MATH 602 (Homological Algebra)
Assignment #4: Derived Functors

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Let G be a group. A *G-module* is an abelian group M equipped with an action of G, i.e., a group homomorphism $\alpha\colon G \to \mathrm{Aut}(M)$. (Usually we just write $g \cdot m$ for $\alpha(g)(m)$.)

1. Show that a G-module is the same thing as a left module for the group ring $R = \mathbb{Z}G$. (R is the free abelian group with generators the elements of G, equipped with the ring structure coming from the multiplication in G.) Show that $\mathbb{Z}G$ (viewed as a left R-module) is a projective object in the category of G-modules, and deduce that G-**Mod** $= \mathbb{Z}G$-**Mod** is an abelian category with enough projectives.

2. Show that the *coinvariants functor* $\mathrm{Coinv}\colon G\text{-}\mathbf{Mod} \to \mathbf{Ab}$, defined by

$$M \mapsto \mathrm{Coinv}(M) = M/\langle g \cdot m - m \mid g \in G,\, m \in M\rangle,$$

   is right exact but not necessarily left exact.

3. For M a G-module, define $H_i(G, M)$ to be $L_i \,\mathrm{Coinv}(M)$. Suppose that $M = \mathbb{Z}$ with the trivial G-action. Show that there is a short exact sequence of G-modules

$$0 \to I \to \mathbb{Z}G \xrightarrow{\varepsilon} M = \mathbb{Z} \to 0, \qquad (*)$$

   where ε is the *augmentation* defined by $g \mapsto 1$ for all $g \in G$, and I is the *augmentation ideal* generated by all $g - 1$, $g \in G$. *A priori*, ε is just a map of left G-modules, and thus I is a left ideal, but show that ε is in fact a ring homomorphism and I is a 2-sided ideal.

4. Consider the special case where G is cyclic of order 2, and thus $\mathbb{Z}G = \mathbb{Z}[x]/(x^2 - 1)$. Show that I may be identified (as a G-module) with $\mathbb{Z}$ with the *sign representation*, on which the generator x of G acts by -1. Deduce that $\mathrm{Coinv}(I)$ is cyclic of order 2.

5. Use the long exact homology sequence of $(*)$ and the fact that $\mathbb{Z}G$ is projective in G-**Mod** to show (for any G) that $H_0(G, \mathbb{Z}) = \mathbb{Z}$ and that $H_1(G, \mathbb{Z}) \cong G_{\mathrm{ab}}$, the maximal abelian quotient of G. (Hint: You need to compute $\mathrm{Coinv}(I)$. The calculation in (4) is a special case.)