Lecture 13: Direct Sums and Projections

This lecture comes from the text, pages 195-198.

Direct Sums

Definition

Let V be a vector space and U and W be subspaces. Then V is said to be the **direct sum** of U and W, written $V = U \oplus W$, if every vector $v \in V$ has the unique expression

$$v = u + w,\ u \in U,\ w \in W.$$

Some more definitions:

Definition

V is said to be the sum of U and W is every vector $v \in V$ has at least on expression

$$v = u + w,\ u \in U,\ w \in W.$$

In this case, we write $V = U + W$.

Definition

U and W are said to be independent if $U \cap W = \{0\}$.

Lemma

Let V be a vector space, $U, W \subseteq V$ subspaces. Then $V = U \oplus W$ if and only if

(i) $V = U + W$

(ii) $U \cap W = \{0\}$.

Proof. ($\Longrightarrow$) (i) is clear since every $v \in V$ can be written (uniquely) as $v = u + w$ with $u \in U$, $w \in W$.

Now for for (ii). Let $v \in U \cap W$. Then since $v \in U$ and $v \in W$, we can write:

$$v = v + 0 \text{ (where } v \in U, 0 \in W)$$

and

$$v = 0 + v \text{ (where } 0 \in U, v \in W)$$

.

But the expression $v = u + w$ is unique, hence $v = 0$.

($\Longleftarrow$) Since $V = U + W$, we must only check uniqueness. So suppose $v = u_1 + w_1$ and $v = u_2 + w_2$, where $u_i,\, u_2 \in U$ and $w_1,\, w_2 \in W$. Then

$$u_1 + w_1 = u_2 + w_2$$

and thus

$$u_1 - u_2 = w_2 - w_1$$

.

Put $x = u_1 - u_2 = w_2 - w_1$.

Then $x \in U$ and $x \in W$, so $x \in U \cap W = \{0\}$ and hence $x = 0$. Thus $u_1 = u_2$ and $w_1 = w_2$. □

Idempotence

Definition

Let $p \in L(V, V)$. Then p is said to be idempotent if $p^2 = p$.

Lemma

Let $p \in L(V, V)$ be idempotent, $W = R(p)$ and $U = N(p)$. Then $V = U \oplus W$.

Proof.

We first show that $V = U + W$. Let $v \in V$. Then $v = (v - p(v)) + p(v)$. By definition, $p(v) \in R(p) = W$. Also, $p(v - p(v)) = p(v) - p^2(v)$ $= p(v) - p(v) = 0$. Hence $v - p(v) \in N(p) = U$.

Now we show $U \cap W = N(p) \cap R(p)$. Then since $v \in R(p)$ we have $v = p(v')$ for some $v' \in V$. And, since $v \in N(p)$ we have $p(v) = 0$.
$p^2(v') = 0$.
But $0 = p^2(v') = p(v') = v$ and thus $v = 0$. □

Direct Sum and Idempotence

Proposition

Every direct sum decomposition arises in this way.

Proof. Let $V = U \oplus W$.
Define $p : V \longrightarrow V$ by

$$p(u + w) \longmapsto w.$$

The $p^2 = p$ and $R(p) = W$, $N(p) = U$. $\square$

Note: We can also define $q : V \longrightarrow V$ by

$$q(u + w) \longmapsto u.$$

Lemma

(i) $p \circ q = 0$

(ii) $p + q = I$

Proof.

(i) Let $v = u + w$. Then

$$(p \circ q)(v) = (p \circ q)(u + w) = p(q(u + w)) = p(u) = 0.$$

(ii) Let $v = u + w$. Then

$$(p + q)(v) = (p + circq)(u+w) = p(u + w)+q(u + w) = w+u = v. \quad \square$$

p and q are called the projections associated to the direct sum decomposition $V = U \oplus W$.

Orthogonal Direct Sums

Suppose now $(V, (\,,\,))$ is an inner product space and $U \subset V$ is a subspace. Define the orthogonal complement:

$$U^{\perp} := \{v \in V : (u,\, v) = 0,\ \text{all } u \in U\}.$$

Lemma

$U^{\perp}$ is a subspace.

