Lecture 7

The Five Basic Discrete Random Variables

1. Binomial
2. Hypergeometric
3. Geometric
4. Negative Binomial
5. Poisson

Remark

On the handout "The basic probability distributions" there are six distributions. I did not list the Bernoulli distribution above because it is too simple

In this lecture we will do 1. and 2. above

The Binomial Distribution

Suppose we have a Bernoulli experiment with $P(S) = p$, for example, a weighted coin with $P(H) = p$. As usual we put $q = 1-p$.

Repeat the experiment (flip the coin). Let X = # of successes (# of heads).

We want to compute the probability distribution of X. Note, we did the special case $n=3$ in Lecture 6, pages 4 and 5.

Clearly the set of possible values for X is $0, 1, 2, 3, \ldots, n$.

Also th...

$$P(X=0) = P(TT \cdots T) = qq \cdots q = q^n$$

Explanation

Here we assume the outcomes of each of the repeated experiments are independent so

$$P((T \text{ on } 1^{st}) \cap (T \text{ on } 2^{nd}) \cap \cdots \cap (T \text{ on } n\text{-th}))$$

$$= P(T \text{ on } 1^{st})\, P(T \text{ on } 2^{nd}) \cdots P(T \text{ on } n\text{-th})$$

$$= q \; q \cdots q = q^n$$

Note T on 2^{nd} means T on 2^{nd} with no other information so

$$P(T \text{ on } 2^{nd}) = q.$$

Also

$$P(X=n) = P(HH\cdots H) = p^n$$

Now we have to work

What is $P(X=1)$?

<u>Another standard mistake</u>

The events $(X=1)$ and $H\underbrace{TT\cdots T}_{n-1}$

are <u>NOT</u> equal.

Why – <u>the head doesn't have to come on the first toss</u>

So in fact

$$(X=1) = HTT\cdots T \cup THT\cdots T \cup \cdots \cup TTT\cdots TH$$

All of the n events on the right have the same probability namely pq^{n-1} and they are mutually exclusive. There are n of them so

$$P(X=1) = npq^{n-1}$$

Similarly

$$P(X=n-1) = np^{n-1}q$$

(exchange H and T above)

The general formula

Now we want $P(X=k)$

First we note

$$P(\underbrace{H \cdots H}_{k}\ \underbrace{TT \cdots T}_{n-k}) = p^k q^{n-k}$$

But again the heads don't have to come first. So we need to

(1) Count all the words of length n in H and T that involve k H's and $n-k$ T's.

(2) Multiply the number in (1.) by $p^k q^{n-k}$.

So how do we solve 1.

Think of filling n slot's with k H's and $n-k$ T's

$$\underbrace{\_\quad\_\quad\_\quad\_\quad\_\quad\_\quad\_}$$

<u>Main Point</u> Once you decide where the k H's go you have no choice with the T's. They have to go in the remaining $n-k$ slots.

So choose the k-slots where the heads go. So we have to make a choice of k things from n things so $\binom{n}{k}$..

So

$$P(X=k)=\binom{n}{k}p^k q^{n-k}$$

So we have motivated the following definiton

<u>Definition.</u>

A discrete random variable X is said to have binomial distribution with parameters n and p (abbreviated $X \sim Bin(n,p)$) if X takes values $0, 1, 2, \ldots, n$ and

$$(*) \quad P(X=k)=\binom{n}{k}p^k q^{n-k}, \quad 0 \le k \le n.$$

Remark

The text uses x insteed of k for the independent (i.e input) variable. So this would be written

$$P(X=x) = \binom{n}{x} p^x q^{n-x}$$

I like to save x for the case of continuous random variables.

Finally we may write

$$(**) \quad p(k) = \binom{n}{k} p^k q^{n-k}, \quad 0 \le k \le n$$

The text uses $b(\cdot; n, p)$ for $p(\cdot)$ so would write for $(**)$

$$b(k; n, p) = \binom{n}{k} p^k q^{n-k}$$

The Expected Value and Variance of a Binomial Random Variable

Proposition

Suppose $X \sim Bin(n,p)$. Then $E(X) = np$ and $V(X) = npq$

so σ = standard deviation $= \sqrt{npq}$.

Remark

The formula for $E(X)$ is what you might expect. If you toss a fair coin 100 times the $E(X)$ = expected number of heads $np = (100)(\frac{1}{2}) = 50$

However if you toss it 51 times then $E(X) = \frac{51}{2}$ — not what you "expect".

Using the binomial tables

Table A1 in the text pg 664 - 666 tabulates the cdf $B(x; n, p)$ for $n = 5, 10, 15, 20, 25$ and selected values of p.

