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# M-File to Introduce Volumes via Double Integrals

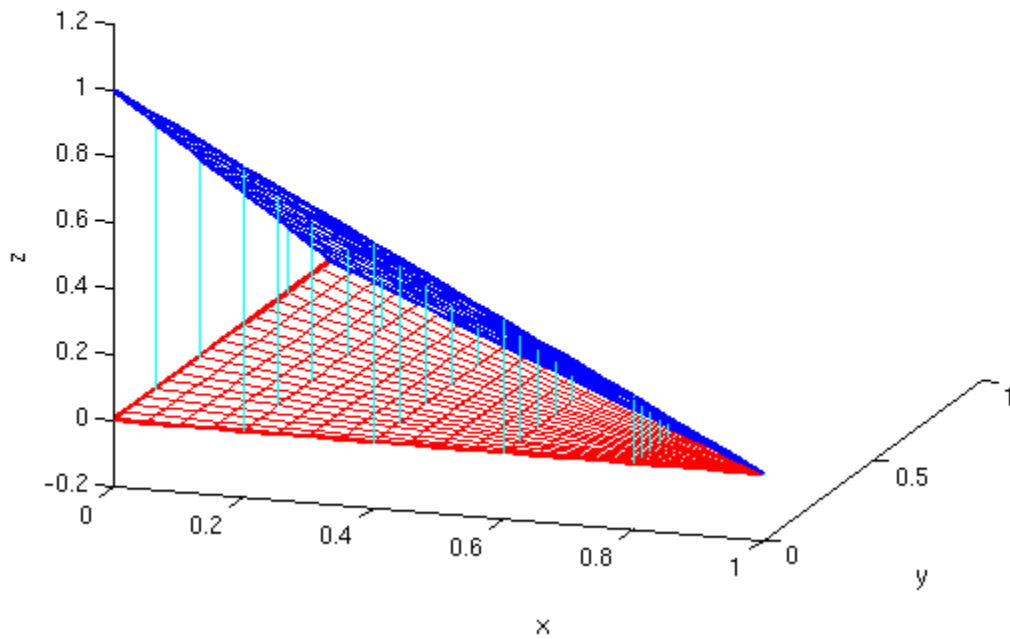
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## Pyramid Problem

A sample problem to get started: computation of the volume of a pyramid. For simplicity we cut the pyramid in quarters and just look at the part in the 1st octant. The base is an isosceles right triangle, say with sides of length 1, and we pick a value for the height, say, 1. (In general the volume clearly scales as the product of the square of the sides  $a$  of the base and the height  $h$ .) Here's a picture:

```
syms x y z real
viewSolid(z, 0+eps*x+eps*y, 1-x-y, y, eps*x, 1-x, x, 0, 1)
view(18.5, 20)
```



The iterated integral that computes the volume (in the 1st octant) is:

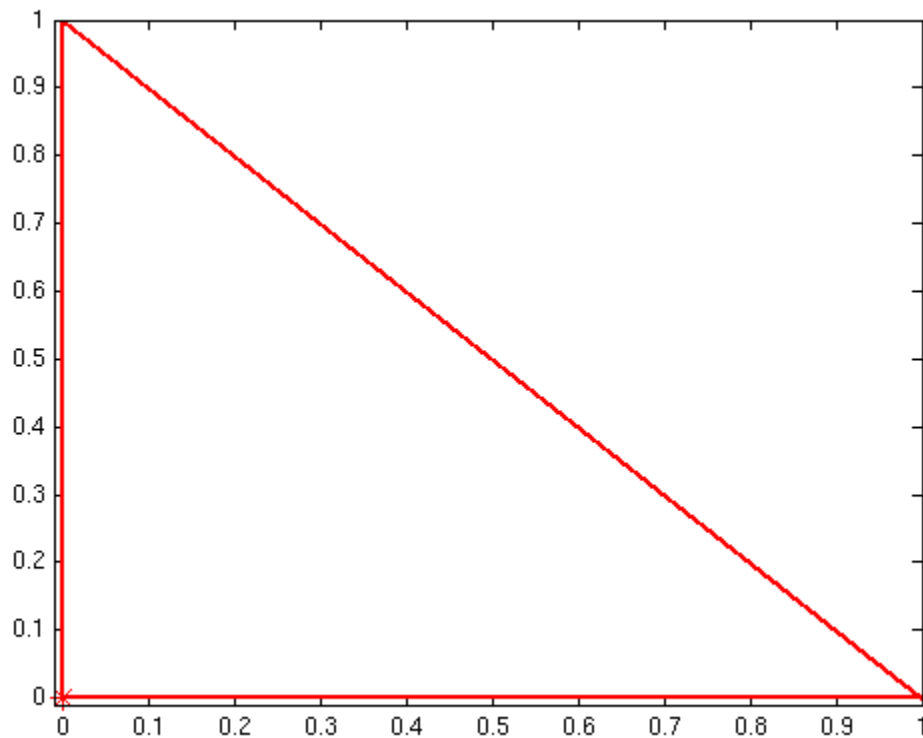
```
int(int(1-x-y, y, 0, 1-x), x, 0, 1)
```

```
ans =
```

```
1/6
```

