MATH 602 (Homological Algebra)
Final Assignment Solutions

Prof. Jonathan Rosenberg

May 17, 2007

1. [5 pts.] **(Complete reducibility and group cohomology)** In this problem, G is a group, k is a field, and all G-modules are assumed to be k-vector spaces (i.e., we are considering kG-modules). Recall that a G-module M is called *simple* or *irreducible* if its only G-submodules are 0 and M itself. By the Jordan-Hölder Theorem, every finite-dimensional G-module V has a filtration $0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$ with all the subquotients V_j/V_{j-1} irreducible. These *composition factors* are unique up to isomorphism and reordering. V is called *completely reducible* if V is a *direct sum* of irreducible G-modules. The aim of this problem is to prove:

   Theorem 1 *Let G be a group, k a field. Then every finite-dimensional kG-module is completely reducible if and only if $H^1(G, W) = 0$ for every finite-dimensional kG-module W.*

   To handle the "only if" direction, suppose $H^1(G, W) \neq 0$. Use the relationship between group cohomology and Ext, as well as the connection between Ext^1 and classification of extensions, to construct a finite-dimensional kG-module that is not completely reducible.

   For the other direction, suppose $H^1(G, W) = 0$ for every finite-dimensional kG-module W. Given any short exact sequence

   $$0 \to V_1 \xrightarrow{\alpha} V \xrightarrow{\beta} V_2 \to 0$$

   of finite-dimensional kG-modules with V_1 irreducible, apply $\mathrm{Hom}_k(\_\_, V_1)$ and then group cohomology (you only need H^0 and H^1). Deduce that $H^0(G, \mathrm{Hom}_k(V, V_1)) \neq 0$, and thus that there is a G-equivariant splitting to α. Then use induction.

   Solution. Suppose $H^1(G, W) \neq 0$ for some finite-dimensional kG-module W. We have $H^1(G, W) \cong \mathrm{Ext}^1_{kG}(k, W)$, which corresponds to classes of extensions of kG-modules

   $$0 \to W \to X \xrightarrow{q} k \to 0,$$

where k has the trivial G-action. We claim X is not completely reducible. For otherwise, $X = \bigoplus_i X_i$ for some simple finite-dimensional kG-modules X_i, and since q is surjective and X is the direct sum of the X_i's, q must be non-zero on some X_i. Since $q\colon X_i \to k$ is non-zero and X_i and k are both simple, q restricted to X_i must be an isomorphism. Then there is a splitting map $k \xrightarrow{\cong} X_i \hookrightarrow X$, contradicting the fact that the extension X of k by W is non-trivial in Ext.

For the other direction, suppose given a short exact sequence

$$0 \to V_1 \xrightarrow{\alpha} V \xrightarrow{\beta} V_2 \to 0$$

of finite-dimensional kG-modules with V_1 irreducible. Since k is a field, $\mathrm{Hom}_k(\_\_, V_1)$ is exact and gives a short exact sequence of finite-dimensional kG-modules

$$0 \to \mathrm{Hom}_k(V_2, V_1) \xrightarrow{\beta^*} \mathrm{Hom}_k(V, V_1) \xrightarrow{\alpha^*} \mathrm{Hom}_k(V_1, V_1) \to 0.$$

Take the corresponding long exact sequence in group cohomology. Since, by assumption, H^1 vanishes on all finite-dimensional kG-modules, this degenerates to a short exact sequence

$$0 \to \mathrm{Hom}_k(V_2, V_1)^G \xrightarrow{\beta^*} \mathrm{Hom}_k(V, V_1)^G \xrightarrow{\alpha^*} \mathrm{Hom}_k(V_1, V_1)^G \to 0.$$

