Lecture 5 : Independence §2.5

Definition

Two events A and B are independent if

$$P(A|B) = P(A) \qquad (\sharp)$$

otherwise they are said to be dependent.

The equation $P(A|B) = P(A)$ says that the knowledge that B has occurred does not effect the probability A will occur.

$\mathcal{Z}$ Remember $P(A|B)$ is defined only if $P(B) \neq 0$

($\sharp$) appears to be assymetric but we have
(assuming $P(A) \neq 0$ so $P(B|A)$ is defined and $P(B) \neq 0$ so $P(A|B)$ is defined)

Proposition

$P(A|B) = P(A) \Leftrightarrow P(B|A) = P(B)$

Proof.

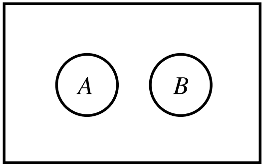
$$P(A \cap B) = P(A)P(B|A)$$
$$P(B \cap A) = P(B)P(A|B)$$

But $A \cap B = B \cap A$ (this is the point)
So
$$P(A)P(B|A) = P(B)P(A|B)$$
So
$$\frac{P(B|A)}{P(B)} = \frac{P(A|B)}{P(A)}$$
Then
$$\text{LHS } = 1 \Leftrightarrow \text{RHS } = 1$$

$\square$

The Standard Mistake

The English language can trip us up here.
Suppose A and B are *mutually exclusive* events ($A \cap B = \emptyset$) with $P(A) \neq 0$ and $P(B) \neq 0$



Are A and B independent?

NO

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(\emptyset)}{P(B)} = \frac{0}{P(B)} = 0$$

So $P(A|B) \neq P(A)$.
In this case if you know B has occurred *then* A *cannot occur* at all.
This is the *opposite* of independence.

Two Contrasting Example

1. Our favorite example

$$A = \heartsuit \text{ on } 1^{\text{st}}$$

$$B = \heartsuit \text{ on } 2^{\text{nd}}$$

$$P(\heartsuit \text{ on } 2^{\text{nd}} \mid \heartsuit \text{ on } 1^{\text{st}}) = \frac{12}{51}*$$

$P(\heartsuit \text{ on } 2^{\text{nd}}$ with no other information) = $13/52$
So $P(B|A) \neq P(B)$ So A and B are *not* independent.

2. Our very first example

Flip a fair coin twice

$$A = H \text{ on } 1^{\text{st}}$$

$$B = H \text{ on } 2^{\text{nd}}$$

$$P(H \text{ on } 2^{\text{nd}} \mid H \text{ on } 1^{\text{st}}) = \frac{1}{2} \qquad (**)$$

(prove the formula immediately above)

$$P(H \text{ on } 2^{\text{nd}}) = \frac{1}{2}$$

So $\quad P(B|A) = P(A)$

So A and B *are* independent.

Hence

$$\begin{aligned} P(A \cap B) &= P(A)P(B) \\ &= \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4} \end{aligned}$$

as we saw in Lecture 1.

Remark (don't worry about this)

Actually in some sense we decided in advance that A and B were independent. *When I give you problems you will told whether or not A and B are independent.*

When we do "real-life" problems we have to decide on a model. In this case in Example 1 it is clear that we require a model so that A and B are not independent and in Example 2 in which they *are*. So we already know the answer to the independence question before doing any mathematics. Again there is a reality beyond the mathematics.

Independence of more than two elements

Definition

The events $A_1, A_2, \ldots, A_n$ are independent if for every k and for every collection of k distinct indices $i_1, i_2, \ldots, i_k$ drawn from $1, 2, \ldots, n$ we have

$$(b) \quad (A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}) = P(A_{i_1}) \ldots P(A_{1_k})$$

Z *So in particular $(k = n)$ we have*

$$(\sharp) \quad P(A_1 \cap A_2 \cap \cdots \cap A_n) = P(A_1)P(A_2) \ldots P(A_n)$$

however there are examples where $(\sharp)$ holds but (b) fails for some $k < n$ so the events are not independent.

Example $n = 3$

Special case of the definition

Three events A, B, C are independent if

$$(\sharp) \quad P(A \cap B \cap C) = P(A)P(B)P(C)$$

and

(b_1) $P(A \cap B) = P(A)P(B)$

(b_2) $P(A \cap C) = P(A)P(C)$

(b_3) $P(B \cap C) = P(B)P(C)$

Z To specialize what I said before there are example where $(\sharp)$ holds but one of the (b)'s foils so $(\sharp)$ *does not* imply independence.

Now we can easily do the problem from Lecture 1.
P(Exactly one head in 100 flips)
Technically we write

$$A_i = H \text{ on } i\text{-th flip}$$

So we want

$$P(A_1 \cap A_2 \cap \ldots \cap A_{100})$$

by independence

$$= \underbrace{P(A_1)P(A_2)\ldots P(A_{100})}_{100}$$

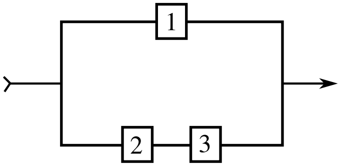
$$= \underbrace{\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\cdots\left(\frac{1}{2}\right)}_{100}$$

It is move efficient to.

One of my favorite types of problems (they of to turn up on my tests)

(see Example 2.35. pg. 79 and problems 80 and 87. pg. 81)

System/Component Problems



Consider the following system S. Suppose each of the three components has probability p of working. Suppose all components function independently. What is the probability the system will work i.e. an input signal on the left will come out on the right.

Solution

It is important that you follow the format below - don't skip steps. Skipping steps is fatal in mathematics (as in almost everything).

1. Define events

S = System works.
A_i = i-th component works $i = 1, 2, 3$.

2. (The hard part)

Express the set S in terms of the sets A_1, A_2, A_3 using the geometry of the system.

$$S = A_1 \cup (A_2 \cap A_3)$$

The signal gets through $\Leftrightarrow$ A_1 works *or* (both A_2 *and* A_3 work) through.

3. Use how P interacts with $\cup$ and $\cap$ independence.

$$P(S) = P(A_1 \cup (A_2 \cap A_3))$$

$\cup$ rule

$$= P(A_1) + P(A_2 \cap A_3) - P(A_1 \cap (A_2 \cap A_3)$$

independence

$$= P(A_1) + P(A_2)P(A_3) - P(A_1)P(A_2)P(A_3)$$
$$= p + p^2 - p^3$$

In a harder problem it is use to group some of the components together in a "block" - For example in this problem we could have grouped A_2 and A_3 into C so then

$$S = A_1 \cup C \qquad \text{etc.}$$

you should do a lot of these.

When you form the blocks, the blocks will be independent as long as *on two blocks have a common component*. So in the example choose A_1 and C are still independent.