

**Mathematics 241
Final Examination
Friday, May 16, 2003**

Instructions. Answer each question **on a separate answer sheet**. *Be sure your name, section number, and problem number are on each answer sheet. Also make sure to copy out and sign the honor pledge on answer sheet #1.* Show all your work. A correct answer with no work to justify it may receive little or no credit. Each problem is worth 25 points. The exam is worth a total of 200 points. In problems with multiple parts, whether the parts are related or not, the parts are graded independently of one another. Be sure to go on to subsequent parts even if there is some part you cannot do. Please leave answers such as $5\sqrt{2}$ in terms of radicals and **do not convert to decimals**.

You are allowed use of a non-programmable calculator and one sheet of notes.

1. Let $f(x, y) = 3x^4 - 8x^3 - 6x^2 + 24x + xy^2 + 3$. Find and classify (relative max, relative min, saddle, or degenerate) all the (real) critical points of f . If you wish you may use results from the following MATLAB session.

```
>> syms x y
>> f = 3*x^4 - 8*x^3 - 6*x^2 + 24*x + x*y^2 + 3;
>> [a b] = solve(diff(f,x), diff(f,y));
>> [a b 36*a.^2-48*a-12]
```

ans =

```
[          -1,           0,           72]
[           1,           0,          -24]
[           2,           0,           36]
[           0,  2*i*6^(1/2),          -12]
[           0, -2*i*6^(1/2),          -12]
```

2. Set up, but do not evaluate, a multiple integral (with explicit limits of integration and explicit integrand) for the volume of the solid region inside the sphere $x^2 + y^2 + z^2 = 4$ and outside the cylinder $x^2 + y^2 = 1$.

3. Evaluate $\iint_{\Sigma} \mathbf{F} \cdot \mathbf{n} \, dS$, where Σ is the surface bounding the cube with vertices $(1, 0, 2)$, $(4, 0, 2)$, $(1, 3, 2)$, $(1, 0, 5)$, $(4, 0, 5)$, $(1, 3, 5)$, $(4, 3, 2)$, and $(4, 3, 5)$,

$$\mathbf{F}(x, y, z) = (z - y^2)\mathbf{i} + (y + z^2)\mathbf{j} - 3z\mathbf{k},$$

and \mathbf{n} is the the outward-pointing unit normal vector to Σ .

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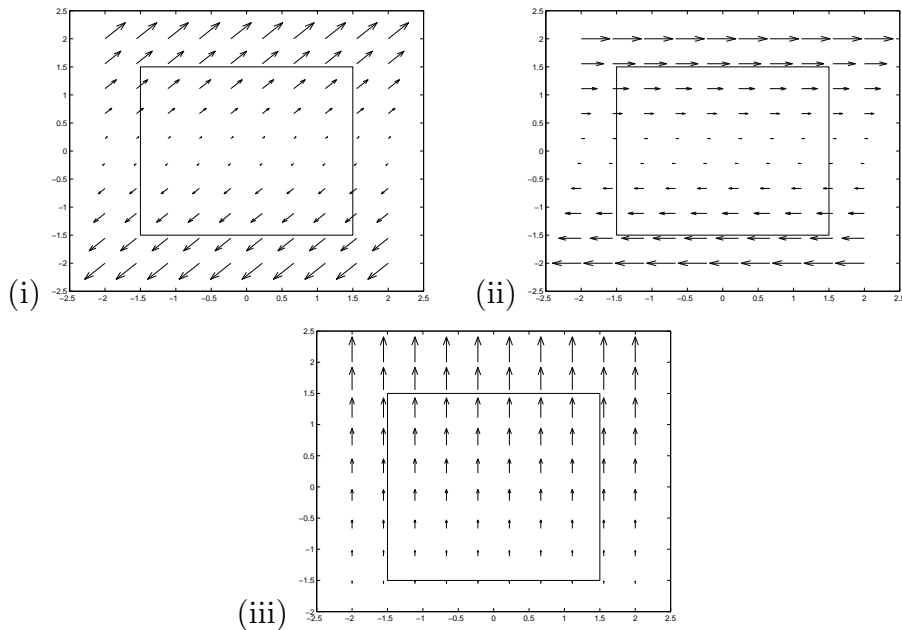
4. Set up, but do not evaluate, an integral (with explicit limits of integration and explicit integrand) for the arclength of the cardioid with the polar equation $r = \cos \theta + 1$.

5. Find a unit vector perpendicular to both of the lines

$$x = y = z \quad \text{and} \quad x + 1 = \frac{y - 2}{2} = -z,$$

and explain why your answer is unique up to multiplication by ± 1 .

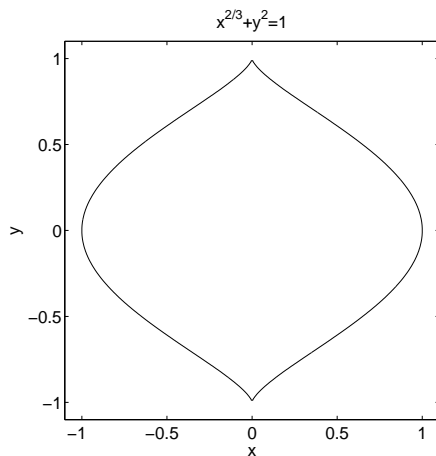
6. The following three pictures show vector fields \mathbf{F} in the plane, along with a closed (square) curve C , which is oriented counterclockwise. For at least one of the vector fields the work integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is zero. For at least one of the vector fields the flux integral $\int_C \mathbf{F} \cdot \mathbf{n} ds$ is zero, where \mathbf{n} is the outward pointing normal.



- For which of the vector fields (i), (ii) or (iii) is $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$?
- For which of the vector fields (i), (ii) or (iii) is $\int_C \mathbf{F} \cdot d\mathbf{r} > 0$?
- For which of the vector fields (i), (ii) or (iii) is $\int_C \mathbf{F} \cdot d\mathbf{r} < 0$?
- For which of the vector fields (i), (ii) or (iii) is $\int_C \mathbf{F} \cdot \mathbf{n} ds = 0$?
- For which of the vector fields (i), (ii) or (iii) is $\int_C \mathbf{F} \cdot \mathbf{n} ds > 0$?
- For which of the vector fields (i), (ii) or (iii) is $\int_C \mathbf{F} \cdot \mathbf{n} ds < 0$?

Continue next page.

7. Find the area of the region (see picture) bounded by the curve $x^{2/3} + y^2 = 1$. (Make the change of variables $x = u^3$, $y = v$ to convert to something more manageable.)



8. Let $f(x, y, z) = (x + z)^2 - x\sqrt{y}$, and let Σ be the surface with equation $f(x, y, z) = 0$.
- Find $\mathbf{grad} f$.
 - Find an equation for the tangent plane to Σ at the point $(1, 1, -2)$.
 - Find the directional derivative $D_{\mathbf{u}}f$ at the point $(1, 1, -2)$, where \mathbf{u} is the unit vector in the direction of $(1, -2, 2)$.
 - Find the largest value of $D_{\mathbf{u}}f$ at the point $(1, 1, -2)$ (as \mathbf{u} ranges over all unit vectors).