Lecture 9: The Change of Basis Formula for the Matrix of a Linear Transformation

The Second Change of Basis Formula

Recall that $P_{\mathscr{C} \longleftarrow \mathscr{B}}$, the change of basis matrix from the basis $\mathscr{B}$ to the basis $\mathscr{C}$ (so read from right to left) is the matrix whose columns are the basis vectors of $\mathscr{B}$ written out in terms of $\mathscr{C}$.

Theorem (The second change of basis formula)

Suppose $T : V \longrightarrow V$ is a linear transformation and $\mathscr{B}$ and $\mathscr{C}$ are bases of V.
Then

$$_{\mathscr{C}}[T]_{\mathscr{C}} = P_{\mathscr{C} \longleftarrow \mathscr{B}} \; {}_{\mathscr{B}}[T]_{\mathscr{B}} \, P_{\mathscr{B} \longleftarrow \mathscr{C}} \qquad (*)$$

The whole point of the notation is to make this formula easy to remember. Mnemonic–keep the $\mathscr{B}$'s together.

Proof. By Proposition (1) of Lecture 7, we have

$$P_{\mathscr{B}\longleftarrow\mathscr{C}} = {}_{\mathscr{B}}[I_V]_{\mathscr{C}}$$

and

$$P_{\mathscr{C}\longleftarrow\mathscr{B}} = {}_{\mathscr{C}}[I_V]_{\mathscr{B}}\,.$$

Hence, the right-hand side of $(*)$ becomes

$$\mathrm{RHS} = {}_{\mathscr{C}}[I_V]_{\mathscr{B}} \bullet {}_{\mathscr{B}}[T]_{\mathscr{B}} \bullet {}_{\mathscr{B}}[I_V]_{\mathscr{C}}\,.$$

Here $\bullet$ is matrix multiplication.

But by Proposition (1) of Lecture 6 (applied twice) we have

$$\begin{aligned} \text{RHS} &= {}_{\mathscr{C}}[I_V \circ T \circ I_V]_{\mathscr{C}} \\ &= {}_{\mathscr{C}}[T]_{\mathscr{C}} . \end{aligned}$$

□

We now do two examples.

Problem 1
Let T be the linear transformation for $\mathbb{R}^2$ to $\mathbb{R}^2$ whose matrix ${}_{\mathscr{E}}[T]_{\mathscr{E}}$ relative to the standard basis $\mathscr{E} = \{(1,\, 0),\, (0,\, 1)\}$ is

$$ {}_{\mathscr{E}}[T]_{\mathscr{E}} = A = \begin{pmatrix} a & b \\ c & c \end{pmatrix} $$

Let $\mathscr{C} = \{f_1,\, f_2\}$ be the new basis for $\mathbb{R}^2$ given by $f_1 = \dfrac{e_1 + e_2}{\sqrt{2}}$, $f_2 = \dfrac{-e_1 + e_2}{\sqrt{2}}$ (so the old basis e_1, e_2 rotated by $45°$).
Find the matrix C of T relative to f_1, f_2. So we want

$$ C = {}_{\mathscr{C}}[T]_{\mathscr{C}} \,. $$

Problem 1

Solution

There are two steps

1. Compute the change of basis matrices $P_{\mathscr{C}\longleftarrow\mathscr{B}}$ and $P_{\mathscr{B}\longleftarrow\mathscr{C}}$.
2. Apply the Second Change of Basis Formula from Theorem 2 (pg 13).

Step 1

Put $\mathscr{E}$ = standard basis = $\{e_1, e_2\}$ (we will use $\mathscr{E}$ instead of $\mathscr{B}$ so you have to replace $\mathscr{B}$ by $\mathscr{E}$ in the formulas) amd $\mathscr{C} = \{f_1, f_2\}$.

Basic Principle

It is easy to compute $P_{\mathscr{E}\longleftarrow\mathscr{C}}$ for any basis $\mathscr{C} \in \mathbb{R}^n$. Then you compute $P_{\mathscr{C}\longleftarrow\mathscr{E}}$ by inverting $P_{\mathscr{E}\longleftarrow\mathscr{C}}$.

