MATH 602 (Homological Algebra)
Assignment #6: $\lim^1$ and Tor (Corrected)

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due Wednesday, April 11, 2007

In this assignment, $R$ is a ring, $A$ is the category $R$-$\text{Mod}$ of $R$-modules, and $\text{Pro-A} = A^{\text{op}}$ is the category of “towers” or inverse systems in $A$.

1. For the case $R = \mathbb{Z}$, consider the object $\{A_i\}$ in $\text{Pro-A}$ given by

$$\mathbb{Z} \leftarrow \mathbb{Z} \leftarrow \mathbb{Z} \leftarrow \mathbb{Z} \leftarrow \cdots,$$

where $p$ is a prime. Compute $\lim_{i \to \infty} \{A_i\}$ and $\lim_{i \to \infty}^1 \{A_i\}$.

2. Show that in general, tensor product does not commute with infinite direct products, by showing that

$$(\prod_{i = 0}^{\infty} \mathbb{Z}) \otimes \mathbb{Q} \neq \prod_{i = 0}^{\infty} (\mathbb{Z} \otimes \mathbb{Q}) = \prod_{i = 0}^{\infty} \mathbb{Q}.$$

3. (For this part $R$ is arbitrary.) Suppose $\{A_i, \phi_i\}$ is an object in $\text{Pro-A}$, and $B$ is a right $R$-module of finite type (i.e., having a finite projective resolution $P_\bullet$ by finitely generated projective $R$-modules). Let $C_\bullet$ be the chain complex with

$$C_0 = C_{-1} = \prod_{i = 0}^{\infty} A_i, \quad d(a_0, a_1, \cdots) = (a_0 - \phi_1(a_1), a_1 - \phi_2(a_2), \cdots),$$

$$C_q = 0 \text{ for } q \neq 0, -1,$$

which has $H_q(C_\bullet) = \lim_{i \to \infty}^{-q} A_i$. Tensoring $P_\bullet$ by $C_\bullet$, you get a double complex $P_\bullet \otimes_R C_q$ whose entries are non-zero only for $p \geq 0, q = 0, -1$. Show that the two spectral sequences attached to this complex converge to the same thing (the homology of the total complex). One of them has

$$E_2^{p,q} = \text{Tor}_p^R(B, \lim_{i \to \infty}^{-q} A_i), \quad q = -1, 0,$$

the other has

$$E_2^{p,q} = \lim_{i \to \infty}^{-p} \text{Tor}_q^R(B, A_i), \quad p = -1, 0.$$
(This theorem is due to Roos, in the paper [Roos] cited in the bibliography of Weibel’s book.) Be careful; where do you use the finite type assumption?

4. Now specialize again to the case $R = \mathbb{Z}$. Show that for $B$ finitely generated, you get (from the results of part (3)) a commuting diagram with exact rows and columns:

\[
\begin{array}{cccccccc}
0 & \to & (\lim A_i) \otimes B & \to & \lim \Tor_1(A_i, B) & \to & H & \to & \lim(A_i \otimes B) & \to & 0 \\
& & \downarrow & \downarrow & \downarrow & & \downarrow & & \\
& & 0 & \to & \Tor_1(\lim A_i, B) & \to & 0
\end{array}
\]

where $H$ is $H_0$ of the total complex.

5. Let $R = \mathbb{Z}$, let $\{A_i\}$ be as in part (1), and let $B = \mathbb{Q}$. Compute all the groups in the sequences of part (4) for this case, and show that exactness must fail. Thus the finite generation hypothesis on $B$ is necessary. Explain how this is related to part (2).

6. Again let $\{A_i\}$ be as in part (1), this time with $B = \mathbb{Z}/p$. Make precise what you get for all the exact sequences of part (4).