Mathematics 241
First Exam Solutions
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1. a) Show that the points $P_1 = (-1,-2,6)$, $P_2 = (0,4,1)$, $P_3 = (1,0,1)$ determine a unique plane $\mathcal{P}$, and determine the equation of $\mathcal{P}$.
Solution: To show they determine a unique plane, it's enough to show that the line segments $\overline{P_1P_2}$ and $\overline{P_1P_3}$ are not parallel. But

$$\begin{aligned}(P_2 - P_1) \times (P_3 - P_1) &= P_2 \times P_3 - P_1 \times P_3 + P_2 \times P_1 \\ &= (4,1,-4) + (-26,2,-4) - (2,7,2) \\ &= (-20,-5,-10) = -5(4,1,2) \neq (0,0,0).\end{aligned}$$

So the points are not collinear and $(4,1,2)$ is perpendicular to $\mathcal{P}$. Then the equation of $\mathcal{P}$ is

$$(4,1,2) \cdot (x,y,z) = (4,1,2) \cdot (-1,-2,6)$$

or

$$4x + y + 2z = 6.$$

b) Find the area of the triangle with vertices P_1, P_2, and P_3.
Solution:

$$\frac{1}{2}\left\|(P_2 - P_1) \times (P_3 - P_1)\right\| = \frac{5}{2}\left\|(4,1,2)\right\| = \frac{5}{2}\sqrt{21}.$$

2. (10 points) Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be the planes with equations

$$3x - 4y + z = 2, \qquad 3x + 2y - z = 7.$$

Find symmetric equations of the line $\mathcal{L}_1$ where $\mathcal{P}_1$ and $\mathcal{P}_2$ intersect. You may use the fact that $(3,-4,1) \times (3,2,-1) = (2,6,18)$.
Solution: $\mathcal{L}_1$ is perpendicular to the normals to both $\mathcal{P}_1$ and $\mathcal{P}_2$, hence is parallel to

$$(3,-4,1) \times (3,2,-1) = (2,6,18) = 2(1,3,9).$$

So we want symmetric equations of a line in the direction of $(1,3,9)$ and just need one point on the line. If we let $x = 0$ and solve the simultaneous equations for y and z, we find that $\left(0, -\frac{9}{2}, -16\right)$ lies on $\mathcal{L}_1$. So we obtain the symmetric equations

$$\frac{x}{1} = \frac{y + \frac{9}{2}}{3} = \frac{z + 16}{9}.$$

3. (20 points) a) Find a vector form of the equation of the line $\mathcal{L}_2$ through $P_4 = (7,0,1)$ perpendicular to the plane $\mathcal{P}_3$ with equation $x + y + z = 11$.

Solution: $\mathcal{L}_2$ is parallel to a normal to $\mathcal{P}_3$, hence the desired vector form is

$$(x, y, z) = (7, 0, 1) + t(1, 1, 1).$$

b) Determine the *closest point* to P_4 on $\mathcal{P}_3$.
Solution: This is simply the point where $\mathcal{L}_2$ intersects $\mathcal{P}_3$. So we solve

$$(7 + t) + (0 + t) + (1 + t) = 11$$

and find $3t + 8 = 11$, or $3t = 3$, so $t = 1$ and the point is $(7, 0, 1) + 1(1, 1, 1) = (8, 1, 2)$.

4. (40 points) Answer the following questions about the curve with parameterization $\mathbf{r}(t) = (3\cos t, 3\sin t + \cos(t/2))$.
a) The curve is *periodic*; that is, there is a positive value of T such that $\mathbf{r}(t) = \mathbf{r}(t+T)$ for all t. What is the smallest period, i.e., the smallest value of T for which this holds?
Solution: Since the smallest period of the cosine function is 2π, the answer must be a multiple of 2π (because of the first coordinate). But 2π is not a period of the second coordinate because of the $\cos(t/2)$ term, so the answer is $T = 4\pi$.
b) The curve has two *nodes*, i.e., self-intersection points. Find their exact coordinates (using the MATLAB session).
Solution: From the picture and reflection symmetry across the x-axis, both self-intersection points must lie on the x-axis, hence must be obtained by setting the second coordinate of the curve equal to 0. From the MATLAB session, we get a solution when $t = \pm\pi$, yielding the point $(-3, 0)$. The other solution is when t takes the other two funny values, and this point is at $t = -2\arctan\frac{1}{\sqrt{35}}$, or at

$$(3\cos\left(-2\arctan\frac{1}{\sqrt{35}}\right), 0) \approx (2.833, 0).$$

c) What are the approximate values of the minimum and maximum radius of curvature, and roughly where on the curve are the points where the minimum and maximum curvature occur?
Solution: From the graph of the curvature, the maximum curvature occurs around $t = 3\pi$ and the minimum curvature occurs around $t = \pi$. This makes sense, since $t = \pi$ and $t = 3\pi$ both correspond to the self-intersection point $(-3, 0)$, where one can see from the picture that there is a strongly curved inner lobe and a weakly curved outer lobe. This also matches our intuition that the maxima and minima of the curvature should be symmetric around the x-axis (since the curve has reflection symmetry around the x-axis). The *minimum* radius of curvature is around $\frac{1}{0.48} \approx 2$ and the *minimum* radius of curvature is thus around $\frac{1}{0.24} \approx 4$; both occur at the self-intersection point $(-3, 0)$.
d) Does the curve have any inflection points? If so, where are they located? If not, explain why not.
Solution: There are no inflection points, since from the graph of the curvature, the curvature never vanishes.