Lecture 4 : Conditional Probability and Bayes' Theorem

The conditional sample space Motivating examples

1. Roll a fair die once

$$S = \boxed{\begin{matrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{matrix}}$$

$$\begin{aligned} \text{Let} \quad A &= \text{6 appears} \\ B &= \text{an even number appears} \end{aligned}$$

So

$$P(A) = \frac{1}{6}$$

$$P(B) = \frac{1}{2}$$

Now what about

$$P\left(\begin{array}{c}\text{6 appears } \textit{given} \text{ an even} \\ \text{number appears}\end{array}\right)$$

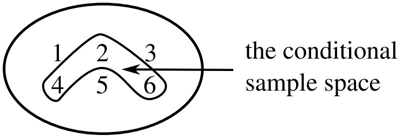
Philosophical Remark

(Ignore this remark unless you intend to be a scientist)
At present the above probability does not have a formal mathematical definition but *we can still compute it*. Soon we will give the formal definition and our computation "will be justified". This is the mysterious way mathematics works. Somehow there is a deeper reality underlying the formal theory.

Back to Stat 400

The above probability will be written written $P(A|B)$ to he read $P(A \text{ given } B)$.

Now we know an even number occurred so the sample space changes



So there are only 3 possible outcomes **given an even number occurred** so

$$P(6 \text{ given an even number occurred}) = \frac{1}{3}$$

The new sample space is called the *conditional sample space*.

2. Very Important example

Suppose you deal two cards (in the usual way without replacement). What is $P(\heartsuit\heartsuit)$ i.e., P(two hearts in a row).

Well, $P(\text{first heart}) = \dfrac{13}{52}$.

Now *what about the second heart*?

Many of you will come up with $^{12}/_{51}$ and

$$P(\heartsuit\heartsuit) = (^{13}/_{52})(^{12}/_{51})$$

There are *TWO* theoretical points hidden in the formula.
Let's first look at

$$P(\underbrace{\heartsuit \text{ on } 2^{\text{nd}}}_{\text{this isn't really correct}}) = {}^{12}/_{51}$$

What we really computed was the *conditional probability*

$$P(\heartsuit \text{ on } 2^{\text{nd}} \text{ deal} \mid \heartsuit \text{ on first deal}) = {}^{12}/_{51}$$

Why? Given we got a heart on the first deal the conditional sample space is the "new deck" with 51 cards and 12 hearts so we get

$$P(\heartsuit \text{ on } 2^{\text{nd}} \mid \heartsuit \text{ on } 1^{\text{st}}) = {}^{12}/_{51}$$

The second theoretical point we used was that in the following formula we **multiplied** the two probablities $P(\heartsuit \text{ on } 1^{\text{st}})$ and $P(\heartsuit \text{ on } 2^{\text{nd}} \mid \heartsuit \text{ on } 1^{\text{st}})$ together. This is a special case of a the formula for the probability of the **intersection** of two events that we will state below.

$$P(\heartsuit\heartsuit) = P(\heartsuit \text{ on } 1^{\text{st}})P(\heartsuit \text{ on } 2^{\text{nd}} \mid \heartsuit \text{ on } 1^{\text{st}})$$
$$= \binom{13}{52}\binom{12}{51}$$

The general formula, the multiplicative formula, we will give as a definition shortly is

$$P(A \cap B) = P(A)PB|A)$$

Compare this to the additive formula which we already proved

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Three Basic Questions

These three examples will occur repeatedly in today's lecture. The first is the example we just discussed, the second is the reverse of what we just discussed and the third is a tricky variant of finding the probability of a heart on the first with no other information.

