the composition

$$\widetilde{\Psi} : \mathrm{End}(V) \otimes \mathrm{Sym}^3(V) = V \otimes V^* \otimes \mathrm{Sym}^3(V) \to V \otimes \mathrm{Sym}^2(V) \to \mathrm{Sym}^3(V)$$

where the first map is the partial derivative map and the second map is the multiplication map. One easily verifies that $\widetilde{\Psi}$ is just the map which expresses the action of the Lie algebra $\mathfrak{gl}(V) = \mathrm{End}(V)$ on the third symmetric power of its standard representation V. We are trying to show that $\widetilde{\Psi}$ induces an injection

$$\Psi : \mathrm{End}(V) \otimes ks \to I^{(3)}.$$

(We know that Ψ maps $\mathrm{End}(V) \otimes ks$ into $I^{(3)}$ since we know that the partial derivatives of s belong to $I^{(2)}$.) Indeed, the kernel of Ψ is the space of linear syzygies between the partial derivatives of s. Now, the kernel of Ψ is equal to $\mathfrak{g} \otimes ks$, where $\mathfrak{g}$ is the annihilator in $\mathfrak{gl}(V)$ of s. Thus Proposition 5.1 is equivalent to the following:

Proposition 5.2. *We have $\mathfrak{g} = 0$.*

We know two important things about $\mathfrak{g}$: first, $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{gl}(V)$, as it is the annihilator of some element in a representation of $\mathfrak{gl}(V)$; and second, $\mathfrak{g}$ is stable under the group G, as the action map Ψ is G-equivariant and ks is stable under G. We will prove Proposition 5.2 by first classifying the G-stable Lie subalgebras of $\mathfrak{gl}(V)$ and then proving that $\mathfrak{g}$ cannot be any of them except zero.

Before continuing, we note a few results:

Proposition 5.3. *The skew-invariant cubic s belongs to $I^{(3)}$.*

Proof. We have already remarked that any element of the Lie algebra $\mathfrak{gl}(V)$ takes s into $I^{(3)}$. Now, the identity matrix in $\mathfrak{gl}(V)$ acts by multiplication by 3 on $\mathrm{Sym}^3(V)$, and thus $3s$, and thus s, belongs to $I^{(3)}$. An alternate argument, using the description of s as a signed sum of cubes of matchings (proof of Lemma 3.2) was given in §1.2. □

Proposition 5.4. *The Lie algebra $\mathfrak{g}$ is contained in $\mathfrak{sl}(V)$.*

Proof. The trace map $\mathfrak{gl}(V) \to k$ is G-equivariant, where G acts trivially on the target. Thus if $\mathfrak{g}$ contained an element of non-zero trace it would have to contain a copy of the trivial representation. Thanks to Proposition 2.2, we know that $\mathfrak{gl}(V) \cong V^{\otimes 2}$ is multiplicity free as a representation of G. Thus the one-dimensional space spanned by the identity matrix is the only copy of the trivial representation in $\mathfrak{gl}(V)$. Therefore, if $\mathfrak{g}$ were not contained in $\mathfrak{sl}(V)$ then it would contain the center of $\mathfrak{gl}(V)$. However, we know that the identity matrix does not annihilate s. Thus $\mathfrak{g}$ must be contained in $\mathfrak{sl}(V)$. □

Proposition 5.5. *Proposition 5.1 implies that $I^{(2)}$ generates $I^{(3)}$.*

Proof. The image of Ψ is exactly the subspace of $I^{(3)}$ generated by $I^{(2)}$. Thus $I^{(2)}$ generates $I^{(3)}$ if and only if Ψ is surjective. Now, V being 14 dimensional, the dimension of $\mathrm{End}(V)$ is 196. It happens that this is exactly the dimension of $I^{(3)}$ as well. Thus the domain and target of Ψ have the same dimension, and so surjectivity is equivalent to injectivity. □

As remarked in the introduction one can prove Theorem 1.1 by using Proposition 5.5 and a result from [HMSV3] which states that I_n is generated in degrees two and three for all n. We will not take this route, however, and instead give an alternate proof in §8 that $I^{(2)}$ generates I.

6. Interlude: G-stable Lie subalgebras of $\mathfrak{sl}(V)$

In this section G will denote an arbitrary finite group and V an irreducible representation of G over an algebraically closed field k of characteristic zero. We investigate the following general problem:

Problem 6.1. *Determine the G-stable Lie subalgebras of $\mathfrak{sl}(V)$.*

We do not obtain a complete answer to this question, but we do prove strong enough results to determine the answer in our specific situation. We will use the term *G-subalgebra* to mean a G-stable Lie subalgebra.

6.1. Some structure theory.

Our first result is the following:

Proposition 6.2. *Let V be an irreducible representation of G. Then every solvable G-subalgebra of $\mathfrak{sl}(V)$ is abelian and consists solely of semi-simple elements.*

Proof. Let $\mathfrak{g}$ be a solvable subalgebra of $\mathfrak{sl}(V)$. By Lie's theorem, $\mathfrak{g}$ preserves a complete flag $0 = V_0 \subset \cdots \subset V_n = V$. The action of $\mathfrak{g}$ on each one-dimensional space V_i/V_{i-1} must factor through $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$; thus $[\mathfrak{g}, \mathfrak{g}]$ acts by zero on V_i/V_{i-1} and so carries V_i into V_{i-1}. The space $[\mathfrak{g}, \mathfrak{g}]V$ is therefore not all of V. On the other hand, $[\mathfrak{g}, \mathfrak{g}]$ is G-stable and therefore so is $[\mathfrak{g}, \mathfrak{g}]V$. From the irreducibility of V we conclude $[\mathfrak{g}, \mathfrak{g}]V = 0$, from which it follows that $[\mathfrak{g}, \mathfrak{g}] = 0$. Thus $\mathfrak{g}$ is abelian.