Computation of $P_{\mathscr{E} \longleftarrow \mathscr{C}}$

The change of basis matrix from $\mathscr{C}$ to $\mathscr{E}$ is the matrix whose i-th column is the coordinates of f_1 relative to e_1, e_2 so

$$P_{\mathscr{E} \longleftarrow \mathscr{C}} = \begin{matrix} e_1 \\ e_2 \end{matrix} \begin{pmatrix} f_1 & f_2 \\ \downarrow & \downarrow \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

So "writing $\mathscr{C}$ in terms of $\mathscr{E}$".

Computation of $P_{\mathscr{C} \longleftarrow \mathscr{E}}$

$$P_{\mathscr{C} \longleftarrow \mathscr{E}} = (P_{\mathscr{E} \longleftarrow \mathscr{C}})^{-1} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

Step 2 Apply the second change of basis formula.

$$_{\mathscr{C}}[T]_{\mathscr{C}} = P_{\mathscr{C}\longleftarrow\mathscr{E}}\ _{\mathscr{E}}[T]_{\mathscr{E}}\ P_{\mathscr{E}\longleftarrow\mathscr{C}}$$

So,

$$\begin{aligned}
C &= P^{-1}AP \\
&= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{a+b}{\sqrt{2}} & \frac{-a+b}{\sqrt{2}} \\ \frac{c+d}{\sqrt{2}} & \frac{-c+d}{\sqrt{2}} \end{pmatrix} \\
&= \begin{pmatrix} \frac{a+b+c+d}{2} & \frac{-a-c+b+d}{2} \\ \frac{-a-b+c+d}{2} & \frac{-b-c+a+d}{2} \end{pmatrix}
\end{aligned}$$

Problem 2
Let L be the (oriented) line that makes an angle of θ radians with the x-axis. Let r be reflection in the line L. Find the matrix of r relative to the standard basis $\mathscr{E}$.

Solution
The unit vector $t = (\cos\theta, \sin\theta)$ lies in L and has correction orientation. Use trigonometry.

Problem 2

The vector $n = (-\sin\theta, \cos\theta)$ perpendicular to the line L (see the picture). I use t for "tangent" and n for "normal".
The reflection r leave the line L fixed and carries the normal vector $(-\sin\theta, \cos\theta)$ to its negative. Think of L as the mirror for r.

Hence

$$r(t) = t$$

and

$$r(n) = -n.$$

Put $\mathscr{C} = \{t, n\}$ (t and n are orthogonal so they are independent).

Hence

$$
{}_{\mathscr{C}}[r]_{\mathscr{C}} = \begin{matrix} & t & n \\ t & \multicolumn{2}{c}{} \\ n & & \end{matrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
$$

We now apply the Second Change of Basis Formula to compute ${}_{\mathscr{E}}[r]_{\mathscr{E}}$.

Step 1 Compute the change of basis matrices $P_{\mathscr{E} \longleftarrow \mathscr{C}}$ and $P_{\mathscr{C} \longleftarrow \mathscr{E}}$.

$$
P_{\mathscr{E} \longleftarrow \mathscr{C}} = \begin{matrix} e_1 \\ e_2 \end{matrix} \overset{\quad t \qquad\qquad n}{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}
$$

Hence

$$
P_{\mathscr{C} \longleftarrow \mathscr{E}} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}^{-1} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}
$$

This matrix has determinant $\det = 1$.

Problem 2

Step 2 By the Second Change of Basis Formula we have:

$$
\begin{aligned}
[r]_{\mathscr{E}} &= P_{\mathscr{E}\longleftarrow\mathscr{C}}\ {}_{\mathscr{C}}[r]_{\mathscr{C}}\ P_{\mathscr{C}\longleftarrow\mathscr{E}} \\
&= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \\
&= \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \\
&= \begin{pmatrix} \cos^2\theta - \sin^2\theta & 2\cos\theta\sin\theta \\ 2\cos\theta\sin\theta & -(\cos^2\theta - \sin^2\theta) \end{pmatrix} \\
&= \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix} \\
&= \begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\end{aligned}
$$

In the last line, the first matrix is a rotation matrix and the second matrix is a reflection accross the x-matrix.