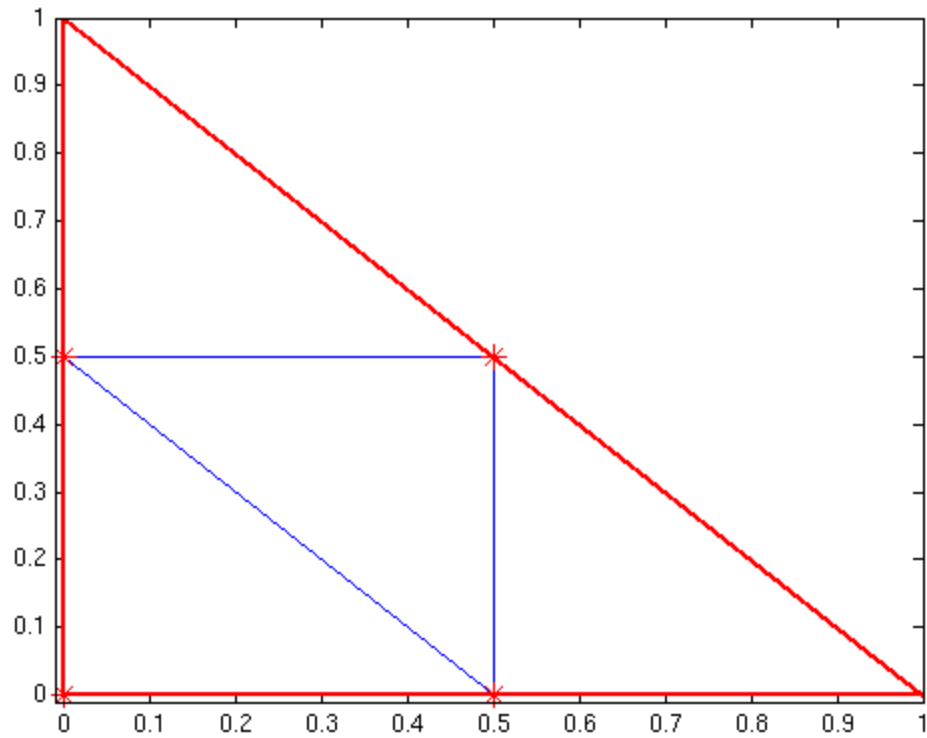
Now we calculate some Riemann sums. Here's the projection  $R$  into the  $x$ - $|y|$  plane. We've marked the vertex of the triangle where the right angle is located with a star.

```
plot([0,0], [0,1], 'r', 'LineWidth', 2), hold on  
plot([0,1], [0,0], 'r', 'LineWidth', 2)  
plot([1,0], [0,1], 'r', 'LineWidth', 2)  
plot(0, 0, 'r*', 'MarkerSize', 10)  
axis([-0.01, 1, -0.01, 1])
```

