MATH 602 (Homological Algebra)
Final Assignment (in lieu of an exam)

Prof. Jonathan Rosenberg

due Wednesday, May 16, 2007

N.B.: This assignment is worth 20 points instead of 10. It is cumulative, though with more emphasis on the second half of the course.

1. [5 pts.] **(Complete reducibility and group cohomology)** In this problem, G is a group, k is a field, and all G-modules are assumed to be k-vector spaces (i.e., we are considering kG-modules). Recall that a G-module M is called *simple* or *irreducible* if its only G-submodules are 0 and M itself. By the Jordan-Hölder Theorem, every finite-dimensional G-module V has a filtration $0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$ with all the subquotients V_j/V_{j-1} irreducible. These *composition factors* are unique up to isomorphism and reordering. V is called *completely reducible* if V is a *direct sum* of irreducible G-modules. The aim of this problem is to prove:

   Theorem 1 *Let G be a group, k a field. Then every finite-dimensional kG-module is completely reducible if and only if $H^1(G, W) = 0$ for every finite-dimensional kG-module W.*

   To handle the "only if" direction, suppose $H^1(G, W) \neq 0$. Use the relationship between group cohomology and Ext, as well as the connection between Ext^1 and classification of extensions, to construct a finite-dimensional kG-module that is not completely reducible.

   For the other direction, suppose $H^1(G, W) = 0$ for every finite-dimensional kG-module W. Given any short exact sequence

   $$0 \to V_1 \xrightarrow{\alpha} V \xrightarrow{\beta} V_2 \to 0$$

   of finite-dimensional kG-modules with V_1 irreducible, apply $\mathrm{Hom}_k(\underline{\ \ }, V_1)$ and then group cohomology (you only need H^0 and H^1). Deduce that $H^0(G, \mathrm{Hom}_k(V, V_1)) \neq 0$, and thus that there is a G-equivariant splitting to α. Then use induction.

2. [4 pts.] Suppose G is finite group and the characteristic of k is either 0 or relatively prime to $|G|$. Verify the cohomology vanishing criterion in Problem 1 and deduce *Maschke's Theorem*, that every finite-dimensional kG-module is completely reducible. **Hint:** Given a 1-cocycle $f\colon G \to W$, "average" its values to get an element $w \in W$ with $f = dw$.

3. [6 pts.] **(Grothendieck, Borel-Serre)** Let X and Y be topological spaces, $f\colon X \to Y$ a continuous map, and $\mathcal{F}$ a sheaf of abelian groups over X. Recall (Weibel, Exercise 2.6.2) that the push-forward functor f_* is a right adjoint and is therefore left exact. Its derived functors are denoted $R^j f_*$.

   (a) Show that $R^j f_* \mathcal{F}$ is the sheaf associated to the presheaf
   $$U \mapsto H^j(f^{-1}(U),\, \mathcal{F}).$$

   (b) Deduce from (a) that if, for a point $y \in Y$, every neighborhood of $f^{-1}(y)$ in X contains a neighborhood of the form $f^{-1}(U)$, U a neighborhood of y in Y (this condition is satisfied if, for example, X and Y are locally compact Hausdorff and f is proper), then the stalk of $R^j f_* \mathcal{F}$ at $y \in Y$ is cohomology group $H^j(f^{-1}(y),\, \iota^{-1}\mathcal{F})$, where $\iota\colon f^{-1}(y) \hookrightarrow X$ is the inclusion.

   (c) Show that there is a spectral sequence with E_2 term
   $$E_2^{p,q} = H^p(Y, R^q f_* \mathcal{F})$$
   converging to $H^{p+q}(X,\, \mathcal{F})$. (**Hint:** Factor the unique map $X \to \mathrm{pt}$ as $X \xrightarrow{f} Y \to \mathrm{pt}$ and use a composition-of-functors spectral sequence. Check that all the requirements for this are satisfied. You may use the result of Weibel, Exercise 2.6.3.)

   (d) Suppose the topological condition in (b) is satisfied, e.g., X and Y are locally compact Hausdorff and f is proper, and that $\mathcal{F}$ and f have the property that $H^q(f^{-1}(y),\, \iota^{-1}\mathcal{F}) = 0$ for all $q > 0$ and for all $y \in Y$. Deduce from (b) that $R^q f_* \mathcal{F} = 0$ for all $q > 0$, and then deduce from (c) that there are natural isomorphisms $H^p(X,\, \mathcal{F}) \cong H^p(Y,\, f_* \mathcal{F})$ for all p.

4. [5 pts.] (See Weibel Exercises 9.1.4 and 9.6.4.) Let k be a field of characteristic 0, and let R be the truncated polynomial algebra $k[x]/(x^{n+1})$. Show that R has a periodic resolution as an $R \otimes R^{\mathrm{op}}$-module, and use this to compute $HH_*(R)$. Then compute $HC_*(R)$.