Lecture 2: Spanning Sets and Independent Sets

Span

Definition

Let v_1, v_2, ..., v_n be the vectors in a vector space V. A vector $u \in V$ is said to be a linear combination of v_1, v_2, ..., v_n if there exist scalars c_1, c_2, ..., c_n such that

$$u = c_1 v_1 + c_2 v_2 + \ldots + c_n v_n.$$

The set of all linear combinations of v_1, v_2, ..., v_n is said to be the **span** of v_1, v_2, ..., v_n written

$$S\left(v_1, v_2, \ldots, v_n\right) \text{ or span}\left(v_1, v_2, \ldots, v_n\right)$$

Examples

(1) The set of the solutions to the differential equation

$$\frac{d^2y}{dx^2} - y = 0 \qquad (*)$$

is a vector space V under the rules of $+$ and $\bullet$ for functions, Example II of the last lecture.
Then this vector space is spaned by e^x and e^{-x}. If y is a solution of $(*)$, then there are constants (scalars) $c_1,\, c_2 \in \mathbb{R}$ such that

$$y(x) = c_1 e^x + c_2 e^{-x}.$$

Examples (continued)

(2) $V = \mathbb{R}^3$, $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$. Then $S(e_1, e_2) =$ the xy-plane .

Subspaces

Suppose X is a set with a binary operation $\circ$ so given any pair x_1, $x_2 \in X$ we have $x_1 \circ x_2 \in X$.
Let Y be a subset of X. We say Y is **closed** under $\circ$ if for any pair $y_1, y_2 \in Y$ we have $y_1 \circ y_2 \in Y$.

Definition

Let $(V, +, \bullet)$ be a vector space and $U \subset V$ be a subset. Then U is said to be a **subvector space** or **subspace** of V if

(i) U is closed under $+$,

(ii) U is closed under $\bullet$.

Subspaces

Proposition

If U is closed under $+$ and $\bullet$, then U equipped with the restrictions of $+$ and $\bullet$ satisfies the 8 vector space axioms $A1$, $A2$, $A3$, $A4$ and $S1$, $S2$, $S3$, $S4$. Hence $(U, +, \bullet)$ is a vector space. U is said to be a subspace of V.

Examples

(1) The xy-plane $\subset \mathbb{R}^3$.

(2) The space of polynomial functions of one variable of degree n $\text{Pol}_n(\mathbb{R}) \in \mathcal{F}_{\mathbb{R}}(\mathbb{R})$.

Proposition

Suppose v_1, v_2, …, v_n *are vectors in* V. *Then* $S\left(v_1, v_2, \ldots, v_n\right)$ *is a subspace of* V.

Proof.

S is closed under $+$

$$c_1v_1 + c_2v_2 + \ldots + c_nv_n + d_1v_1 + d_2v_2 + \ldots + d_nv_n =$$
$$= (c_1 + d_1)v_1 + (c_2 + d_2)v_2 + \ldots + (c_n + d_n)v_n.$$

S is closed under $\bullet$

$$c(c_1v_1 + c_2v_2 + \ldots + c_nv_n) = (cc_1)v_1 + (cc_2)v_2 + \ldots + (cc_n)v_n. \quad \square$$

$S\left(v_1, v_2, \ldots, v_n\right)$ is said to be the subspace of V spanned by $v_1, v_2, \ldots, v_n$

Proposition

$S(v_1, v_2, \ldots, v_n)$ is a subspace of V is the smallest subspace of V containing v_1, v_2, ..., v_n.

Proof.
Suppose U contains v_1, v_2, ..., v_n. Then since U is closed under $+$ and $\bullet$, any linear combination $c_1v_1 + c_2v_2 + \ldots + c_nv_n$ must be in U. Hence

$$S(v_1, v_2, \ldots, v_n) \subseteq U.$$

But U is the smallest subspace of V containing v_1, v_2, ..., v_n, so

$$U \subseteq S(v_1, v_2, \ldots, v_n),$$

so

$$U = S(v_1, v_2, \ldots, v_n). \quad \square$$

Spanning Sets

Definition

Let V be a vector space. Then a collection of vectors $\{v_1, v_2, \ldots, v_n\} \subset V$ is said to be a spanning set for V if

$$V = S\left(v_1, v_2, \ldots, v_n\right).$$

Examples

(1) (e_1, e_2, e_3) is a spanning set for $\mathbb{R}^3$.

(2) $(e_1, e_2, \ldots, e_n)$ is a spanning set for $\mathbb{R}^n$.

(3) $\left(1, x, x^2, \ldots, x^n\right)$ is a spanning set for $\mathrm{Pol}_n(\mathbb{R})$.

There are inefficient (too big) spanning sets for a vector space V. For example $\{(1,\,0),\,(0,\,1),\,(1,\,1)\}$ is a spanning set for $\mathbb{R}^2$ but any two of the three vectors still spans.

Dependence Relation

Let $v_1,\,v_2,\,\ldots,\,v_n \in V$. Then a dependence relation between $v_1,\,v_2,\,\ldots,\,v_n$ is an equation

$$c_1v_1 + c_2v_2 + \ldots + c_nv_n = 0, \quad c_1,\,c_2,\ldots c_n \in \mathbb{R}.$$

The dependence relation is said to the trivial dependence relation if all the $c_i's$ are zero.

So in the example from the top of the page

$$1\bullet(1,\,0) + 1\bullet(0,\,1) - 1\bullet(1,\,1) = 0.$$

is a (non-trivial) dependence relation.

Linear Independence

Definition

(1) If $v_1, v_2, \ldots, v_n$ satisfy a nontrivial dependence relation then they are said to be linearly **dependent**.

(2) $v_1, v_2, \ldots, v_n$ are said to be linearly **independent** if they are not linearly dependent.
So, $v_1, v_2, \ldots, v_n$ are linearly independent if

$$c_1 v_1 + c_2 v_2 + \ldots + c_n v_n = 0$$

$\implies$ all the c_i's are zero.

Exercise: Show $e_1, e_2, \ldots, e_n$ are linearly independent in $\mathbb{R}^n$.