Proof. First $0 \in U^{\perp}$ since $(u,\, 0) = 0$ for all $u \in U$. Now suppose v_1, $v_2 \in U^{\perp}$. Then

$$(u,\, v_1 + v_2) = (u,\, v_1) + (u,\, v_2) = 0 + 0 \text{ for all} u \in U$$

and thus $v_1 + v_2 \in U^{\perp}$.
Finally, if $c \in \mathbb{R}$ and $v \in U^{\perp}$ then

$$(u,\, cv) = c\,(u,\, v) = c \cdot 0 = 0 \text{ for all } u \in U.$$

and thus $cv \in U^{\perp}$. □

Orthogonal Direct Sums

Proposition

Let $(V, (\ ,\ ))$ be an inner product space and $U \subseteq V$ a subspace. The given an orthogonal basis $\mathscr{B}_U = \{u_1, \ldots, u_k\}$ for U, it can be extended to an orthonormal basis $\mathscr{B} = \{u_1, \ldots, u_n\}$ for V.

Proof. First, extend $\mathscr{B}_U$ to a basis for V,
$\mathscr{B}' = \mathscr{B}_U = \{u_1, \ldots, u_k, v_1, \ldots, v_{n-k}\}$. Now apply Gram-Schmidt. Since $\{u_1, \ldots, u_k\}$ is already an orthonormal set, they are left fixed (this is a property of Gram-Schmidt). Thus the resulting basis is an orthonormal extension of the basis $\mathscr{B}_U$.

The following lemmas are a consequence of this proposition.

Lemma

Given $U \subset V$, there is an orthonormal basis for V,
$\mathscr{B} = \{u_1, \ldots, u_k, u_{k+1}, \ldots, u_n\}$ *so that*

$$u_i \in U \text{ for } 1 \leq k$$

$$u_i \in U^{\perp} \text{ for } k+1 \leq n.$$

Proof. Take a basis for U, $\mathscr{B}' = \{v_1, \ldots, v_k\}$ and apply Gram-Schmidt to get an orthonormal basis $\mathscr{B}_U = \{u_1, \ldots, u_k\}$ for U. □

Lemma

Given $U \subset V$ *and a basis* $\mathscr{B} = \{u_1, \ldots, u_k, u_{k+1}, \ldots, u_n\}$ *as above,* $\mathscr{B}_{U^\perp} = \{u_{k+1}, \ldots, u_n\}$ *is a basis for* $U^\perp$*.*

Proof. First, $S(u_{k+1}, \ldots, u_n) \subset U^\perp$ is clear. Now we claim $U^\perp \cap U = \{0\}$. Suppose $U^\perp \cup U$. Then $(u, u) = 0$, hence $u = 0$. Thus, $U^\perp \cap S(u_1, \ldots, u_k) = \{0\}$ and so $S(u_{k+1}, \ldots, u_n) = U^\perp$. Finally, any subset of a basis is linearly independent. □

Corollary

$U \cap U^{\perp} = \{0\}$.

Lemma

$V = U \oplus U^{\perp}$.

Proof. It remains to show that $V = U + U^{\perp}$. But since we have a basis for V, $\mathscr{B} = \{u_1, \ldots, u_k, u_{k+1}, \ldots, u_n\}$ with

$$u_i \in U \text{ for } 1 \leq k$$

$$u_i \in U^{\perp} \text{ for } k + 1 \leq n,$$

this is clear.

Corollary

$\dim V = \dim U + \dim U^{\perp}$.

Lemma

$(U^{\perp})^{\perp} = U$.

Proof. $U \subset (U^{\perp})^{\perp}$ is clear and since they are both subspaces of V, with the same dimension, $U = \subset (U^{\perp})^{\perp}$.