In particular, the identity map $V_1 \to V_1$, which is certainly G-equivariant, has a G-equivariant lifting $V \to V_1$ under α^*. That means precisely that our original exact sequence splits in the category of kG-modules. So $V \cong V_1 \oplus V_2$. Now we argue by induction on the number of simple composition factors that all finite-dimensional kG-modules V are completely reducible. If there is only one composition factor, V is simple and this is obvious. Otherwise, V contains a simple submodule V_1 and the quotient V_2 has shorter length, hence is completely reducible by inductive hypothesis. Since we've just shown that $V \cong V_1 \oplus V_2$, that takes care of the inductive step. □

2. [4 pts.] Suppose G is finite group and the characteristic of k is either 0 or relatively prime to $|G|$. Verify the cohomology vanishing criterion in Problem 1 and deduce *Maschke's Theorem*, that every finite-dimensional kG-module is completely reducible. **Hint:** Given a 1-cocycle $f\colon G \to W$, "average" its values to get an element $w \in W$ with $f = dw$.

   Solution. Let $f\colon G \to W$ be a 1-cocycle. Let $w = \frac{1}{|G|} \sum_{g \in G} f(g)$, the "average" of the values of f. This makes sense since k is a field and $|G|$ is (by the assumption on the characteristic) non-zero in k. Then

$$\begin{aligned}
dw(h) &= h \cdot w - w \\
&= -w + \frac{1}{|G|} \sum_{g \in G} h \cdot f(g) \\
&= -w + \frac{1}{|G|} \sum_{g \in G} \bigl(f(hg) - f(h)\bigr) \\
&\quad \text{(cocycle identity)} \\
&= -w + \left(\frac{1}{|G|} \sum_{\tilde{g} \in G} f(\tilde{g}) \right) - \frac{|G|}{|G|} f(h) \\
&= -w + w - f(h) = -f(h).
\end{aligned}$$

In other words, $f = d(-w)$ and f is a coboundary. Thus the condition of (1) holds and Maschke's Theorem follows. □

3. [6 pts.] **(Grothendieck, Borel-Serre)** Let X and Y be topological spaces, $f\colon X \to Y$ a continuous map, and $\mathcal{F}$ a sheaf of abelian groups over X. Recall (Weibel, Exercise 2.6.2) that the push-forward functor f_* is a right adjoint and is therefore left exact. Its derived functors are denoted $R^j f_*$.

   (a) Show that $R^j f_* \mathcal{F}$ is the sheaf associated to the presheaf

   $$U \mapsto H^j(f^{-1}(U), \mathcal{F}).$$

   Solution. First of all, a comment about the fact that I said "presheaf" here. The gluing condition is satisfied when $j = 0$, since $\mathcal{F}$ is a sheaf and not just a presheaf. But for higher values of j, suppose $U = U_1 \cup U_2$ with U_j open. Then we have a Mayer-Vietoris sequence

   $$\begin{aligned}
   H^{j-1}(f^{-1}(U_1 \cap U_2), \mathcal{F}) &\xrightarrow{\partial} H^j(f^{-1}(U), \mathcal{F}) \\
   &\to H^j(f^{-1}(U_1), \mathcal{F}) \oplus H^j(f^{-1}(U_2), \mathcal{F}) \\
   &\qquad \to H^j(f^{-1}(U_1 \cap U_2), \mathcal{F})
   \end{aligned}$$

   The map ∂ is thus an obstruction to the gluing condition for this value of j; i.e., if $\partial \neq 0$, then a class in $H^{j-1}(f^{-1}(U), \mathcal{F})$ is not necessarily determined by its restrictions to $f^{-1}(U_1)$ and $f^{-1}(U_2)$. However, the other direction of the gluing condition is satisfied, i.e., classes in $H^{j-1}(f^{-1}(U_k), \mathcal{F})$ with the same restriction to $f^{-1}(U_1 \cap U_2)$ do come from a class over $f^{-1}(U)$.

