Lecture 19

The Characteristic Polynomial of a Linear Transformation

Eigenvalues and Eigenvectors

Let A be an $n \times n$ matrix. We define the Characteristic polynomial $h(x)$ by

$$h_A(x) = \det(xI_n - A)$$

$$= \det \begin{pmatrix} x-a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & x-a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & \cdots & & x-a_{nn} \end{pmatrix}$$

Example 2×2 case:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad h_A(x) = \det \begin{pmatrix} x-a & b \\ c & x-d \end{pmatrix}$$

$$h_A(x) = (x-a)(x-d) - bc$$

$$= x^2 - (a+d)x + (ad-bc)$$

$$= x^2 - tr(A)x + \det(A).$$

Proposition

$h_A(x)$ is an (inhomogeneous) polynomial of degree n. The top term is x^n. The constant term is $(-1)^n \det A$.

We need

Proposition

Suppose $T \in L(V,V)$, $\mathcal{B} = \{b_1, \ldots, b_n\}$, $\mathcal{C} = \{c_1, \ldots, c_n\}$ are ordered bases for V, and

$$A = {}_{\mathcal{B}}[T]_{\mathcal{B}}, \quad B = {}_{\mathcal{C}}[T]_{\mathcal{C}}$$

Then $\det A = \det B$.

proof

$$B = {}_{\mathcal{C}}[T]_{\mathcal{C}} = P_{\mathcal{C}\leftarrow\mathcal{B}} \, {}_{\mathcal{B}}[T]_{\mathcal{B}} \, P_{\mathcal{B}\leftarrow\mathcal{C}} = C^{-1}AC$$

(where C is the matrix of $\mathcal{C}$ relative to $\mathcal{B}$)

Then $\det B = \det(C^{-1}AC) = \det C^{-1} \det A \det C = \det A$

$\square$

We may accordingly define $\det T$ for $T \in L(V,V)$ as follows. Choose an ordered basis $\mathcal{B}$ for V and define

$$\det T := \det {}_{\mathcal{B}}[T]_{\mathcal{B}}$$

This is well-defined because $\det {}_{\mathcal{B}}[T]_{\mathcal{B}}$ doesn't depend on the choice of basis $\mathcal{B}$.

We can follow a similar road to define the characteristic polynomial of a linear transformation T. Choose a basis $\mathcal{B}$ for V. Define

$$h_T(x) = \det\left(x I_n - {}_{\mathcal{B}}T_{\mathcal{B}}\right).$$

Why doesn't this depend on $\mathcal{B}$?

Let $\mathcal{B}$ and $\mathcal{C}$ be bases for V. Let $C = P_{\mathcal{B} \leftarrow \mathcal{C}}$, $A = {}_{\mathcal{B}}[T]_{\mathcal{B}}$

and $B = {}_{\mathcal{C}}[T]_{\mathcal{C}}$ so $B = CAC^{-1}$.

Then

$$\det(xI_n - B) = \det(xI_n - CAC^{-1})$$
$$= \det(C(xI_n - A)C^{-1})$$
$$= \det(xI_n - A)$$

So the $\det(xI - {}_{\mathcal{B}}[T]_{\mathcal{B}})$ does not depend on the choice of basis $\mathcal{B}$.

So we can define

$$h_T(x) = h_A(x)$$

The Hamilton - Cayley Theorem

(to be proved in Lecture 22)

Theorem

$$h_T(T) = 0$$

The proof is hard. We will prove this after we have proved we can put a ~~matrix~~ in triangular form by making a change of basis. The "naive" proof of plugging A in for X in

$\det(xI - A)$ is not correct...

"$f(A) = \det(A - AI) = \det(0) = 0$." No!

But this is 0 the number. We need 0 the matrix. In the 2×2 case we need to show

$$f(A) = A^2 - (\operatorname{tr} A)A + (\det A)I = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

<u>Corollary</u>

$m_T(x) \mid h_T(x)$ So $\deg m_T \leq n$.

