MATH 602 (Homological Algebra)
Assignment #5: Derived Functors, Ext, and Tor

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due Friday, March 9, 2007

1. **Pushouts**. Let I be the small category with 3 objects given by the partially ordered set $\{x, y, z\}$ with $x < y$, $x < z$, morphisms corrsponding to the relations $<$ and $=$. Schematically, I looks like

$$y \longleftarrow x \longrightarrow z.$$

Thus if $\mathcal{A}$ is a category, the functor category $\mathcal{A}^I$ consists of all diagrams $B \xleftarrow{\alpha} A \xrightarrow{\alpha'} B'$ in $\mathcal{A}$, morphisms between them given by commutative diagrams

$$\begin{array}{ccccc} B & \xleftarrow{\alpha} & A & \xrightarrow{\alpha'} & B' \\ \downarrow & & \downarrow & & \downarrow \\ D & \xleftarrow{\beta} & C & \xrightarrow{\beta'} & D'. \end{array}$$

The colimit of such a diagram, assuming it exists, is called the *pushout* of the diagram. It is the universal object D sitting in a diagram

$$\begin{array}{ccc} A & \xrightarrow{\alpha'} & B' \\ {\scriptstyle\alpha}\downarrow & & \vdots \\ B & \cdots\rightarrow & D. \end{array}$$

Show that the pushout of a diagram $B \xleftarrow{\alpha} A \xrightarrow{\alpha'} B'$ in $\mathbf{Ab} = \mathbb{Z}$-**Mod** is given by the quotient group $(B \oplus B')/D$, where $D = \{(\alpha(a), -\alpha'(a)) : a \in A\}$.

2. Do Exercise 2.6.4 in Weibel, page 54. Hint: Consider the commuting

diagram with exact rows:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z} & \xrightarrow{2} & \mathbb{Z} & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle 2} & & \uparrow{\scriptstyle 2} & & \uparrow{\scriptstyle 0} & & \\
0 & \longrightarrow & \mathbb{Z} & \xrightarrow{2} & \mathbb{Z} & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 2} & & \downarrow{\scriptstyle 2} & & \downarrow{\scriptstyle 0} & & \\
0 & \longrightarrow & \mathbb{Z} & \xrightarrow{2} & \mathbb{Z} & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & 0.
\end{array}
$$

3. Compute $L_* \operatorname{colim}\big(\mathbb{Z}/2 \xleftarrow{0} \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2 \big)$. (Hint: if you did the last part of the problem correctly, you are almost done.)

4. The rest of the assignment just has to do with Ext and Tor. Do Exercises 3.1.2 and 3.3.1 in Weibel.