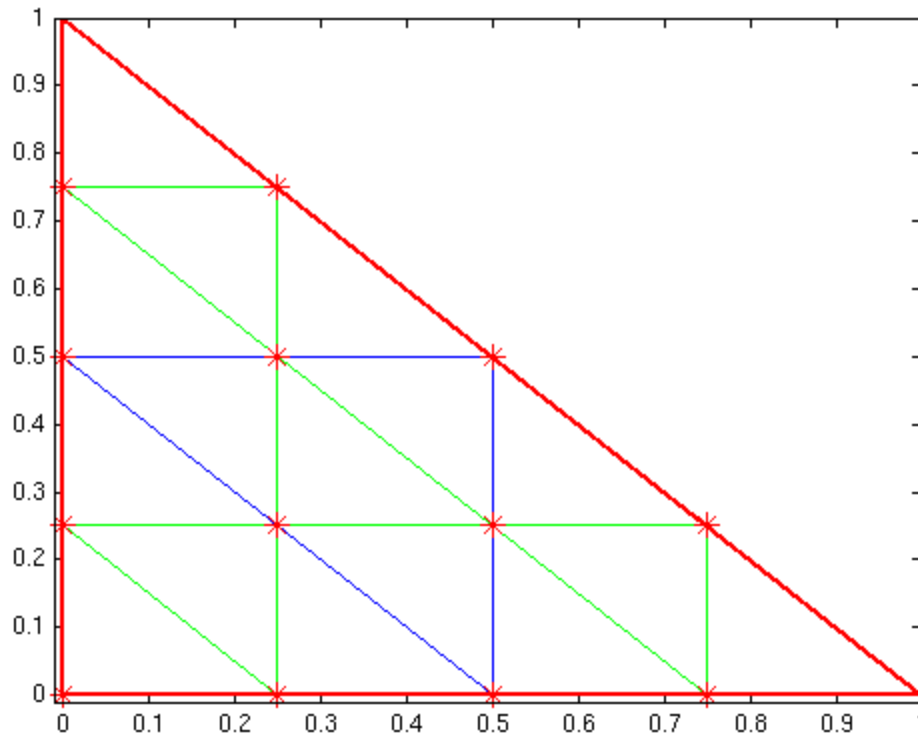


We chop up  $R$  into 4 smaller triangles, and iterate. At each step, we again mark the right-angle vertex of each triangle with a star.

```
plot([0, 0.5], [0.5, 0], 'b', 'LineWidth', 1)  
plot([0, 0.5], [0.5, 0.5], 'b', 'LineWidth', 1)  
plot([0.5, 0.5], [0.5, 0], 'b', 'LineWidth', 1)  
plot([0, 0.5, 0.5], [0.5, 0, 0.5], 'r*', 'MarkerSize', 10)
```



```
plot([0, 0.75], [0.25, 0.25], 'g', 'LineWidth', .7)
plot([0.25, 0.25], [0, 0.75], 'g', 'LineWidth', .7)
plot([0.75, 0.75], [0,.25], 'g', 'LineWidth', .7)
plot([0, 0.25], [0.75, 0.75], 'g', 'LineWidth', .7)
plot([0, 0.75], [0.75, 0], 'g', 'LineWidth', .7)
plot([0, 0.25], [0.25, 0], 'g', 'LineWidth', .7)
plot([0, 0, 0.25, 0.25, 0.25, 0.25, 0.5, 0.75, 0.75], ...
     [0.25, 0.75, 0, 0.25, 0.5, 0.75, 0.25, 0, 0.25], 'r*', 'MarkerSize',10)
```



Note that at each stage we multiply the number of pieces (triangles) by 4. Recall that the formula for one of the Riemann sums is

$$\sum_j f(x_j, y_j) \text{area}(R_j).$$

Let's choose  $(x_j, y_j)$  to be the vertex in  $R_j$  marked with a star. Since  $\text{area}(R) = \frac{1}{2}$ , the first Riemann sum is

$$1 * (1/2)$$

*ans* =

$$0.5000$$

In subsequent iterations, all the  $R_j$  have area 1/4 time what it was in the previous step, and  $f(x_j, y_j) = 1 - x_j - y_j$ . So we get in turn, after grouping together all  $R_j$  with the same value of  $f(x_j, y_j)$ :

$$(1 + 2*(1/2) + 1*0)*(1/8)$$

$$(1 + 2*(3/4) + 4*(1/2) + 6*(1/4) + 3*0)*(1/32)$$

```
ans =  
  
0.2500
```

```
ans =  
  
0.1875
```

In general, in the  $n$ -th step, there are  $2^{2*(n-2)}$  triangles each with  $\text{area}(R_j) = 2^{1-2*n}$  and we get

```
syms n j real  
vol = simplify(2^(1-2*n)*(1*1 + symsum(2*j*(1 - j*2^(1 - n)), ...  
j, 1, 2^(n - 1) - 1) + (2^(n - 1) - 1)*0))
```

```
vol =  
  
(4/2^(2*n))/3 + 1/6
```

Here is this evaluated as a function of  $n$ :

```
disp('      n      approximation')  
for j=1:10, disp([j, subs(vol, n, j)]), end
```

```
      n      approximation  
1.0000      0.5000  
  
2.0000      0.2500  
  
3.0000      0.1875  
  
4.0000      0.1719  
  
5.0000      0.1680  
  
6.0000      0.1670  
  
7.0000      0.1667  
  
8.0000      0.1667  
  
9.0000      0.1667  
  
10.0000     0.1667
```

Note the convergence to  $1/6$ .