   Now let's check the assertion. Let $(\mathcal{I}^\bullet, d)$ be an injective resolution of $\mathcal{F}$ over X. By definition, $R^q f_*(\mathcal{F})$ is the cohomology of the complex of sheaves $(f_*(\mathcal{I}^\bullet), f_* d)$ over Y, i.e.,

$$\begin{aligned} R^q f_*(\mathcal{F}) &= H^q\big(f_*(\mathcal{I}^\bullet), f_* d\big) \\ &= \ker\big(f_* d\colon\ f_*(\mathcal{I}^q) \to f_*(\mathcal{I}^{q+1})\big) \\ &\qquad / \operatorname{image}\big(f_* d\colon\ f_*(\mathcal{I}^{q-1}) \to f_*(\mathcal{I}^q)\big). \end{aligned}$$

But a quotient sheaf is, by definition, the sheafification of the quotient of the corresponding presheaves. So $R^q f_*(\mathcal{F})$ is the sheafification of the presheaf

$$\begin{aligned} U \mapsto \ker&\big(f_* d\colon\ f_*(\mathcal{I}^q)(U) \to f_*(\mathcal{I}^{q+1})(U)\big) \\ &/ \operatorname{image}\big(f_* d\colon\ f_*(\mathcal{I}^{q-1})(U) \to f_*(\mathcal{I}^q)(U)\big). \end{aligned}$$

But by the recipe for computing cohomology,

$$H^q(f^{-1}(U), \mathcal{F}) = H^q\big(\Gamma\left(f^{-1}(U), \mathcal{I}^\bullet\right), d\big).$$

By the definition of f_*, this is the same as

$$H^q\big(\Gamma\left(U, f_*\mathcal{I}^\bullet\right), f_* d\big),$$

which is precisely what we had for the presheaf defining $R^q f_*(\mathcal{F})$. $\square$

(b) Deduce from (a) that if, for a point $y \in Y$, every neighborhood of $f^{-1}(y)$ in X contains a neighborhood of the form $f^{-1}(U)$, U a neighborhood of y in Y (this condition is satisfied if, for example, X and Y are locally compact Hausdorff and f is proper), then the stalk of $R^j f_* \mathcal{F}$ at $y \in Y$ is cohomology group $H^j(f^{-1}(y), \iota^{-1}\mathcal{F})$, where $\iota\colon f^{-1}(y) \hookrightarrow X$ is the inclusion.

Solution. First note that if a sheaf $\mathcal{G}^\sharp$ is the sheafification of a presheaf $\mathcal{G}$, then the stalk of $\mathcal{G}^\sharp$ at a point y is the same as

$$\varinjlim_{y \in U} \mathcal{G}(U).$$

(See standard books on sheaf theory, e.g., Godement II.1.2.) Apply this with $\mathcal{G}^\sharp = R^j f_* \mathcal{F}$ and with $\mathcal{G}$ the presheaf of (a). That gives

$$\left(R^j f_* \mathcal{F}\right)_y = \varinjlim_{y \in U} H^j(f^{-1}(U), \mathcal{F}).$$

We want to identify this (under certain topological conditions) with $H^j(f^{-1}(y), \iota^{-1}\mathcal{F})$. Now for each open U containing y, the inclusion $f^{-1}(y) \to f^{-1}(U)$ induces a map

$$H^j(f^{-1}(U), \mathcal{F}) \to H^j(f^{-1}(y), \iota^{-1}\mathcal{F}),$$

and as U varies, these satisfy an obvious compatibility condition. So by the universal property of the colimit, we get a map

$$\left(R^j f_* \mathcal{F}\right)_y = \varinjlim_{y \in U} H^j(f^{-1}(U), \mathcal{F}) \to H^j(f^{-1}(y), \iota^{-1}\mathcal{F})$$

which under favorable circumstances should be an isomorphism.