Now let R be the subalgebra of $\mathrm{End}(V)$ generated (under the usual multiplication) by $\mathfrak{g}$. Let R_s (resp. R_n) denote the set of semi-simple (resp. nilpotent) elements of R. Then R_s is a subring of R, R_n is an ideal of R and $R = R_s \oplus R_n$. As $R_n^m = 0$ for some m, the space R_nV is not all of V. As it is G-stable it must be zero, and so $R_n = 0$. We thus find that $R = R_s$ and so all elements of R, and thus all elements of $\mathfrak{g}$, are semi-simple. □

Let V be a representation of G. We say that V is *imprimitive* if there is a decomposition $V = \bigoplus_{i \in I} V_i$ of V into non-zero subspaces, at least two in number, such that each element of G carries each V_i into some V_j. We say that V is *primitive* if it is not imprimitive. Note that primitive implies irreducible. An irreducible representation is imprimitive if and only if it is induced from a proper subgroup.

Proposition 6.3. *Let V be an irreducible representation of G. Then V is primitive if and only if the only abelian G-subalgebra of $\mathfrak{sl}(V)$ is zero.*

Proof. Let V be an irreducible representation of G and let $\mathfrak{g}$ be a non-zero abelian G-subalgebra of $\mathfrak{sl}(V)$. We will show that V is imprimitive. By Proposition 6.2 all elements of $\mathfrak{g}$ are semi-simple. We thus get a decomposition $V = \bigoplus V_\lambda$ of V into eigenspaces of $\mathfrak{g}$ (each λ is a linear map $\mathfrak{g} \to k$). As $\mathfrak{g}$ is G-stable, each element of G must carry each V_λ into some $V_{\lambda'}$. Note that if $V = V_\lambda$ for some λ then $\mathfrak{g}$ would consist of scalar matrices, which is impossible as $\mathfrak{g}$ is contained in $\mathfrak{sl}(V)$. Thus there must be at least two non-zero V_λ and so V is imprimitive.

We now establish the other direction. Thus let V be an irreducible imprimitive representation of G. We construct a non-zero abelian G-subalgebra of $\mathfrak{sl}(V)$. Write $V = \bigoplus V_i$ where the elements of G permute the V_i. Let p_i be the endomorphism of V given by projecting onto V_i and then including back into V and let $\mathfrak{g}$ be the subspace of $\mathfrak{gl}(V)$ spanned by the p_i. Then $\mathfrak{g}$ is an abelian subalgebra of $\mathfrak{gl}(V)$ since $p_i p_j = 0$ for $i \neq j$. Furthermore, $\mathfrak{g}$ is G-stable since for each i we have $g p_i g^{-1} = p_j$ for some j. Intersecting $\mathfrak{g}$ with $\mathfrak{sl}(V)$ gives a non-zero abelian G-subalgebra of $\mathfrak{sl}(V)$ (the intersection is non-zero because $\mathfrak{g}$ has dimension at least two and $\mathfrak{sl}(V)$ has codimension one). □

We have the following important consequence of Proposition 6.3:

Corollary 6.4. *Let V be a primitive representation of G. Then every G-subalgebra of $\mathfrak{sl}(V)$ is semi-simple.*

Proof. Let $\mathfrak{g}$ be a G-subalgebra of $\mathfrak{sl}(V)$. The radical of $\mathfrak{g}$ is then a solvable G-subalgebra and therefore vanishes. Thus $\mathfrak{g}$ is semi-simple. □

Proposition 6.3 can also be used to give a criterion for primitivity.

Corollary 6.5. *Let V be an irreducible representation of G such that each non-zero G-submodule of $\mathfrak{sl}(V)$ has dimension at least that of V. Then V is primitive.*

Proof. Let $\mathfrak{g}$ be a abelian G-subalgebra of $\mathfrak{sl}(V)$. We will show that $\mathfrak{g}$ is zero. By Proposition 6.2 $\mathfrak{g}$ consists of semi-simple elements and is therefore contained in some Cartan subalgebra of $\mathfrak{sl}(V)$. This shows that $\dim \mathfrak{g} < \dim V$. Thus, by our hypothesis, $\mathfrak{g} = 0$. □

Let V be a primitive G-module and let $\mathfrak{g}$ be a G-subalgebra. As $\mathfrak{g}$ is semi-simple it decomposes as $\mathfrak{g} = \bigoplus \mathfrak{g}_i$ where each $\mathfrak{g}_i$ is a simple Lie algebra. The $\mathfrak{g}_i$ are called the *simple factors* of $\mathfrak{g}$ and are unique. As the simple factors are unique, G must permute them. We call $\mathfrak{g}$ *prime* if the action of G on its simple factors is transitive. Note that in this case the $\mathfrak{g}_i$'s are isomorphic and so $\mathfrak{g}$ is "isotypic." Clearly, every G-subalgebra of $\mathfrak{sl}(V)$ breaks up into a sum of prime subalgebras and so it suffices to understand these.

