Mathematics 241
Second Exam
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Friday, April 4, 2003

Instructions. Answer each question **on a separate answer sheet**. *Be sure your name, section number, and problem number are on each answer sheet.* The point value of each problem is indicated. The exam is worth a total of 100 points. In problems with multiple parts, whether the parts are related or not, the parts are graded independently of one another. Be sure to go on to subsequent parts even if there is some part you cannot do. Please leave answers such as $5\sqrt{2}$ in terms of radicals and **do not convert to decimals**.
You are allowed use of a non-programmable calculator and one sheet of notes.

1. (20 points) Find the equation of the tangent plane to the surface $z = \sin(x)e^{-y} + e^{-2y}$ at the point $(0, 0, 1)$.

2. (30 points) Suppose f is a smooth function of x and y and

$$\frac{\partial f}{\partial x} = 3, \qquad \frac{\partial f}{\partial y} = -1$$

when $x = 2$, $y = 0$.
a) What is the maximum value of the directional derivative $D_{\mathbf{u}}f$ at $(2, 0)$ (with $\mathbf{u}$ allowed to vary over all unit vectors), and what is the value of $\mathbf{u}$ for which it is achieved?
b) If you switch into polar coordinates, what are

$$\frac{\partial f}{\partial r}, \qquad \frac{\partial f}{\partial \theta}$$

at $r = 2$, $\theta = 0$? Explain.

3. (50 points) Answer the following questions about the function f of three variables defined by

$$f(x, y, z) = x^4 + 2xy + 4xz + y^2 + z^2.$$

Refer to the MATLAB session on the other sheet.
a) Give the values of x, y, and z at each critical point of f.
b) Classify each of the critical points from (a) as a local maximum, local minimum, saddle point, or degenerate critical point. (Use the MATLAB calculations and the second derivative test.)
c) **Write down explicitly, but do not solve**, the four Lagrange multiplier equations (in variables x, y, z, and λ) for the *constrained* critical points of f on the level surface $g(x, y, z) = x^2+y^2+z^2 = 1$. (These equations are solved by MATLAB in the MATLAB session.)
d) What are the maximum and minimum values of $f(x, y, z)$ on the ball where $x^2 + y^2 + z^2 \leq 1$, and where are the maximum and minimum attained? Explain your answer.

```
>> syms x y z lam
>> f = x^4 + 2*x*y + 4*x*z + y^2 + z^2; pretty(f)

      4                   2    2
     x  + 2 x y + 4 x z + y  + z
>> gradf=jacobian(f,[x,y,z])

gradf =

[ 4*x^3+2*y+4*z,       2*x+2*y,       4*x+2*z]

>> [xc yc zc]=solve(gradf(1), gradf(2), gradf(3)); [xc yc zc]

ans =

[             0,             0,             0]
[  1/2*10^(1/2), -1/2*10^(1/2),     -10^(1/2)]
[ -1/2*10^(1/2),  1/2*10^(1/2),      10^(1/2)]

>> hessf=jacobian(gradf,[x,y,z]), eigs=eig(hessf);

hessf =

[ 12*x^2,      2,      4]
[      2,      2,      0]
[      4,      0,      2]

>> for j=1:3, double(subs(eigs, [x,y,z], [xc(j) yc(j) zc(j)])), end

ans =

2.0000
5.5826
-3.5826
ans =

2.0000
30.6969
1.3031
ans =

2.0000
30.6969
1.3031

>> g = x^2 + y^2 + z^2; gradg = jacobian(g, [x y z]);
```
(over)

```
>> for j=1:3, double(subs(f, [x,y,z], [xc(j) yc(j) zc(j)])), end

ans =
0
ans =
-6.2500
ans =
-6.2500

>> [lamc xbc ybc zbc]=solve(gradf(1)-lam*gradg(1),gradf(2)-lam*gradg(2),...
gradf(3)-lam*gradg(3),g-1); double([xbc ybc zbc])

ans =

0             0.8944            -0.4472
0            -0.8944             0.4472
1.1696 + 0.9317i   0.5231 - 0.4167i   1.0461 - 0.8333i
-1.1696 - 0.9317i  -0.5231 + 0.4167i  -1.0461 + 0.8333i
1.1696 - 0.9317i   0.5231 + 0.4167i   1.0461 + 0.8333i
-1.1696 + 0.9317i  -0.5231 - 0.4167i  -1.0461 - 0.8333i
0.7071             0.3162             0.6325
-0.7071            -0.3162            -0.6325
-0.7071             0.3162             0.6325
0.7071            -0.3162            -0.6325

>> for j=1:10, double(subs(f, [x,y,z], [xbc(j) ybc(j) zbc(j)])), end

ans =
1
ans =
1
ans =
6.0000 - 0.0000i
ans =
6.0000 - 0.0000i
ans =
6.0000 + 0.0000i
ans =
6.0000 + 0.0000i
ans =
2.9861
ans =
2.9861
ans =
-1.4861
ans =
-1.4861
```