1. What is

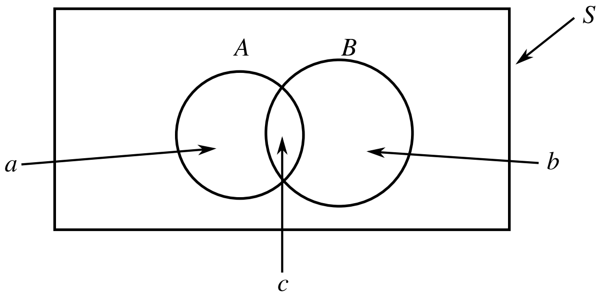
$$P(\underbrace{\heartsuit \text{ on } 2^{\text{nd}} \mid \heartsuit \text{ on } 1^{\text{st}}}_{\text{reverse of pg. 5}})$$

2. What is

$$P(\underbrace{\heartsuit \text{ on } 1^{\text{st}} \mid \heartsuit \text{ on } 2^{\text{nd}}}_{\text{reverse of pg. 5}})$$

3. What is $P(\heartsuit \text{ on } 2^{\text{nd}}$ with no information on what happened on the $1^{\text{st}})$.

The Formal Mathematical Theory of Conditional Probability



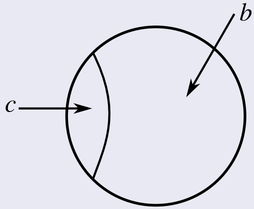
$$\sharp(S) = n,\ \sharp(A) = a,\ \sharp(B) = b,\ \sharp(A \cap B) = C$$

Problem

Let S be a finite set with the equally - likely probability measure and A and B be events with coordinalities shown in the picture.

Problem

Compute $P(A|B)$.
We are given B occurs so the conditional sample space is B



Only part of A is allowed since we know B occurred namely the part $A \cap B$. So counting elements we get

$$P(A|B) = \frac{\sharp(A \cap B)}{\sharp(B)}$$
$$= \frac{c}{b}$$

We can rewrite this as

$$P(A|B) = \frac{c}{b} = \frac{c/n}{b/n} = \frac{P(A \cap B)}{P(B)}$$

so

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \qquad (*)$$

This formula for the equally likely probability measure leads to the following.

Formal Mathematical Definition

Let A and B be any two events in a sample space S with $P(B) \neq 0$. The conditional probability of A given B is written $P(A|B)$ and is *defined* by

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \qquad (*)$$

so, reversing the roles of A and B (so we get the formula that is in the text) if $P(A) \neq 0$ then

$$P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{P(A \cap B)}{P(A)} \qquad (**)$$

Since $A \cap B - B \cap A$.

We won't prove the next theorem but you could do it and it is useful.

Theorem

Fix B with $P(B) \neq 0$. $P(\cdot|B)$, ()so $P(A|B)$ as a function of A), satisfies the axioms (and theorems) of a probability measure - see Lecture 1.

For example

1. $P(A_1 \cup A_2|B) = P(A_1|B) + P(A_2|B) - P(A_1 \cap A_2|B)$
2. $P(A'|B) = 1 - P(A|B)$

¨ $P(A|\cdot)$ *(so $P(A|B)$ as a function of B) does not satisfy the axioms and theorems.*

The Multiplicative Rule for $P(A \cap B)$

Rewrite (**) as

$$P(A \cap B) = P(A)P(B|A)(\sharp)$$

($\sharp$) is very important, more important then (**).
It complement the formula

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Now we know how P interacts with the basic binary operations $\cup$ and $\cap$.

More generally

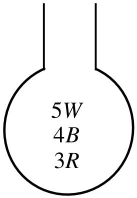
$$P(A \cap B \cap C) = P(A)P(B|A)P(C|A \cap B)$$

Exercise

Write down $P(A \cap B \cap C \cap D)$.

Traditional Example

An urn contains 5 white chips, 4 black chips and 3 red chips.
Four chips are drawn sequentially without replacement. Find $P(WRWB)$.