Proposition

Suppose $(V, (\ ,\ ))$ *is an inner product space and* $V = U \oplus W$ *is a direct sum decomposition (not necessarily orthogonal). Let* p_U *be the associated projection. Then* $W = U^\perp$ *if and only if*

$$(*)\ (p_U v_1,\ v_2) = (v_1,\ p_U v_2) \ \ \textit{all}\ v_1,\ v_2 \in V.$$

Proof. Let $u \in U$, $w \in W$.
($\Longleftarrow$) Suppose ($*$) holds. Then

$$(u,\ w) = (p_{U u},\ w) = (u,\ p_U w) = (u,\ 0) = 0.$$

($\Longrightarrow$) Suppose $V = U \oplus W$ is orthogonal. Let v_1, $v_2 \in V$. Then $v_1 = u_1 + w_1$ and $v_2 = u_2 + w_2$ (uniquely) with u_1, $u_2 \in U$, w_1, $w_2 \in W$.
Then

$$\begin{aligned} (p_U v_1, v_2) &= (P_U(u_1 + w_1), u_2 + w2) = (u_1, u_2 + w_2) \\ &= (u_1, u_2), \end{aligned}$$

and

$$\begin{aligned} (v_1, p_U v_2) &= (u_1 + w1, P_U(u_2 + w_2)) = (u_1 + w_1, u_2) \\ &= (u_1, u_2). \quad \square \end{aligned}$$

M-Fold Direct Sums

Definition

Let U_1, U_2, ..., U_m be subspaces of V. Then V is the direct sum of U_1, U_2, ..., U_m, written

$$V = U_1 \oplus U_2 \oplus \ldots \oplus U_m$$

if every $v \in V$ may be written as

$$v = u_1 + u_2 + \ldots + u_m \text{ with } u_i \in U_i,\ 1 \leq i \leq m.$$

Proposition

$V = U_1 \oplus U_2 \oplus \ldots \oplus U_m$ if and only if

(i) $V = U_1 + U_2 + \ldots + U_m$

(ii) $U_i \cap \left\{U_1 + \ldots + \hat{U}_i + \ldots + U_m\right\} = \{0\}$ *for $i \leq i \leq m$ (where hat signifies that this term has been omitted.)*

Proof.
($\Longrightarrow$) (i) is clear since every $v \in V$ can be expressed

$$v = u_1 + u_2 + \ldots + u_m \text{ where } u_i \in U_i,\ 1 \le i \le m.$$

(ii) Fix i with $1 \le i \le m$. Let $v \in U_i \cap \{u_1 + \ldots + \hat{u}_i + \ldots + u_m\}$. Then

$$v = 0 + \ldots + 0 + \hat{u}_i + 0 + \ldots + 0 = u_1 + \ldots + u_{i-1} + 0 + u_{i+1} + \ldots + u_m$$

and hence $u_j = 0$, $1 \le j \le m$. So $v = 0$.
($\Longleftarrow$) Suppose $u_1 + u_2 + \ldots + u_m = u'_1 + u'_2 + \ldots + u'_m$. Fix i with $1 \le i \le m$. Then

$$u_i - u'_i = (u'_1 - u_1) + \ldots + (u'_i - u_i) + (u'_{i+1} - u_{i+1}) + \ldots + (u'_m - u_m).$$

Set $v = u_i - u'_i$. Then $v \in U_i$ and $v \in U_1 + \ldots + \hat{U}_i + \ldots + U_m$, hence $v = 0$ and $u_i = \hat{u}'_i$. This is true for each i, hence the expression $v = u_1 + \ldots + u_m$ is unique. □

Projection

Definition

Given $V = U_1 \oplus U_2 \oplus \ldots \oplus U_m$, define the projection $P_i \in L(V, V)$ by

$$P_i(u_1 + u_2 + \ldots + u_i + \ldots + u_m) = u_i.$$

Hence

$$\begin{aligned} R(p_i) &= U_i \\ N(p_i) &= U_1 \oplus \ldots \oplus \hat{U}_i \oplus \ldots \oplus U_m \end{aligned}$$

Lemma

(i) $p_i \circ p_j = 0,\ i \neq j.$

(ii) $p_1 + p_2 + \ldots + p_m = I.$

Proof. Same as for when $m = 2$.