6.2. The action of a G-subalgebra on V.

We now consider how a G-stable subalgebra acts on V:

Proposition 6.6. *Let V be a primitive G-module, let $\mathfrak{g}$ be a G-subalgebra of $\mathfrak{sl}(V)$ and let $\mathfrak{g} = \bigoplus_{i \in I} \mathfrak{g}_i$ be the decomposition of $\mathfrak{g}$ into simple factors.*

1. *The representation of $\mathfrak{g}$ on V is isotypic, that is, it is of the form $V_0^{\oplus m}$ for some irreducible $\mathfrak{g}$-module V_0.*
2. *We have a decomposition $V_0 = \bigotimes_{i \in I} W_i$ where each W_i is a faithful irreducible representation of $\mathfrak{g}_i$.*
3. *We have $V_0 \cong V_0^g$ for each element g of G. (Here V_0^g denotes the $\mathfrak{g}$-module obtained by twisting V_0 by the automorphism g induces on $\mathfrak{g}$.)*
4. *If $\mathfrak{g}$ is a prime subalgebra then for any i and j one can choose an isomorphism $f : \mathfrak{g}_i \to \mathfrak{g}_j$ so that W_i and $f^* W_j$ become isomorphic as $\mathfrak{g}_i$-modules.*

Proof. (1) Since $\mathfrak{g}$ is semi-simple we get a decomposition $V = \bigoplus V_i^{\oplus m_i}$ of V as a $\mathfrak{g}$-module, where the V_i are pairwise non-isomorphic simple $\mathfrak{g}$-modules. Each element g of G must take each isotypic piece $V_i^{\oplus m_i}$ to some other isotypic piece $V_j^{\oplus m_j}$ since the map $g : V \to V^g$ is $\mathfrak{g}$-equivariant. As V is primitive for G, we conclude that it must be isotypic for $\mathfrak{g}$, and so we may write $V = V_0^{\oplus m}$ for some irreducible $\mathfrak{g}$-module V_0.

(2) As V_0 is irreducible, it necessarily decomposes as a tensor product $V_0 = \bigotimes_{i \in I} W_i$ where each W_i is an irreducible $\mathfrak{g}_i$-module. Since the representation of $\mathfrak{g}$ on $V = V_0^{\oplus m}$ is faithful so too must be the representation of $\mathfrak{g}$ on V_0. From this, we conclude that each W_i must be a faithful representation of $\mathfrak{g}_i$.

(3) For any $g \in G$ the map $g : V \to V^g$ is an isomorphism of $\mathfrak{g}$-modules and so $V_0^{\oplus m}$ is isomorphic to $(V_0^{\oplus m})^g = (V_0^g)^{\oplus m}$, from which it follows that V_0 is isomorphic to V_0^g.

(4) Since G acts transitively on the simple factors, given i and j we can pick $g \in G$ such that $g\mathfrak{g}_i = \mathfrak{g}_j$. The isomorphism of V_0 with V_0^g then gives the isomorphism of W_i and W_j as $\mathfrak{g}_i$-modules. □

This proposition gives a strong numerical constraint on prime subalgebras:

Corollary 6.7. *Let V be a primitive representation of G and let $\mathfrak{g} = \mathfrak{g}_0^n$ be a prime subalgebra of $\mathfrak{sl}(V)$, where $\mathfrak{g}_0$ is a simple Lie algebra. Then $\dim V$ is divisible by d^n where d is the dimension of some faithful representation of $\mathfrak{g}_0$. In particular, $\dim V \geq d_0^n$ where d_0 is the minimal dimension of a faithful representation of $\mathfrak{g}_0$.*

6.3. **Self-dual representations.** Let V be an irreducible self-dual G-module. Thus we have a non-degenerate G-invariant form $\langle , \rangle : V \otimes V \to k$. Such a form is unique up to scaling, and either symmetric or anti-symmetric. We accordingly call V *orthogonal* or *symplectic*.

Let A be an endomorphism of V. We define the *transpose* of A, denoted A^t, by the formula

$$\langle A^t v, u \rangle = \langle v, Au \rangle.$$

It is easily verified that $(AB)^t = B^t A^t$ and $({}^g A)^t = {}^g(A^t)$. We call an endomorphism A *symmetric* if $A = A^t$ and *anti-symmetric* if $A = -A^t$. One easily verifies that the commutator of two anti-symmetric endomorphisms is again anti-symmetric. Thus the set of all anti-symmetric endomorphisms forms a G-subalgebra of $\mathfrak{sl}(V)$ which we denote by $\mathfrak{sl}(V)^-$. In the orthogonal case $\mathfrak{sl}(V)^-$ is isomorphic to $\mathfrak{so}(V)$ as a Lie algebra and $\bigwedge^2 V$ as a G-module, while in the symplectic case it is isomorphic to $\mathfrak{sp}(V)$ as a Lie algebra and $\mathrm{Sym}^2(V)$ as a G-module. We let $\mathfrak{sl}(V)^+$ denote the space of symmetric endomorphisms.

Proposition 6.8. *Let V be an irreducible self-dual G-module. Assume that:*

- $\mathrm{Sym}^2(V)$ *and* $\bigwedge^2 V$ *have no isomorphic G-submodules; and*
- $\mathfrak{sl}(V)^-$ *has no proper non-zero G-subalgebras.*

Then any proper non-zero G-subalgebra of $\mathfrak{sl}(V)$ other than $\mathfrak{sl}(V)^-$ is commutative. In particular, if V is primitive then the G-subalgebras of $\mathfrak{sl}(V)$ are exactly 0, $\mathfrak{sl}(V)^-$ and $\mathfrak{sl}(V)$.