Example 3.32

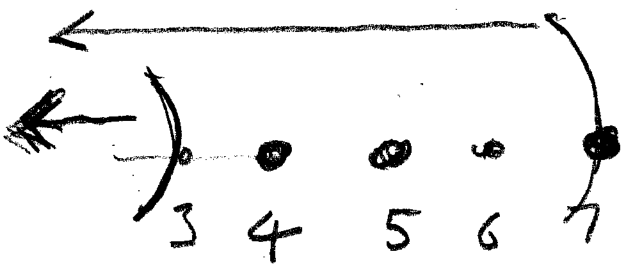
Suppose that 20% of all copies of a particular text book fail a certain binding strength test. Let X denote the number among 15 randomly selected copies that fail the test. Find

$$P(4 \leq X \leq 7)$$

Solution

$X \sim Bin(15, .2)$. We want to compute $P(4 \leq X \leq 7)$ using the table on page 664

So how to we write $P(4 \leq X \leq 7)$ in terms of terms of the form $P(X \leq a)$



Answer (#) $P(4 \leq X \leq 7) = P(X \leq 7) - P(X \leq 3)$

So $P(4 \leq X \leq 7) = B(7; 15, .2) - B(3; 15, .2)$

from table

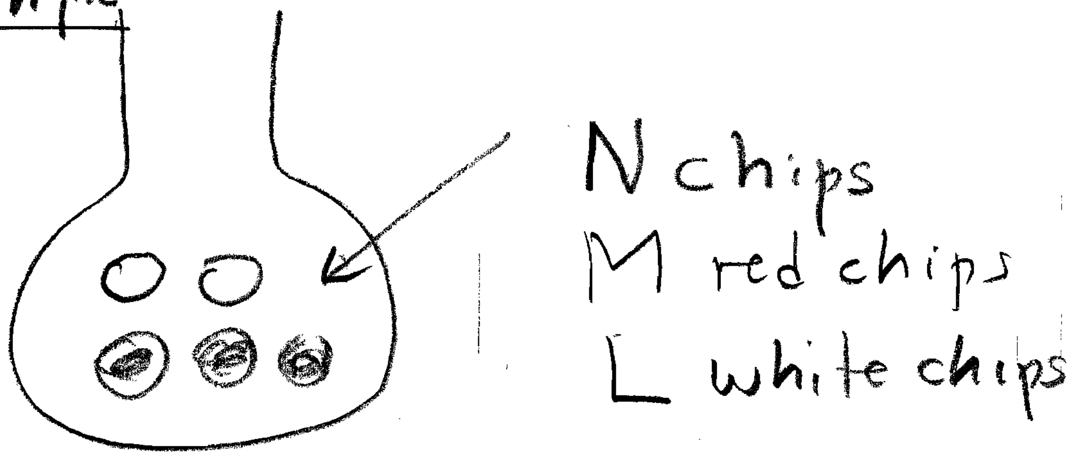
$= .996 - .648$

$= .348$

N.B. Understand (#) This the key using computers and statistical calculators to compute.

2. The hypergeometric distributcon

Example



Consider an urn containing N chips of which M are red and $L = N-M$ are white. Suppose we remove n chips without replacement so $n \leq N$.

Define a random variable X by $X =$ # of red chips we get

Find the probability distribution of X.

<u>Proposition</u>

$$P(X=k) = \frac{\binom{M}{k}\binom{L}{n-k}}{\binom{N}{n}} \quad (*)$$

if

(b) $\underbrace{\max(0, n-L) \le k \le \min(n, M)}$

This means $k \le$ both n and M

and

both 0 and $n-L \le k$.

These are the possible values of k, that is, if k doesn't satisfy (b) then

$$P(X=k) = 0.$$

Proof of the formula (*)

Suppose we first consider the special case where all the chips are red so

$$P(X=n).$$

This is the same problem as the one of finding all hearts in bridge

red chip $\Longleftrightarrow$ heart

white chip $\Longleftrightarrow$ non heart

So we use the principle of restricted choice

$$P(X=n) = \frac{\binom{M}{n}}{\binom{N}{n}}$$

This agrees with (*)

But (*) is harder because we have to consider the case where there are $k < n$ red chips. <u>So we have to choose $n-k$ white chips as well.</u>

So choose k red chips — $\binom{M}{k}$ ways, then for each such choice, choose $n-k$ white chips $\binom{L}{n-k}$ ways.

So

$$\#\left(\begin{array}{l}\text{choices of \underline{exactly}} \\ k \text{ red chips} \\ \text{in the } n \text{ chips}\end{array}\right) = \binom{M}{k}\binom{L}{n-k}$$

Clearly there are $\binom{N}{n}$ ways of choosing n chips from N chips so (*) follows.

Definiton

If X is a discrete random variable with pmf defined by page 14 then X is said to have <u>hypergeometric distribution with parameters n, M, N</u>.

In the text the pmf is denoted

$h(x; n, M, N)$.

