Lecture 26
Random Intervals and Confidence Intervals

1 The definition of a random interval

Let X_1 and X_2 be random variables defined on the same sample space S such that $X_1(s) < X_2(s)$ for all $s \in S$. Then $I = (X_1, X_2)$ is called an (open) random interval. For each $s \in S$ we obtain an ordinary interval $I(s) = (X_1(s), X_2(s))$. Thus we may think of a random interval as an *interval-valued* random variable defined on S. The point of this lecture is that *confidence intervals are random intervals*.

Example Suppose X is a random variable defined on S and a is a positive number then $I = (X - a, X + a)$ is the random interval with random center X and (deterministic) width $2a$. More generally the random interval $I = (X - Y, X + Y)$ has random center X and random width $2Y$.

2 Probabilities connected with random intervals

Now consider a random interval interval $I = (X_1, X_2)$ and a fixed number a. We want to compute the probability that the random interval I will contain (or cover) the fixed number a. But this is just the probability that a will be between X_1 and X_2 hence we have

$$P(a \in (X_1, X_2)) = P(X_1 < a, a < X_2) = P(X_1 < a, X_2 > a). \tag{1}$$

The probability on the right is the probability that the random variable X_1 will be less than the number a *and* the random variable X_2 will be greater than the number a. This is just a random variable computation of the type we have done many times in the course already. Technically we should think of the formula (1) as the *definition* of the probability that a will be inside I but this is a technical point - it is the only reasonable definition.

Warning The probability $P(X_1 < a, X_2 > a)$ in equation (1) is almost never equal to the product probability $P(X_1 < a) \cdot P(X_2 > a)$ because X_1 and X_2 are almost never independent. For example in the above problem $X_1 = Z - 1$ and $X_2 = Z + 1$ so $X_2 = X_1 + 2$ so X_1 and X_2 are perfectly correlated and in particular not independent.

We will conclude with an example of how to compute such probabilities.

Problem Suppose that Z has standard normal distribution. Compute $P(0 \in (Z - 1, Z + 1))$.

Solution According to the equation (1) we have

$$P(0 \in (Z-1, Z+1)) = P(Z - 1 < 0, 0 < Z + 1).$$

But

$$P(Z - 1 < 0, 0 < Z + 1) = P(Z < 1, -1 < Z) = P(-1 < Z < 1) = P(-1 \leq Z \leq 1).$$

By the "handy formula" we have

$$P(-1 \leq Z \leq 1) = 2\Phi(1) - 1 = 2(.8413) - 1 = .6826$$

3 In which we go completely random

In the first part of the course we were given a random variable X and we computed probabilities like $P(a \leq X \leq b)$. But we have an equality of events

$$(a \leq X \leq b) = (X \in (a, b))$$

so

$$P(a \leq X \leq b) = P(X \in (a, b)).$$

In other words we were computing the probability that a *random variable* was in an *ordinary interval*. We have just learned how to compute the probability that a fixed *real number* is in a *random interval* e.g. $P(0 \in (Z - 1, Z + 1))$. It remains to "go completely random" and learn how to compute the probability that a *random variable* is in a *random interval*. Actually we can already do this. Let's do an example.

Problem. *Suppose Z has standard normal distribution. Compute $P(2Z \in (Z-1, Z+1))$.*

Solution. *We have*

$$P(2Z \in (Z-1, Z+1) = P(Z-1 \leq 2Z,\ 2Z \leq Z+1) = P(-1 \leq Z,\ Z \leq 1)$$
$$= P(-1 \leq Z \leq 1) = 2\Phi(1) - 1 = .6826.$$

Remark. *In the above we had to do a little manipulation of inequalities, namely $Z - 1 \leq 2Z \iff -1 \leq Z$ (subtract Z from each side or "bring the Z from the left-hand side to the right-hand side") and $2Z \leq Z+1 \iff Z \leq 1$ (again subtract Z from each side or bring the Z from the right-hand side to the left-hand side").*

4 The definition of a confidence (random) interval

Suppose now that $X_1, X_2, \ldots, X_n$ is a random sample from a population whose probability mass function (or probability density function) depends on an unknown parameter θ. Let α be a real number between 0 and 1. Then a $100(1-\alpha)\%$ confidence interval for the unknown parameter α is a *random interval* $I = (W_1, W_2)$ where $W_1 = h(X_1, X_2, \ldots, X_n)$ and $W_2 = g(X_1, X_2, \ldots, X_n)$ are *statistics* such that

$$P(\theta \in (W_1, W_2)) = 1 - \alpha. \tag{2}$$

If we hadn't given the definition in Equation (1) we wouldn't have been able to make the correct definition in Equation (2). If we have an actual sample $x_1, x_2, \ldots, x_n$ then we plug $x_1, x_2, \ldots, x_n$ into the functions h and g to get numbers w_1 and w_2 and an ordinary interval (w_1, w_2). This ordinary interval is the *observed value* of the confidence interval $I = (W_1, W_2)$ on the sample $x_1, x_2, \ldots, x_n$. This actual interval is also called a confidence interval for θ. It is important to keep the difference between the confidence random interval and its obseved value on a sample firmly in mind.

The rest of the course will be concerned with finding formulas for confidence intervals in various situations - e.g. a 90% confidence interval for the mean in a normal distribution. In each case we will verify that the equation (2) is satisfied. It is imperative that you all learn how to do these verifications - these will be "good citizen" problems.