MATH 141, Integrating the Secant: An Application of Partial Fractions

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One of the most mysterious integration formulas is the formula for the integral of the secant function: $\int \sec x \, dx = \ln(\sec x + \tan x) + C$ (for $|x| < \frac{\pi}{2}$). The standard derivation of this, found in Ellis and Gulick section 5.7, seems to come out of nowhere. Another approach uses partial fractions.

$$\begin{aligned}
\int \sec x \, dx &= \int \frac{dx}{\cos x} = \int \frac{\cos x \, dx}{\cos^2 x} \\
&\text{(substitute } u = \sin x,\ \cos^2 x = 1 - \sin^2 x = 1 - u^2,\ du = \cos x \, dx) \\
&= \int \frac{du}{1 - u^2} \\
&= \int \frac{1}{2} \left(\frac{du}{1 - u} + \frac{du}{1 + u} \right) \\
&= \frac{1}{2} \left(-\ln(1 - u) + \ln(1 + u) \right) + C \\
&= \frac{1}{2} \ln \left(\frac{1 + \sin x}{1 - \sin x} \right) + C \\
&= \frac{1}{2} \ln \left(\frac{(1 + \sin x)^2}{(1 + \sin x)(1 - \sin x)} \right) + C \\
&= \frac{1}{2} \ln \left(\frac{(1 + \sin x)^2}{1 - \sin^2 x} \right) + C \\
&= \frac{1}{2} \ln \left(\frac{(1 + \sin x)^2}{\cos^2 x} \right) + C \\
&= \ln \left(\frac{1 + \sin x}{\cos x} \right) + C \\
&= \ln\bigl(\sec x + \tan x\bigr) + C.
\end{aligned}$$