What about the conditions

$$\max(0, n-L) \leq k \leq \min(n, M) \quad (b)$$

This really means

$$k \leq \text{both } n \text{ and } M \quad (b_1)$$

and

$$\text{both } 0 \text{ and } n-L \leq k \quad (b_2)$$

(b_1) says

$k \leq n$ $\longleftrightarrow$ we can't choose more than n red chips because we are only choosing n chips in toto

$k \leq M$ $\longleftrightarrow$ because ~~there~~ are only M red chips to choose from

$b_2)$

$k \geq 0$ is obvious.

So the above three inequalities 19
are necessary. At first glance they look sufficient because if k satisfies the above three inequalities you can certainly go ahead and choose k red chips. <u>But what about the white chips?</u> We aren't done yet, you have to choose $n-k$ white chips and there are only L white chips available so if $n-k > L$ we are sunk so we must have

$$n-k \le L \iff k \ge n-L$$

This is the second inequality of (b_2). If it is satisfied we can go ahead and choose the $n-k$ white chips so the inequalities in (b) are necessary and sufficient.

Proposition

Suppose X has hypergeometric distribution with parameters n, M, N. Then

(i) $$E(X) = n\frac{M}{N}$$

(ii) $$V(X) = \left(\frac{N-n}{N-1}\right) n \frac{M}{N}\left(1-\frac{M}{N}\right)$$

If you put

$p = \frac{M}{N}$ = the probability of getting a red disk on the first draw

then we may rewrite the above formulas as

$$\left.\begin{aligned} E(X) &= np \\ V(X) &= \left(\frac{N-n}{N-1}\right) npq \end{aligned}\right\} \text{reminiscent of the binomial distribution}$$

Another Way to Derive (*)

There is another way to derive (*) - the way we derived the binomial distribution. It is way harder.

Example Take $n=2$

$$P(X=0) = \frac{L}{N}\,\frac{L-1}{N-1}$$

$$P(X=2) = \frac{M}{N}\,\frac{M-1}{N-1}$$

$$P(X=1) = P(RW) + P(WR)$$

$$= \frac{M}{N}\,\frac{L}{N-1} + \frac{L}{N}\,\frac{M}{N-1}$$

$$= 2\,\frac{M}{N}\,\frac{L}{N-1}$$

In general, we claim that all the words with k R's and $n-k$ W's have the same probability. Indeed each of these probabilities are fractions with the same denominator

$$N(N-1)\cdots(N-n-1)$$

and they have the same factors in the numerator scrambled up

$M(M-1)$ $(M-k+1)$ and $L(L-1)\cdots,(L-n-k+1)$

But the order of the factors doesn't matter so

$$P(X=k)=\binom{n}{k}P(\overbrace{R\cdots R}^{k}\,W\cdots W)$$

$$=\binom{n}{k}\frac{M(M-1)\cdots(M-k+1)\,L(L-1)\cdots(L-n-k+1)}{N(N-1)\cdots N(-n+1)}$$

Why is $(*)$ equal to this?

$$(*) = \frac{\binom{M}{k}\binom{L}{n-k}}{\binom{N}{n}}$$

cancelling $(M-k)!$

cancelling $(L-(n-k))!$

$$= \frac{\dfrac{M(M-1)\cdots(M-k+1)}{k!}\;\dfrac{L(L-1)\cdots(L-n-k+1)}{(n-k)!}}{\dfrac{N(N-1)\cdots(N-n+1)}{n!}}$$

goes on top

exercise in fractions

$$= \frac{n!}{k!\,(n-k)!}\;\frac{M(M-1)\cdots(M-k+1)\,L(L-1)\cdots(L-n-k+1)}{N(N-1)\cdots(N-n+1)}$$

$$= \binom{n}{k}\frac{M(M-1)\cdots(M-k+1)\,L(L-1)\cdots(L-n-k+1)}{N(N-1)\cdots(N-n+1)}.$$

Obviously, the first way (*) is easier so if you are doing a real-world problem and you start getting things that look like (**) step back and see if you can use the first method instead. You will tend to try the second method first. I will test you on this later.

<u>Prediction</u> (I was wrong before)

Most of you will use the second (wrong) method.

An Important General Problem

Suppose you draw n chips with replacement and let X be the number of red chips you get. What distribution does X have?

This explains (a little) the formulas on page 21.

Note that if N is far bigger than n then it is almost like drawing with replacement. "The urn doesn't notice that any chips have been removed because so few (relatively) have been removed".

In this case

$$\frac{N-n}{N-1} = \frac{N(1-\frac{n}{N})}{N(1-\frac{1}{N})} \approx \frac{N}{N} = 1$$

(because N is huge $\frac{1}{N}$ and $\frac{n}{N} \approx 0$)

So $V(X) \approx npq$ (see the bottom of pg 21)

This is what is going on in page 118 of the text.
The number $\frac{N-n}{N-1}$ is called

the "finite population correction factor".