The condition that every neighborhood of $f^{-1}(y)$ in X contains a neighborhood of the form $f^{-1}(U)$, U a neighborhood of y in Y, implies that open sets of the form $f^{-1}(U)$ are cofinal in the family of all open neighborhoods of $f^{-1}(y)$. Thus we get

$$\begin{aligned}\varinjlim_{y\in U} H^j(f^{-1}(U),\,\mathcal{F}) &= \varinjlim_{f^{-1}(y)\subseteq f^{-1}(U)} H^j(f^{-1}(U),\,\mathcal{F})\\ &= \varinjlim_{f^{-1}(y)\subseteq V} H^j(V,\,\mathcal{F}).\end{aligned}$$

This maps isomorphically to $H^j(f^{-1}(y),\,\iota^{-1}\mathcal{F})$ under mild conditions (see Godement, Théorème II.4.11.1), for example, if X is paracompact and $f^{-1}(y)$ is closed (the latter is automatic if Y is Hausdorff, or if X is metrizable). $\square$

(c) Show that there is a spectral sequence with E_2 term

$$E_2^{p,q} = H^p(Y, R^q f_*\mathcal{F})$$

converging to $H^{p+q}(X,\,\mathcal{F})$. (**Hint:** Factor the unique map $X \to \mathrm{pt}$ as $X \xrightarrow{f} Y \to \mathrm{pt}$ and use a composition-of-functors spectral sequence. Check that all the requirements for this are satisfied. You may use the result of Weibel, Exercise 2.6.3.)

Solution. As explained in the hint, let $c_X\colon\ X \to \mathrm{pt}$ and $c_Y\colon\ Y \to \mathrm{pt}$ be the "collapse" maps. Then $c_X = c_Y \circ f$. Furthermore,

$$R^j(c_X)_*(\mathcal{F}) = H^j(X,\mathcal{F}) \text{ and } R^j(c_Y)_*(\mathcal{G}) = H^j(Y,\mathcal{G}).$$

So the result will follow from the composition-of-functors spectral sequence if the conditions of Theorem 5.8.3 in Weibel are satisfied. We need to check that f_* sends injective sheaves over X to acyclic sheaves over Y. But f_* is right adjoint to f^{-1} (Weibel, Exercise 2.6.2), and f^{-1} is exact (Weibel, Exercise 2.6.6), so f_* preserves injectives (Weibel, Proposition 2.3.10), and the condition holds. $\square$

(d) Suppose the topological condition in (b) is satisfied, e.g., X and Y are locally compact Hausdorff and f is proper, and that $\mathcal{F}$ and f have the property that $H^q(f^{-1}(y),\,\iota^{-1}\mathcal{F}) = 0$ for all $q > 0$ and for all $y \in Y$. Deduce from (b) that $R^q f_*\mathcal{F} = 0$ for all $q > 0$, and then deduce from (c) that there are natural isomorphisms $H^p(X,\,\mathcal{F}) \cong H^p(Y,\,f_*\mathcal{F})$ for all p.

Solution. Apply the spectral sequence from (c),

$$H^p(Y, R^q f_*\mathcal{F}) \Rightarrow H^{p+q}(X,\,\mathcal{F}).$$

If the conditions in (b) are satisfied, then for $q > 0$, the stalk of $R^q f_*\mathcal{F}$ at y is $H^q(f^{-1}(y),\,\iota^{-1}\mathcal{F})$, which by assumption vanishes. Thus every stalk of $R^q f_* \mathcal{F}$ vanishes, and so $R^q f_* \mathcal{F}$ vanishes. Hence E_2 of the spectral sequence is concentrated on the p-axis, so all differentials must be zero, and in addition, there cannot be any extension issues (since $E_2^{p,q} = 0$ for $q > 0$). Thus the spectral sequence collapses at E_2 to give natural isomorphisms $H^p(X, \mathcal{F}) \cong H^p(Y, f_*\mathcal{F})$ for all p. $\square$

4. [5 pts.] (See Weibel Exercises 9.1.4 and 9.6.4.) Let k be a field of characteristic 0, and let R be the truncated polynomial algebra $k[x]/(x^{n+1})$. Show that R has a periodic resolution as an $R \otimes R^{\mathrm{op}}$-module, and use this to compute $HH_*(R)$. Then compute $HC_*(R)$.

