Mathematics 241
Second Exam Solutions
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1. (20 points) Find the equation of the tangent plane to the surface $z = \sin(x)e^{-y} + e^{-2y}$ at the point $(0, 0, 1)$.
Solution: Write the surface as $g(x, y, z) = \sin(x)e^{-y} + e^{-2y} - z = 0$. Thus the tangent plane is perpendicular to

$$\nabla g = (\cos(x)e^{-y}, -\sin(x)e^{-y} - 2e^{-2y}, -1)$$

evaluated at $(0, 0, 1)$, or $(1, -2, -1)$. Hence the tangent plane is $(1, -2, -1) \cdot (x, y, z) = (1, -2, -1) \cdot (0, 0, 1)$, or $x - 2y - z = -1$.

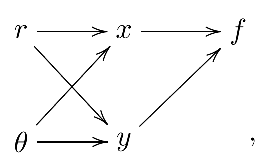
2. (30 points) Suppose f is a smooth function of x and y and

$$\frac{\partial f}{\partial x} = 3, \qquad \frac{\partial f}{\partial y} = -1$$

when $x = 2$, $y = 0$.
a) What is the maximum value of the directional derivative $D_{\mathbf{u}}f$ at $(2, 0)$ (with $\mathbf{u}$ allowed to vary over all unit vectors), and what is the value of $\mathbf{u}$ for which it is achieved?
Solution: The maximal directional derivative is the norm of the gradient vector, or $\|(3, -1)\| = \sqrt{3^2 + (-1)^2} = \sqrt{10}$. It occurs when $\mathbf{u}$ is a unit vector in the direction of the gradient, or $\mathbf{u} = \frac{1}{\sqrt{10}}(3, -1)$.
b) If you switch into polar coordinates, what are

$$\frac{\partial f}{\partial r}, \qquad \frac{\partial f}{\partial \theta}$$

at $r = 2$, $\theta = 0$? Explain.
Solution: By the chain rule, based on the diagram



we have

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r}$$

and

$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta}.$$

Since $x = r\cos\theta$ and $y = r\sin\theta$,

$$\frac{\partial x}{\partial r} = \cos\theta = \frac{x}{r},$$
$$\frac{\partial x}{\partial \theta} = -r\sin\theta = -y,$$

and similarly

$$\frac{\partial y}{\partial r} = \sin\theta = \frac{y}{r},$$
$$\frac{\partial y}{\partial \theta} = r\cos\theta = x.$$

So at $(2,0)$, where $r=2$, $\theta=0$, we have

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} = 3\cdot 1 + (-1)\cdot 0 = 3,$$

and

$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} = 3\cdot 0 + (-1)\cdot 2 = -2.$$

3. (50 points) Answer the following questions about the function f of three variables defined by

$$f(x,y,z) = x^4 + 2xy + 4xz + y^2 + z^2.$$

Refer to the MATLAB session.

a) Give the values of x, y, and z at each critical point of f.

Solution: The critical points are computed by MATLAB as $(0,0,0)$,

$$\left(\frac{1}{2}\sqrt{10}, -\frac{1}{2}\sqrt{10}, -\sqrt{10}\right), \quad \left(-\frac{1}{2}\sqrt{10}, \frac{1}{2}\sqrt{10}, \sqrt{10}\right).$$

b) Classify each of the critical points from (a) as a local maximum, local minimum, saddle point, or degenerate critical point. (Use the MATLAB calculations and the second derivative test.)

Solution: One needs to look at the signs of the eigenvalues of the Hessian. At $(0,0,0)$, the signs are mixed, so this is a saddle point. At the other two critical points, the signs are all positive, so these are local minimum points.

c) **Write down explicitly, but do not solve**, the four Lagrange multiplier equations (in variables x, y, z, and λ) for the *constrained* critical points of f on the level surface $g(x,y,z) = x^2+y^2+z^2 = 1$. (These equations are solved by MATLAB in the MATLAB session.)

Solution: The equations are $\nabla f = \lambda \nabla g$ and $g = 1$, or

$$\frac{\partial f}{\partial x} = 4x^3 + 2y + 4z = \lambda\frac{\partial g}{\partial x} = 2\lambda x,$$
$$\frac{\partial f}{\partial y} = 2x + 2y = \lambda\frac{\partial g}{\partial y} = 2\lambda y,$$
$$\frac{\partial f}{\partial z} = 4x + 2z = \lambda\frac{\partial g}{\partial z} = 2\lambda z,$$
$$g = x^2 + y^2 + z^2 = 1.$$

d) What are the maximum and minimum values of $f(x,y,z)$ on the ball where $x^2+y^2+z^2 \le 1$, and where are the maximum and minimum attained? Explain your answer.

Solution: Since f has no local maximum points, the maximum value of f must occur on the boundary surface $g = 1$. The minimum value must also occur on the boundary, since the local minima of f do not satisfy the constraint:

$$(\frac{1}{2}\sqrt{10})^2 + (\frac{1}{2}\sqrt{10})^2 + (\sqrt{10})^2 = (3/2)\cdot(10) = 15 > 1.$$

There are many constrained critical points (solutions of the Lagrange multiplier equations), but only 6 of these are real. From the MATLAB session, we see that the maximum value of f (subject to the constraint) is 2.9861 at $\pm(0.7071, 0.3162, 0.6325)$, and the minimum value is -1.4861 at $\pm(-0.7071, 0.3162, 0.6325)$.