## Ball problem

Here's another example, the volume of a spherical ball of radius 1. This projects to  $R =$  disk of radius 1 centered at the origin. The distance from the bottom to the top at a point  $(x, y)$  is  $2\sqrt{1-x^2-y^2}$ , so the volume as an iterated integral is

```
int(int(2*sqrt(1-x^2-y^2), y, -sqrt(1-x^2), sqrt(1-x^2)), x, -1, 1)
```

```
Warning: Explicit integral could not be found.
```

```
Warning: Explicit integral could not be found.
```

```
ans =
```

```
int(int(2*(- x^2 - y^2 + 1)^(1/2), y = -(1 - x^2)^(1/2)..(1 - x^2)^(1/2)),
```

Of course we know the answer is  $4\pi/3$ , but the integral is just too complicated for MATLAB to do without help. We can suggest a substitution for the inner integral.

```
syms a
```

```
int(2*sqrt(a^2-y^2), y, -a, a)
```

```
ans =
```

```
pi*a^2
```

Now substitute back  $a = \sqrt{1-x^2}$  and do the next integration:

```
int(subs(ans, a^2, 1-x^2), x, -1, 1)
```

```
ans =
```

```
(4*pi)/3
```

The numerical value of the volume is

```
double(ans)
```

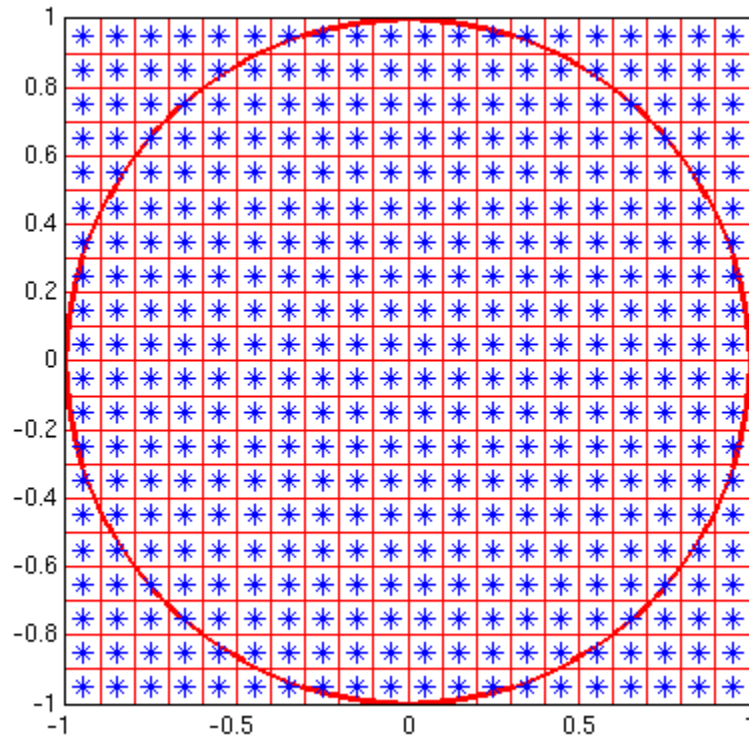
```
ans =
```

```
4.1888
```

We can try computing a Riemann sum to get an approximation. Since  $R$  is irregular in shape, we will integrate over a circumscribed square, but take the function  $f$  to be 0 outside the disk. Let's chop the square  $-1 \leq x, y \leq 1$  into 400 smaller squares each of width .1. If the center of one of the small squares is inside the disk, we multiply the area of the square by the value of the function evaluated at the center. Otherwise, we multiply by 0. Here's a picture of the grid and chosen points:

```
close
```

```
tt = 2*pi*(0:0.01:1);
plot(cos(tt), sin(tt), 'r', 'LineWidth', 2), axis equal tight, hold on
[xx, yy] = meshgrid(-1:0.1:1, -1:0.1:1);
plot(xx, yy, 'r', 'LineWidth', 1)
plot(yy, xx, 'r', 'LineWidth', 1)
[xx, yy] = meshgrid(-.95:0.1:.95, -.95:0.1:.95);
plot(xx, yy, 'b*', 'MarkerSize', 8)
```



```
clear x y
sum = 0;
for x = -.95:.1:.95,
    for y = -.95:.1:.95,
        sum = sum + 0.01*(1 >= x^2 + y^2)*2*sqrt(1 - x^2 - y^2);
    end,
end;
```

The computed value is

sum

```
sum =
    4.1982
```

Not so bad! Let's try using squares half as wide. This increases the total number of squares by a factor of 4.

```
sum = 0;
for x = -.975:.05:.975,
    for y = -.975:.05:.975,
        sum = sum + 0.0025*(1 - x^2 - y^2)*sqrt(1 - x^2 - y^2);
    end,
end;
sum
```

*sum =*

*4.1920*

a little better!

*Published with MATLAB® 7.13*