   Solution. Observe that in this case $R^e = R \otimes R^{\mathrm{op}} = k[x, y]/(x^{n+1}, y^{n+1})$ has dimension $(\dim R)^2 = (n+1)^2$ over k, and the monomials $x^i y^j$ with $0 \le i,\ j \le n$ are a basis. Following in the hint in Weibel, consider the sequence of R^e-modules

$$\cdots \xrightarrow{v} R^e \xrightarrow{u} R^e \xrightarrow{v} R^e \xrightarrow{u} R^e \xrightarrow{\varepsilon} R \to 0,$$

   where ε is the ring homomorphism induced by sending both x and y to x, u is multiplication by $x - y$, and v is multiplication by $\sum_{i+j=n} x^i y^j$. Since $uv = vu = x^{n+1} - y^{n+1} = 0$ and $\varepsilon(uf(x,y)) = (x-x)f(x,x) = 0$, this is a complex, periodic to the left with period 2. Since ε is clearly split by the map sending a polynomial in x to itself (i.e., without any terms involving y), ε is surjective, and $\ker\varepsilon$ has dimension $(n+1)^2 - (n+1) = n(n+1)$ over k. Now the image of u in total degree $m = i + j + 1$ is spanned by all $x^{i+1}y^j - x^i y^{j+1}$, $0 \le i,\ j \le n$, $i + j + 1 = m$. Now we can count dimensions: the subspace of R^e of total degree m in x and y has dimension $\min(m+1, 2n-m+1)$ and the dimension of the image of u in total degree $m = i + j + 1$ has dimension

$$\min\bigl(\min(m+1, 2n-m+1),\ \min(m, 2n-m+2)\bigr) = \begin{cases} m-1, & 1 \le m \le n+1, \\ 2n-m+1, & n+1 \le m \le 2n. \end{cases}$$

   Summing up, one finds that $\dim \operatorname{image} u = \frac{n(n+1)}{2} \cdot 2 = n(n+1) = \dim\ker\varepsilon$. Thus $\dim\ker u = (n+1)^2 - n(n+1) = n+1$. However, the image of v contains

$$\sum_{i+j=n} x^{i+m} y^j = \sum_{i=0}^{n-m} x^{i+m} y^{n-i},$$

   $0 \le m \le n$, and these elements are obviously linearly independent (since they have different degrees), so $\dim \operatorname{image} v \ge n+1$. Since $\operatorname{image} v \subseteq \ker u$, we must have equality, and thus our complex is a resolution of R by free R^e-modules.

Now we can compute $HH_*(R)$. This is $\operatorname{Tor}_*^{R^e}(R, R)$, so we compute it by tensoring our resolution (over R^e) with R and taking homology. Since $R \otimes_{R^e} R^e = R$, we get the complex

$$\cdots \xrightarrow{v_*} R \xrightarrow{u_*} R \xrightarrow{v_*} R \xrightarrow{u_*} R,$$

where u_* and v_* are induced by u and v after mapping both x and y to x. So u_* is multiplication by $x - x = 0$, and v_* is multiplication by $\sum_{i+j=n} x^i x^j = (n+1)x^n$. Since we are in characteristic 0, $n+1$ is invertible, so v_* can be replaced by multiplication by x^n and our complex becomes

$$\cdots \xrightarrow{x^n} R \xrightarrow{0} R \xrightarrow{x^n} R \xrightarrow{0} R,$$

Thus we see

$$HH_m(R) = \begin{cases} R, & m = 0, \\ R/(x^n) = k[x]/(x^n), & m \geq 1 \text{ odd}, \\ \ker x^n = xR, & m \geq 2 \text{ even}. \end{cases}$$

As k-vector spaces, we thus see that $\dim HH_m = n$ for $m \geq 1$.