12 chips

$$P(WRWB) = \left(\frac{5}{12}\right)\left(\frac{3}{11}\right)\left(\frac{4}{10}\right)\left(\frac{4}{9}\right)$$

What did we do formally? The answer is we used the following formula for the intersection of four events

$$P(WRWB) = P(W) \cdot P(R|W) \cdot P(W|W \cap R) \cdot P(B|W \cap R \cap W)$$

Now we make a computation that reverses the usual order, namely, we compute

$$P(\heartsuit\text{on first}|\heartsuit\text{on second})$$

By Definition

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

so

$$P(\heartsuit\text{on first}|\heartsuit\text{on second}) = \frac{P(\heartsuit\heartsuit)}{P\left(\begin{array}{c}\heartsuit \text{ on } 2^{\text{nd}} \text{ with no} \\ \text{other information}\end{array}\right)}$$

Now we know from pg. 5.

$$P(\heartsuit\heartsuit) = (13/52)(12/51)$$

Now we need

$$P(\heartsuit \text{ on } 2^{\text{nd}} \text{ with no other information}) = 13/52$$

By Definition

We will prove this later (to some people this is intuitively clear). In fact if you write down any probability statement in this situation, take that statement and everywhere you see “first” write “second” and everywhere you see “second” write “first” then the resulting event will have the same probability as the event we started with.

So back to our problem we have

$$P(\heartsuit \text{ on } 1^{\text{st}} \mid \heartsuit \text{ on } 2^{\text{nd}}) = \frac{\cancel{(13/52)}(\cancel{12/51})}{\cancel{(13/52)}} = \frac{12}{51}$$

$$= P(\underbrace{\heartsuit \text{ on } 2^{\text{nd}} \mid \heartsuit \text{ on } 1^{\text{st}}}_{\text{pg. 5}})$$

This is another instance of the symmetry (in “first” and “second ”) stated three lines above.

Bayes' Theorem (pg. 72)

Bayes' Theorem is a truly remarkable theorem. It tells you "how to compute $P(A|B)$ if you know $P(B|A)$ and a few other things".
For example - we will get a new way to compute are favorite probability $P(\heartsuit \text{ as } 1^{\text{st}} \mid \heartsuit \text{ on } 2^{\text{nd}})$ because we know $P(\heartsuit \text{ on } 2^{\text{nd}} \mid \heartsuit \text{ on } 1^{\text{st}})$.
First we will need on preliminary result.

The Law of Total Probability

Let $A_1, A_2, \ldots, A_k$ be mutually exclusive
($A_i \cap A_j = \emptyset$) and exhaustive.
($A_1 \cup A_2 \cup \ldots \cup A_k = S =$ the whole space)
Then for any event B

$$\begin{aligned} P(B) = &P(B|A_1)P(A_1) + P(B|A_2)P(A_2) \\ &+ \cdots + P(B|A_k)P(A_k) \end{aligned} \tag{b}$$

Prove this

First prove $P(B|S) = 1$ then use the $P(B, C)$ is satisfies the additivity rule for a probability measure as function of C.

Special case $k = 2$ so we have A and A'

$$P(B) = P(B|A)P(A) + P(B|A')P(A') \tag{bb}$$

Now we can *prove*
$P(\heartsuit$ on 2^{nd} with no other information$) = {}^{13}/_{52}$

$$\text{Put} \quad B = \heartsuit \text{ on } 2^{\text{nd}}$$
$$A = \text{heart on } 1^{\text{st}}$$
$$A' = \text{a nonheart on } 1^{\text{st}}$$

Lets write $\not\heartsuit$ for nonheart.
So,

$$P(\not\heartsuit \text{ on } 1^{\text{st}}) = {}^{39}/_{52}$$
$$P(\heartsuit \text{ on } 2^{\text{nd}} / \not\heartsuit \text{ on first}) = {}^{13}/_{51}$$

Now

$$\begin{aligned} P(B) &= P(B|A)P(A) + P(B|A')P(A') \\ &= P(\heartsuit \text{ on } 2^{\text{nd}} \mid \heartsuit \text{ on } 1^{\text{st}})P(\heartsuit \text{ on } 1^{\text{st}}) \\ &\quad + P(\heartsuit \text{ on } 2^{\text{nd}} \mid \cancel{\heartsuit} \text{ on } 1^{\text{st}})P(\cancel{\heartsuit} \text{ on } 1^{\text{st}}) \\ &= (12/51)(13/52) + (13/51)(39/52) \end{aligned}$$

add fractions

$$= \frac{(12)(13) + (13)(39)}{(51)(52)}$$

factor out 13 add to get 51

$$= \frac{(13)(12+39)}{(51)(52)} = \frac{(13)(\cancel{51})}{(\cancel{51})(52)}$$

$$= (13)/(52) \qquad \text{Done!}$$

Now we can state Bayes' Theorem.