Proof. Let $\mathfrak{g}$ be a non-zero G-subalgebra of $\mathfrak{sl}(V)$. The intersection of $\mathfrak{g}$ with $\mathfrak{sl}(V)^-$ is a G-subalgebra of $\mathfrak{sl}(V)^-$ and therefore either 0 or all of $\mathfrak{sl}(V)^-$. First assume that the intersection is zero. Since the spaces of symmetric and anti-symmetric elements of $\mathfrak{sl}(V)$ have no isomorphic G-submodules, it follows that $\mathfrak{g}$ is contained in the space of symmetric elements of $\mathfrak{sl}(V)$. However, two symmetric elements bracket to an anti-symmetric element. It thus follows that all brackets in $\mathfrak{g}$ vanish and so $\mathfrak{g}$ is commutative. Now assume that $\mathfrak{g}$ contains all of $\mathfrak{sl}(V)^-$. It is then a standard fact that $\mathfrak{sl}(V)^-$ is a maximal subalgebra of $\mathfrak{sl}(V)$ and so $\mathfrak{g}$ is either $\mathfrak{sl}(V)^-$ or $\mathfrak{sl}(V)$. (To see this, note that $\mathfrak{sl}(V) = \mathfrak{sl}(V)^- \oplus \mathfrak{sl}(V)^+$ and so to prove the maximality of $\mathfrak{sl}(V)^-$ it suffices to show that $\mathfrak{sl}(V)^+$ is an irreducible representation of $\mathfrak{sl}(V)^-$. In the orthogonal case this amounts to the fact that, as a representation of $\mathfrak{so}(V)$, the space $\mathrm{Sym}^2(V)/W$ is irreducible, where W is the line spanned by the orthogonal form on V. The symplectic case is similar.) □

7. The partial derivatives of s have no linear syzygies — completion of proof

We now complete the proof of Proposition 5.1. We return to our previous notation. We begin with the following:

Proposition 7.1. *Assume k is algebraically closed. The G-subalgebras of $\mathfrak{sl}(V)$ are exactly 0, $\mathfrak{so}(V)$ and $\mathfrak{sl}(V)$.*

Proof. We begin by noting that any irreducible representation of the symmetric group is defined over the reals (in fact, the rationals) and is therefore orthogonal self-dual. Thus $\mathfrak{so}(V) = \mathfrak{sl}(V)^-$ makes sense as a G-subalgebra.

For our particular representation V, Proposition 2.2 shows that $\mathrm{Sym}^2(V)$ has five irreducible submodules of dimensions 1, 14, 14, 20 and 56, while $\bigwedge^2 V$ has two irreducible submodules of dimensions 35 and 56. Furthermore, none of these seven irreducibles are isomorphic. As all irreducible submodules of $\mathfrak{sl}(V)$ have dimension at least that of V (which in this case is 14), we see from Corollary 6.5 that V is primitive. (Note that the one-dimensional representation occurring in $\mathrm{Sym}^2(V)$ is the center of $\mathfrak{gl}(V)$ and does not occur in $\mathfrak{sl}(V)$.)

As V is primitive, multiplicity free and self-dual, we can apply Proposition 6.8. This shows that to prove the present proposition we need only show that $\mathfrak{so}(V)$ has no proper non-zero G-subalgebras. Thus assume that $\mathfrak{g}'$ is a proper non-zero G-subalgebra of $\mathfrak{so}(V)$. As $\mathfrak{so}(V) = \bigwedge^2 V$ has two irreducible submodules we see that $\mathfrak{g}'$ must be one of these two irreducibles. In particular, this shows that $\mathfrak{g}'$ must be prime and so therefore isotypic. Now, by examining the list of all simple Lie algebras, we see that there are exactly four isotypic Lie algebras of dimension either 35 or 56:

$$\mathfrak{g}_2^4, \qquad \mathfrak{so}(8)^2, \qquad \mathfrak{sl}(3)^7, \qquad \mathfrak{sl}(6).$$

The minimal dimensions of faithful representations of $\mathfrak{g}_2$, $\mathfrak{so}(8)$ and $\mathfrak{sl}(3)$ are 7, 8 and 3. As 7^4, 8^2 and 3^7 are all bigger than $\dim V$, Corollary 6.7 rules out the first three Lie algebras above. (One can also rule out $\mathfrak{g}_2^4$ and $\mathfrak{sl}(3)^7$ by noting that the alternating group A_8 does not act non-trivially on them.) We rule out $\mathfrak{sl}(6)$ by using Proposition 6.6 and noting that $\mathfrak{sl}(6)$ has no faithful 14 dimensional isotypic representation — this is proved in Lemma 7.2 below. (One can also rule out $\mathfrak{sl}(6)$ by noting that A_8 does not act on it.) This shows that $\mathfrak{g}'$ cannot exist, and proves the proposition. □

Lemma 7.2. *The Lie algebra $\mathfrak{sl}(6)$ has exactly two non-trivial irreducible representations of dimension ≤ 14: the standard representation and its dual. It has no 14-dimensional faithful isotypic representation.*

Proof. For a dominant weight λ let V_λ denote the irreducible representation with highest weight λ. If λ and λ' are two dominant weights then a general fact valid for any semi-simple Lie algebra states

$$\dim V_{\lambda+\lambda'} \geq \max(\dim V_\lambda, \dim V_{\lambda'}).$$

(To see this, recall the Weyl dimension formula:

$$\dim V_\lambda = \prod_{\alpha^\vee > 0} \frac{\langle \lambda + \rho, \alpha^\vee \rangle}{\langle \rho, \alpha^\vee \rangle},$$

where ρ is half the sum of the positive roots and the product is taken over the positive co-roots $\alpha^\vee$. Then note that $\langle \lambda, \alpha^\vee \rangle$ is positive for any dominant weight λ and any positive co-root $\alpha^\vee$. Thus $\dim V_{\lambda+\lambda'} \geq \dim V_\lambda$.)