Finally, we compute $HC_*(R)$ from the SBI sequence. Recall that for any ring, we always have $HC_0(R) = HH_0(R)$, which coincides with R itself if R is commutative. Also, for R commutative, we know that $B\colon HC_0(R) \to HH_1(R)$ can be identified (up to sign) with the exterior differential $d\colon R \to \Omega_{R/k}$. Since the $d(x^{m+1}/(m+1)) = x^m\,dx$ span $HH_1(R)$, this map is surjective. (Here we are using characteristic zero again in order to divide by $m+1$.) From the exact sequence

$$HC_0(R) \xrightarrow{B} HH_1(R) \xrightarrow{I} HC_1(R) \to 0,$$

it follows that $HC_1(R) = 0$. We now show by induction on m that $HC_{2m-1}(R) = 0$ and $\dim HC_{2m}(R) = n+1$ for $m \geq 1$. Indeed we've show already that $HC_1(R) = 0$. Next consider the exact sequence

$$\begin{aligned} 0 = HC_1(R) \xrightarrow{B} HH_2(R) \xrightarrow{I} HC_2(R) & \\ \xrightarrow{S} HC_0(R) \xrightarrow{B} HH_1(R) & \xrightarrow{I} HC_1(R) = 0. \end{aligned}$$

Since all vector spaces are finite dimensional, the rank-nullity formula implies the alternating sum of the dimensions must vanish. So

$$\dim HH_2(R) + \dim HC_0(R) = \dim HC_2(R) + \dim HH_1(R).$$

Since we already know $\dim HH_m(R)$ is independent of m for $m \geq 1$, this gives $\dim HC_2(R) = \dim HC_0(R) = n+1$.

To show $HC_3(R) = 0$, because of the exact sequence

$$HC_2(R) \xrightarrow{B} HH_3(R) \xrightarrow{I} HC_3(R) \xrightarrow{S} HC_1(R) = 0,$$

it's enough to show that $B\colon HC_2(R) \to HH_3(R)$ is surjective. Assume for now that we've checked this. Then consider the exact sequence

$$0 = HC_3(R) \xrightarrow{B} HH_4(R) \xrightarrow{I} HC_4(R) \\ \xrightarrow{S} HC_2(R) \xrightarrow{B} HH_3(R) \xrightarrow{I} HC_3(R) = 0.$$

The same argument as before shows that

$$\dim HH_4(R) + \dim HC_2(R) = \dim HC_4(R) + \dim HH_3(R),$$

and so in this way we find $\dim HC_4(R) = \dim HC_2(R) = n+1$. The rest of the induction is exactly the same.

It remains to show that $B\colon HC_{2m}(R) \to HH_{2m+1}(R)$ is always surjective. Recall we've already proved this for $m = 0$. By the way the induction works, we can assume we've already shown that $\dim HC_{2m}(R) = \dim R = n+1$, and we also know $\dim HH_{2m+1}(R) = n$. As explained in Weibel, Exercise 9.8.2, the map B comes from the d^2 differential in the first-quadrant spectral sequence with $E^2_{p,q} = H_q(R)$, $p \geq 0$ even, and $E^2_{p,q} = 0$, p odd, arising from the cyclic homology double complex when you take first homology along the columns, then homology along the rows. This differential sends $HH_{2m}(R) \cong xR$ in the $(2, 2m)$ position to $HH_{2m+1}(R) \cong R/(x^n)$ in the $(0, 2m+1)$ position. But one can see that this map is compatible with the periodicity of HH_* of period 2 coming from the periodic resolution we constructed earlier, so it agrees with the map $\widetilde{HH}_0(R) \to HH_1(R)$ studied earlier. (Here $\widetilde{HH}_0(R)$ denotes 'reduced homology,' obtained by dividing out by the extra copy of k in HH_0 coming from the scalars.) Since this map is an isomorphism, B is surjective in all degrees.

Incidentally, there is another way to compute the cyclic homology, using $H^\lambda_\bullet$ and reducing to the case $n = 1$, but I won't give it here. $\square$