Bayes' Theorem (pg. 73)

Let $A_1, A_2, \ldots, A_k$ be a collection of n mutually exclusive and exhaustive events with $P(A_i) > 0$
$i = 1, 2, \ldots, k$. Then for any event B with $P(B) > 0$

$$P(A_j|B) = \frac{P(B|A_j)P(A_j)}{\sum\limits_{i=1}^{k} P(B|A_i)P(A_i)}$$

Again we won't give the proof.

Special Case $k = 2$

Suppose we have two events A and B with $P(A) > 0$, $P(A') > 0$ and $P(B > 0)$. Then

$$\sharp) P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A')P(A')}$$

Now we will compute (for the last time)

$$P(\heartsuit \text{ on } 1^{\text{st}} \mid \heartsuit \text{ on } 2^{\text{nd}})$$

Using Bayes' Theorem.
This is the obvious way to

do it since we know the probability "the other way around"

$$P(\heartsuit \text{ on } 2^{\text{nd}} \mid \heartsuit \text{ on } 1^{\text{st}}) = {}^{12}/_{51}$$

So let's do it.

$$\begin{aligned} \text{We put } A &= \heartsuit \text{ on } 1^{\text{st}} \\ \text{so } A' &= \cancel{\heartsuit} \text{ on } 1^{\text{st}} \\ \text{and } B &= \heartsuit \text{ on second} \end{aligned}$$

plugging into ($\sharp$) we get

$$P(\heartsuit \text{ on } 1^{\text{st}} \mid \heartsuit \text{ on } 2^{\text{nd}})$$

$$= \frac{P(\heartsuit \text{ on } 2^{\text{nd}} \mid \heartsuit \text{ on } 1^{\text{st}})P(\heartsuit \text{ on } 1^{\text{st}})}{P(\heartsuit \text{ on } 2^{\text{nd}} \mid \heartsuit \text{ on } 1^{\text{st}})P(\heartsuit \text{ on } 1^{\text{st}}) + P(\heartsuit \text{ on } 2^{\text{nd}} \mid \cancel{\heartsuit} \text{ on } 1^{\text{st}})P(\cancel{\heartsuit} \text{ on } 1^{\text{st}})}$$

$$= \frac{(^{12}/_{51})(^{13}/_{52})}{(^{12}/_{51})(^{13}/_{52}) + (^{13}/_{51})(^{39}/_{52})}$$

swap
numbers
use

$$\left(\frac{a}{b}\right)\left(\frac{c}{d}\right) = \left(\frac{c}{b}\right)\left(\frac{a}{d}\right)$$

something
from
high school

$$= \frac{(^{13}/_{51})(^{12}/_{52})}{(^{13}/_{51})(^{12}/_{52}) + (^{13}/_{51})(^{39}/_{52})}$$

factor this out

$$= \frac{(^{13}/_{51})(^{12}/_{52})}{(^{13}/_{51})\left(^{12}/_{52} + \dfrac{39}{52}\right)}$$

$$= \frac{\cancel{(^{13}/_{51})}(^{12}/_{52})}{\cancel{(^{13}/_{51})}(^{51}/_{52})} = \frac{(^{12}/\cancel{_{52}})}{(^{51}/\cancel{_{52}})}$$

$$= {}^{12}/_{51} \quad \text{once again!}$$

The algebra was hard but the approach was the most natural - a special case of *General Principle*

Compulsory Reading (for your own heath)

In case you or someone you love tests positive for a *rare* (this is the point) disease, read Example 2.31, pg. 81. Misleading (and even bad) statistics is rampant in medicine.