Now, let $\varpi_1, \ldots, \varpi_5$ be the fundamental weights for $\mathfrak{sl}(6)$. The representation V_{ϖ_i} is just $\bigwedge^i V$, where V is the standard representation. For $2 \leq i \leq 4$ the space V_{ϖ_i} has dimension ≥ 15. Furthermore, a simple calculation shows that

$$\dim V_{2\varpi_1} = 21, \qquad \dim V_{\varpi_1+\varpi_5} = 168, \qquad \dim V_{2\varpi_5} = 21.$$

(Note that $V_{2\varpi_1}$ is $\mathrm{Sym}^2(V)$, while $V_{2\varpi_5}$ is its dual. This shows why they are 21-dimensional. To compute the dimension of $V_{\varpi_1+\varpi_5}$ we use the formula for the dimension of the relevant Schur functor, [FH, Ex. 6.4].) Thus only V_{ϖ_1} and V_{ϖ_5} have dimension at most 14, and they each have dimension 6. Since 6 does not divide 14 we find that there are no non-trivial 14-dimensional isotypic representations. □

Remark 7.3. We can prove Proposition 7.1 whenever L has cardinality at most 14. Perhaps it is true for all L.

We now have the following:

Proposition 7.4. *The only element of* $\mathrm{Sym}^3(V)$ *annihilated by* $\mathfrak{so}(V)$ *is zero.*

Proof. As mentioned, V has a canonical non-degenerate symmetric inner product. Pick an orthonormal basis $\{x_i\}$ of V and let $\{x_i^*\}$ be the dual basis of V^*. We can think of $\mathrm{Sym}(V)$ as the polynomial ring in the x_i. The space $\mathfrak{so}(V)$ is spanned by elements of the form $E_{ij} = x_i \otimes x_j^* - x_j \otimes x_i^*$. Recall that, for an element s of $\mathrm{Sym}(V)$, the element $x_i \otimes x_j^*$ of $\mathrm{End}(V)$ acts on s by $x_i \partial_j s$, where ∂_j denotes differentiation with respect to x_j. Thus we see that s is annihilated by E_{ij} if and only if it satisfies the equation

$$x_i \partial_j s = x_j \partial_i s. \tag{2}$$

Therefore s is annihilated by all of $\mathfrak{so}(V)$ if and only if the above equation holds for all i and j.

Let s be an element of $\mathrm{Sym}^3(V)$. We now consider (2) for a fixed i and j. Write

$$s = g_3(x_j) + g_2(x_j)x_i + g_1(x_j)x_i^2 + g_0(x_j)x_i^3$$

where each g_i is a polynomial in x_j whose coefficients are polynomials in the x_k with $k \neq i, j$. Note that g_0 must be a constant by degree considerations. We have

$$\begin{aligned} x_i \partial_j s &= g_3'(x_j) x_i + g_2'(x_j) x_i^2 + g_1'(x_j) x_i^3 \\ x_j \partial_i s &= x_j g_2(x_j) + 2 x_j g_1(x_j) x_i + 3 x_j g_0(x_j) x_i^2. \end{aligned}$$

We thus find

$$g_2 = 0, \quad 2x_j g_1 = g_3', \quad 3x_j g_0 = g_2', \quad g_1' = 0.$$

From this we deduce that $g_0 = g_2 = 0$ and that g_1 is determined from g_3. The constraint on g_3 is that it must satisfy

$$g_3'(x_j) = x_j g_3''(x_j). \tag{3}$$

Putting

$$g_3(x_j) = a + b x_j + c x_j^2 + d x_j^3$$

we see that (3) is equivalent to $b = d = 0$. We thus have

$$g_3(x_j) = a + c x_j^2, \qquad \text{and} \qquad g_1(x_j) = c$$

and so

$$s = a + c(x_i^2 + x_j^2)$$

is the general solution to (2).

We thus see that if s satisfies (2) for a particular i and j then x_i and x_j occur in s with only even powers. Thus if s satisfies (2) for all i and j then all variables appear to an even power. This is impossible, unless $s = 0$, since s has degree three. Thus we see that zero is the only solution to (2) which holds for all i and j. □

Remark 7.5. The above computational proof can be made more conceptual. By considering the equation (2) for a fixed i and j we are considering the invariants of $\mathrm{Sym}^3(V)$ under a certain copy of $\mathfrak{so}(2)$ sitting inside of $\mathfrak{so}(V)$. The representation V restricted to $\mathfrak{so}(2)$ decomposes as $V' \oplus T$ where V' is the standard representation of $\mathfrak{so}(2)$ and T is a 12-dimensional trivial representation of $\mathfrak{so}(2)$. We then have

$$\mathrm{Sym}^3(V)^{\mathfrak{so}(2)} = \bigoplus_{i=0}^{3} \mathrm{Sym}^i(V')^{\mathfrak{so}(2)} \otimes \mathrm{Sym}^{3-i}(T).$$

Finally, our general solution to (2) amounts to the fact that the ring of invariant $\mathrm{Sym}(V')^{\mathfrak{so}(2)}$ is generated by the norm form $x_i^2 + x_j^2$.

We can now prove Proposition 5.2, which will establish Proposition 5.1.

Proof of Proposition 5.2. To prove $\mathfrak{g} = 0$ we may pass to the algebraic closure of k; we thus assume k is algebraically closed. By Proposition 7.1, the Lie algebra $\mathfrak{g}$ must be 0, $\mathfrak{so}(V)$ or $\mathfrak{sl}(V)$. By Proposition 7.4 $\mathfrak{g}$ cannot be $\mathfrak{so}(V)$ or $\mathfrak{sl}(V)$ since it annihilates s and s is non-zero. Thus $\mathfrak{g} = 0$. □

8. Completion of the proof of Theorem 1.1

We now complete the proof of Theorem 1.1. Before doing so, we need to review some commutative algebra. In this section we work over an algebraically closed field k of characteristic zero.

