Mathematics 241
Third Exam Solutions
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1. (25 points) Evaluate by reversing the order of integration:

$$\int_0^1 \int_0^{\tan^{-1}(x)} \cos y \, dy \, dx.$$

Solution: The integration is over the region where $0 \le x \le 1$, $0 \le y \le \tan^{-1}(x)$. The last inequality is the same as saying $x \ge \tan y$. And $\tan^{-1}(1) = \pi/4$. So the integral becomes

$$\int_0^{\pi/4} \int_{\tan y}^1 \cos y \, dx \, dy = \int_0^{\pi/4} (1 - \tan y) \cos y \, dy = \int_0^{\pi/4} (\cos y - \sin y) \, dy,$$

which becomes

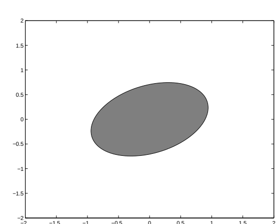
$$\Big[\sin y + \cos y\Big]_0^{\pi/4} = \sqrt{2} - 1.$$

2. (20 points) **Set up** (with explicit limits of integration), but **do not evaluate**, a multiple integral for the volume of the solid inside the hyperboloid $x^2 + y^2 - z^2 = 1$ and between the planes $z = -1$ and $z = 1$.
Solution: The simplest method is to use cylindrical coordinates, in which the equation of the hyperboloid is $r^2 = 1 + z^2$ or $r = \sqrt{1 + z^2}$:

$$\int_{-1}^1 \int_0^{2\pi} \int_0^{\sqrt{1+z^2}} r \, dr \, d\theta \, dz.$$

3. (25 points) **Set up** (with explicit limits of integration), but **do not evaluate**, a multiple integral for the area of the region shown in the picture, bounded by the ellipse

$$(x + 2y)^2 + 4(x - y)^2 = 4.$$



Solution: Make a change of coordinates: $u = x + 2y$, $v = 2(x - y)$. Then the ellipse becomes a circle, $u^2 + v^2 = 4$. Now $u + v = 3x$ and $2u - v = 6y$, so $x = (u + v)/3$, $y = (2u - v)/6$, and the Jacobian is the absolute value of

$$\det \begin{pmatrix} 1/3 & 1/3 \\ 1/3 & -1/6 \end{pmatrix} = -\frac{1}{18} - \frac{1}{9} = -\frac{3}{18} = -\frac{1}{6}$$

or 1/6. So the area is

$$\frac{1}{6}\iint_{u^2+v^2\leq 4} du\,dv = \frac{1}{6}\int_0^{2\pi}\int_0^2 r\,dr\,d\theta = \frac{2}{3}\pi.$$

4. (30 points, 10 for (a) and 20 for (b))
(a) Show from the Divergence Theorem that if D is a solid region in $\mathbb{R}^3$ with smooth boundary Σ, and if $\mathbf{n}$ is the unit outward normal to Σ, then the volume of D is given by the flux integral

$$\iint_\Sigma z\,\mathbf{n}\cdot\mathbf{k}\,dS.$$

Solution: If $\mathbf{F} = z\mathbf{k}$, then $\operatorname{div}\mathbf{F} = \frac{\partial z}{\partial z} = 1$. So

$$\iint_\Sigma z\,\mathbf{n}\cdot\mathbf{k}\,dS = \iint_\Sigma \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_D 1\,dV = \text{vol}(D).$$

(b) Write an explicit iterated integral (with explicit limits of integration) for the flux integral of (a), in the special case where Σ is the unit sphere $x^2+y^2+z^2=1$. You need not evaluate your integral.
Solution: On the sphere, $\mathbf{n} = x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$, so $z\,\mathbf{n}\cdot\mathbf{k} = z^2$, which in spherical coordinates is $\cos^2\phi$. So the flux integral becomes

$$\int_0^{2\pi}\int_0^{\pi}\cos^2\phi\,\sin\phi\,d\phi\,d\theta.$$