<u>The minimal polynomial of 2×2 matrices</u>

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$h_A(x) = x^2 - (a+d)x + (ad - bc)$$

$$= x^2 - \operatorname{Tr} A\, x + \det A.$$

So h_A is the minimal polynomial unless $A = cI = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}$ is a scalar. In this case $h_A(x) = (x-c)^2$ but $m(x) = x - c$.

Eigenvalues and Eigenvectors

Definition:

Let $T \in L(V,V)$. An element $\lambda \in F$ is called an eigenvalue of T if there exists a non-zero vector v such that

$$Tv = \lambda v \quad (*)$$

v satisfying $(*)$ is said to be an eigenvector belonging to λ (or "with eigenvalue λ").

Note that if v satisfies

$$Tv = \lambda v$$

then cv satisfies

$$T(cv) = \lambda (cv)$$

In fact the set of vectors satisfying (*) is a linear subspace of V called the λ-eigenspace.

It is not a priori clear that T has a finite number of eigenvalues but we have

Theorem

λ is an eigenvalue of $T \Leftrightarrow \lambda$ is a root of the characteristic polynomial h_T.

Proof

λ is an eigenvalue of T if and only if there exists a nonzero vector v such that

$$Tv = \lambda v$$

or

$$(T - \lambda I) v = 0 \qquad (*)$$

Note that $(T-\lambda I) : V \to V$ so $v \neq 0$ exists $\Leftrightarrow$ $T-\lambda I$ is not invertible $\Leftrightarrow \det(T-\lambda I) = 0 \Leftrightarrow h_T(\lambda) = 0$

$\square$

An Eigenvalue/Eigenvector Computation

(there is another one in Lecture 21)

Find the two eigenvalues of $A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$ and a nonzero eigenvector belonging to each one.

<u>Step 1</u> Compute the characteristic polynomial $h_A(x)$ of A.

$$h_A(x) = \det\begin{pmatrix} x-1 & -1 \\ 0 & x-2 \end{pmatrix} = (x-1)(x-2)$$

<u>Step 2</u> Compute the roots of the characteristic polynomial. (this step can be very hard/impossible but here it is very easy).

$h_A(x) = 0$ iff and only if $(x-1)(x-2) = 0$

if and only if $x = 1$ or $x = 2$.

So there are two distinct eigenvalues λ_1, λ_2 of A namely $\lambda_1 = 1$ and $\lambda_2 = 2$. (best possible outcome).

Step 3 For each of λ_1 and λ_2 find a nonzero solution v_i to the linear equation

$$A v_i = \lambda_i v_i, \quad i = 1, 2.$$

Then v_i will be an eigenvector belonging to λ_i.

Warning There will always be a one (or more) parameter family of solutions. So you have to pick a value for the parameter

An eigenvector v corresponding to the eigenvalue $\lambda = 1$

Put $v = (x, y)$ so we need to solve

$$A \begin{pmatrix} x \\ y \end{pmatrix} = (1) \begin{pmatrix} x \\ y \end{pmatrix} \quad (*)$$

or

$$\begin{pmatrix} x + y \\ 2y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

Hence, we get two equations

$$\begin{aligned} x + y &= x \iff y = 0 \\ 2y &= y \iff y = 0 \end{aligned} \quad (**)$$

So both equations are equivallent to the single equation $y=0$. So the generall solution to (**) is $(x, 0)$ depending on the parameter x. We will choose $x=1$ so we obtain that $v=(1, 0)$ is a non zero eigenvector belonging to the eigenvalue 1 (this was obvious from the start).

<u>An eigenvector v corresponding to the eigenvalue $\lambda=2$</u>

Now we need to solve the equation

$$A\begin{pmatrix} x \\ y \end{pmatrix} = 2\begin{pmatrix} x \\ y \end{pmatrix} \quad (*)$$

or

$$\begin{pmatrix} x+y \\ 2y \end{pmatrix} = \begin{pmatrix} 2x \\ 2y \end{pmatrix}$$

Hence, we get two equations

$$\left.\begin{aligned} x+y &= 2x \iff y=x \\ 2y &= 2y \iff 0=0 \end{aligned}\right. \quad (**)$$

So the general sollution to (**) is (x, x). We will choose $x=1$ so $v=(1, 1)$ is a non zero eigenvector belonging to $\lambda=2$.