8.1. **Betti numbers of modules over polynomial rings.** Let P be a graded polynomial ring over k in finitely many indeterminates, each of positive degree. Let M be a finite P-module. One can then find a surjection $F \to M$ with F a finite free module having the following property: if $F' \to M$ is another surjection from a finite free module then there is a surjection $F' \to F$ making the obvious diagram commute. We call $F \to M$ a *free envelope* of M. It is unique up to non-unique isomorphism. As an example, if M is generated by its degree d piece then we can take F to be $P[-d] \otimes M^{(d)}$ where the tensor product is over k and $P[-d]$ is the free P-module with one generator in degree d.

Let M be a finite free P-module. We can build a resolution of M by using free envelopes:

$$\cdots \to F_2 \to F_1 \to F_0 \to M \to 0$$

Here F_0 is the free envelope of M and F_{i+1} is the free envelope of $\ker(F_i \to F_{i-1})$. Define integers $b_{i,j}$ by

$$F_i = \bigoplus_{j \in \mathbb{Z}} P[-i-j]^{\oplus b_{i,j}}.$$

These integers are called the *Betti numbers* of M and the collection of them all the *Betti diagram* of M. They are independent of the choice of free envelopes, as $b_{i,j}$ is also the dimension of the jth graded piece of $\mathrm{Tor}_i^P(M, P/I)$, where I is ideal of positive degree elements. The Betti numbers have the following properties:

- (B1) We have $b_{i,j} = 0$ for all but finitely many i and j. This is because each F_i is finitely generated and $F_i = 0$ for i large by Hilbert's theorem on syzygies.
- (B2) We have $b_{i,j} = 0$ for $i < 0$. This follows from the definition.
- (B3) If $b_{i_0,j} = 0$ for $j \le j_0$ then $b_{i,j} = 0$ for all $i \ge i_0$ and $j \le j_0$. This follows from the fact that if d is the lowest degree occurring in a module M and $F \to M$ is a free envelope then $F^{(d)} \to M^{(d)}$ is an isomorphism, and thus the lowest degree occurring in $\ker(F \to M)$ is $d+1$.
- (B4) In particular, if M is in non-negative degrees then $b_{i,j} = 0$ for $j < 0$.
- (B5) Let $f(k) = \dim M^{(k)}$ (resp. $g(k) = \dim P^{(k)}$) denote the Hilbert function of M (resp. P). Then

$$f(k) = \sum_{i,j \in \mathbb{Z}} (-1)^i \cdot b_{i,j} \cdot g(k-i-j).$$

  This follows by taking the Euler characteristic of the kth graded piece of $F_\bullet \to M$.

In particular we see that if M is in non-negative degrees then its Betti diagram is contained in a bounded subset of the first quadrant.

8.2. **Betti numbers of graded algebras.** Let R be a finitely generated graded k-algebra, which we assume for simplicity to be generated by its degree one pice. We let $P = \mathrm{Sym}(R^{(1)})$ be the graded polynomial algbera on the first graded piece. We have a natural surjective map $P \to R$ and so R is a P-module. We can thus speak of the Betti numbers of R as a P-module. We call these the Betti numbers of R.

Assume now that the ring R is Gorenstein and a domain. The canonical module ω_R of R is then naturally a graded module. Furthermore, there exists an integer a, called the *a-invariant* of R, such that ω_R is isomorphic to $R[a]$. We now have the following important property of the Betti numbers of R:

- (B6) We have $b_{i,j} = b_{r-i,d+a-j}$ where $d = \dim R$ is the Krull dimension of R, $r = \dim P - \dim R$ is the codimension of $\mathrm{Spec}(R)$ in $\mathrm{Spec}(P)$ and a is the a-invariant of R.

No doubt this formula appears in the literature, but we will derive it here for completeness. We have $\mathrm{Ext}^i_P(R, \omega_P) \cong \omega_R$ if $i = r$ and 0 if $i \ne r$. If n is the dimension of P, then $\omega_P \cong P[-n]$. Since R is Gorenstein we have $\omega_R \cong R[a]$. Therefore we obtain a minimal free resolution $G_\bullet$ of $R[a]$ by $G_i = \mathrm{Hom}_P(F_{r-i}, P[-n])$. We have $G_\bullet[-a]$ is a minimal free resolution of R, and by uniqueness of the resolution we therefore have $G_i[-a] \cong F_i$ for each i. Now $G_i[-a] \cong \oplus_{j'} P[-n-r+i+j'-a]$, and so

$$\oplus_{j'} P[-n+r-i+j'-a]^{b_{r-i,j'}} \cong \oplus_j P[-i-j]^{b_{i,j}}.$$

Equating components of the same degree gives $-n+r-i+j'-a = -i-j$, or $j' = n-r+a-j$. Hence $b_{i,j} = b_{r-i,n-r+a-j} = b_{r-i,d+a-j}$.

8.3. **Completion of the proof of Theorem 1.1.** We now return to our previous notation. Thus L is a fixed eight element set, $R = R_L$, k is a field of characteristic zero, *etc.* We begin with the following:

Proposition 8.1. *The ring R is Gorenstein with a-invariant -2.*

Proof. We first recall a theorem of Hochster-Roberts [BH, Theorem 6.5.1]: if V is a representation of the reductive group G (over a field of characteristic zero) then the ring of invariants $(\mathrm{Sym}\, V)^G$ is Cohen-Macaulay. As our ring R can be realized in this manner, with V being the space of 2×8 matrices and $G = \mathrm{SL}(2) \times T$, where T is the maximal torus in $\mathrm{SL}(8)$, we see that R is Cohen-Macaulay. We now recall a theorem of Stanley [BH, Corollary 4.4.6]: if R is a Cohen-Macaulay ring generated in degree one with Hilbert series $f(t)/(1-t)^d$, where d is the Krull dimension of R, then R is Gorenstein if and only if the polynomial f is symmetric. Furthermore, if f is symmetric then the a-invariant of R is given by $\deg f - d$. Going back to our situation, the Hilbert series of our ring was given in §1.1. The numerator is symmetric of degree four and the denominator has degree six. We thus see that R is Gorenstein with $a = -2$. □

We can now deduce the Betti diagram of R:

Proposition 8.2. *The Betti diagram of R is given by:*

	0	1	2	3	4	5	6	7	8
0	1	0	0	0	0	0	0	0	0
1	0	14	0	0	0	0	0	0	0
2	0	0	175	512	700	512	175	0	0
3	0	0	0	0	0	0	0	14	0
4	0	0	0	0	0	0	0	0	1

The i-axis is horizontal and the j-axis vertical. All $b_{i,j}$ outside of the above range are zero.

Proof. We first note that (B6) gives $b_{8-i,4-j} = b_{i,j}$ as $r = 8$, $d = 6$ and $a = -2$ in our situation. We thus have the symmetry of the table. Now, by (B2) and (B4) we have $b_{i,j} = 0$ if either i or j is negative. We thus see that $b_{i,j} = 0$ if $i > 8$ or $j > 4$ by symmetry. Next, observe that $P \to R$ is the free envelope of R, where $P = \mathrm{Sym}(V)$. This gives the $i = 0$ column of the table. We now look at the $i = 1$ column. We know that the 14 generators have no linear relations and so $b_{1,0} = 0$. By (B3) we have $b_{i,0} = 0$ for $i \geq 1$. We also know that there are 14 quadric relations and so $b_{1,1} = 14$. We now look at the $i = 2$ column of the table. We have proved (Proposition 5.1) that the 14 quadric relations have no linear syzygies; this gives $b_{2,1} = 0$. Using (B3) again, we conclude $b_{i,1} = 0$ for $i \geq 2$. We have thus completed the first two rows of the table. The last two rows can then be completed by symmetry. The middle row can now be determined from (B5) by evaluating both sides at $k = 2, \ldots, 10$ and solving the resulting upper triangular system of equations for $b_{i,2}$. (In fact, the computation is simpler than that since $b_{i,2} = b_{8-i,2}$ and we know $b_{0,2} = b_{1,2} = 0$, the latter vanishing coming from Proposition 5.5.) $\square$

Proposition 8.2 — in particular, the $i = 1$ column of the table — shows that I_8 is generated by its degree two piece. Thus we have proved Theorem 1.1.

Remark 8.3. The resolution of R as a P-module, without any consideration of grading, is given by Freitag and Salvati Manni [FS2, Lemma 1.3, Theorem 1.5]. It was obtained by computer.

9. Working over $\mathbb{Z}$: Proof and discussion of Theorem 1.2

In this section we take the base ring k to be $\mathbb{Z}$.

We begin with a short discussion of linear algebra over $\mathbb{Z}$. Let M be a finite free $\mathbb{Z}$-module and let N be a submodule. We say that N is *saturated* (in M) if whenever nx belongs to N, with $n \in \mathbb{Z}$ and $x \in M$, we have that x belongs to N. Of course, N is saturated if and only if it is a summand of M. Note that if N and N' are saturated submodules such that $N \otimes \mathbb{Q} = N' \otimes \mathbb{Q}$ then $N = N'$. Finally, we remark that $I^{(n)}$, the nth graded piece of the ideal, is a saturated submodule of $\mathrm{Sym}^n(V)$. This is easily seen as $I^{(n)}$ is the kernel of $\mathrm{Sym}^n(V) \to R^{(n)}$, and $R^{(n)}$ is torsion free.

We begin our discussion proper by giving an explicit formula for s' in terms of the basis of non-crossing matchings (see Figure 2 for a listing of these 14 generators):

$$\begin{aligned} s' \;=\;& x_1x_2(x_1+x_2) + x_1x_2(z_1+z_2+z_3+z_4+z_5+z_6+z_7+z_8) - (x_1y_2y_4 + x_2y_3y_1) \\ &+ (x_1z_2z_6 + x_2z_3z_7 + x_1z_4z_8 + x_2z_5z_1) + (y_1z_2z_6 + y_2z_3z_7 + y_3z_4z_8 + y_4z_5z_1) \\ &- (z_1z_2z_3 + z_2z_3z_4 + z_3z_4z_5 + z_4z_5z_6 + z_5z_6z_7 + z_6z_7z_8 + z_7z_8z_1 + z_8z_1z_2). \end{aligned}$$

(Note that for the formula to be unambiguous we need to specify how the edges of the matchings are directed. Label the vertices from 1 to 8 going clockwise, starting at any vertex. Then the edges are directed to point from smaller to larger numbers. The choice of starting vertex does not affect the above expression for s'.) This formula was found with the aid of a computer by taking the skew-average of a particular element of $\mathrm{Sym}^3(V)$. This element is related to the generalized Segre cubics of [HMSV3] and was chosen because it has a large isotropy subgroup. The right side of the expression for s' is visibly non-zero as we are in the polynomial ring on the x, y and z variables.

Proposition 9.1. *The 14 partial derivatives of s' give a basis for $I^{(2)}$ as a $\mathbb{Z}$-module.*

Proof. The reader may check that each of the 14 partial derivatives of s' contains a monomial with unit coefficient which does not appear in the other 13 partial derivatives. For example $\frac{\partial s'}{\partial x_1}$ contains the monomial

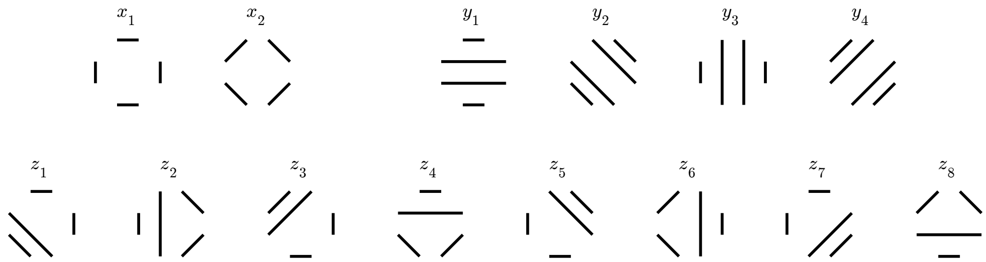

FIGURE 2. The fourteen non-crossing matchings.

term x_2^2 with coefficient $+1$, and this monomial does not appear in any of the other 13 partial derivatives. It follows that the $\mathbb{Z}$-module spanned by these 14 quadrics inside of $\mathrm{Sym}^2(V)$ is saturated. As these 14 quadrics give a basis for $I^{(2)} \otimes \mathbb{Q}$ and $I^{(2)}$ is saturated in $\mathrm{Sym}^2(V)$ we see that they must in fact give a basis for $I^{(2)}$ as a $\mathbb{Z}$-module. Thus the partial derivatives of s' give a basis for $I^{(2)}$ as a $\mathbb{Z}$-module. □

We can now prove Theorem 1.2:

Proof of Theorem 1.2. Let J be the ideal of $\mathrm{Sym}(V)$ generated by s' and its 14 partial derivatives. We must show $I = J$. By the main theorem of [HMSV1] it suffices to show $I^{(2)} = J^{(2)}$, $I^{(3)} = J^{(3)}$, and $I^{(4)} = J^{(4)}$. The previous proposition established the first of these equalities. We must establish the second.

Now, as in the previous proof, we know that $J^{(3)} \otimes \mathbb{Q} = I^{(3)} \otimes \mathbb{Q}$ and that $I^{(3)}$ is saturated in $\mathrm{Sym}^3(V)$. Thus to prove $I^{(3)} = J^{(3)}$ it suffices to show that $J^{(3)}$ is saturated. Unfortunately we do not how to do this by hand without an exorbitant amount of work. However, we can use the computer algebra system Magma [M] to check this; see the website[1] of the first author for the code. We perform this check as follows. Create a 197×560 matrix M, with the columns indexed by the cubic monomials in the 14 non-crossing variables and with the rows corresponding to the 196 possible products $u\frac{\partial s'}{\partial v}$, for $u, v \in \{x_1, x_2, y_1, \dots, y_4, z_1, \dots, z_8\}$ as well as s'. We verify that the elementary divisors of M are all equal to 1 by computing the Smith normal form of M in Magma. Similarly we check by computer that in degree 4 that the $197 \cdot 14 = 2758$ quartics generated by the cubics span a saturated sub-lattice of $\mathrm{Sym}^4(R^{(1)})$ of rank $1295 = \mathrm{rk}\, I^{(4)}$. (We compute the rank of $I^{(4)}$ by expressing it as $\mathrm{rk}(\mathrm{Sym}^4(R^{(1)})) - \mathrm{rk}(R^{(4)})$.)

Now suppose that J_0 is the ideal generated by the 14 partial derivatives alone. Similar to the above, we define a 196×560 matrix M_0, with rows indexed by the $u\frac{\partial s'}{\partial v}$, and we find using Magma that it has an elementary divisor equal to 3. This shows that $I \neq J_0$ and so the cubic s' is necessary when 3 is not invertible. □

9.1. **The Betti diagram in characteristic $p > 0$.** We now explain how the results and proofs in §8.3 can be adapted to work over a field k of positive characteristic. First, in [HMSV3] we will show that R is Cohen-Macaulay. In fact, we will show that R_n, over any field, is Cohen-Macaulay. (One cannot use the Hochster-Roberts theorem to prove this as the group $\mathrm{SL}(2) \times T$ does not have a semi-simple representation category in positive characteristic.) The proof of Proposition 8.1 then carries over to show that R is Gorenstein with $a = -2$. Note that Stanley's theorem is true over any field and that the Hilbert series of $(R)_k$ is independent of the field k as $(R)_{\mathbb{Z}}$ is flat over $\mathbb{Z}$.

We thus see that Proposition 8.1 holds true over any field. We now turn to Proposition 8.2. We first note that the same reasoning used in the proof of Proposition 5.5 shows that $b_{1,2} = b_{2,1}$ over any field. When $\mathrm{char}\, k \neq 3$ Theorem 1.2 gives $b_{1,2} = 0$. The proof of Proposition 8.2 then carries over exactly the same to this situation. Thus the Betti diagram is the same as in characteristic zero. Now consider the case where $\mathrm{char}\, k = 3$. Theorem 1.2 then gives $b_{1,2} = 1$ and so $b_{2,1} = 1$ as well. ¿From this, one may conclude $b_{3,1} = 0$ (as there must be at least two relations to produce a syzygy) and thus that $b_{i,1} = 0$ for $i \geq 3$. We thus have the first two rows of the table and by symmetry get the last two rows. One can again compute the middle row using (B5). However, all that goes into this computation is the alternating sum of the $b_{i,j}$ along diagonals $i + j = n$. Since $b_{1,2}$ still equals $b_{2,1}$, the alternating sum along the $i + j = 3$ diagonal is still the same. From this we see that the middle row is the same as in characteristic zero.

[1]http://www-personal.umich.edu/∼howardbj/8points.html

To sum up, we have proved the following:

Proposition 9.2. *Let k be a field. If* $\operatorname{char} k \neq 3$ *then the Betti diagram of* $(R)_k$ *is the same as that given in Proposition 8.2. If* $\operatorname{char} k = 3$ *then the only change is that* $b_{1,2} = b_{2,1} = 1$ *and, symmetrically,* $b_{7,2} = b_{6,3} = 1$.

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