MATH 602 (Homological Algebra)
Assignment #6: $\varprojlim^1$ and Tor (Corrected)

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due Wednesday, April 11, 2007

In this assignment, R is a ring, $\mathcal{A}$ is the category R-**Mod** of R-modules, and **Pro-**$\mathcal{A} = \mathcal{A}^{\mathbb{N}^{\text{op}}}$ is the category of "towers" or inverse systems in $\mathcal{A}$.

1. For the case $R = \mathbb{Z}$, consider the object $\{A_i\}$ in **Pro-**$\mathcal{A}$ given by

$$\mathbb{Z} \xleftarrow{p} \mathbb{Z} \xleftarrow{p} \mathbb{Z} \xleftarrow{p} \mathbb{Z} \xleftarrow{p} \cdots,$$

where p is a prime. Compute $\varprojlim\{A_i\}$ and $\varprojlim^1\{A_i\}$.

2. Show that in general, tensor product does not commute with infinite direct products, by showing that

$$\left(\prod_0^\infty \mathbb{Z}\right) \otimes_{\mathbb{Z}} \mathbb{Q} \neq \prod_0^\infty (\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}) = \prod_0^\infty \mathbb{Q}.$$

3. (For this part R is arbitrary.) Suppose $\{A_i, \phi_i\}$ is an object in **Pro-**$\mathcal{A}$, and B is a right R-module of finite type (i.e., having a finite projective resolution $P_\bullet$ by finitely generated projective R-modules). Let $C_\bullet$ be the chain complex with

$$C_0 = C_{-1} = \prod_{i=0}^\infty A_i, \quad d(a_0, a_1, \cdots) = (a_0 - \phi_1(a_1), a_1 - \phi_2(a_2), \cdots),$$
$$C_q = 0 \text{ for } q \neq 0,\ -1,$$

which has $H_q(C_\bullet) = \varprojlim^{-q} A_i$. Tensoring $P_\bullet$ by $C_\bullet$, you get a double complex $P_p \otimes_R C_q$ whose entries are non-zero only for $p \geq 0$, $q = 0, -1$. Show that the two spectral sequences attached to this complex converge to the same thing (the homology of the total complex). One of them has

$$E_2^{p,q} = \operatorname{Tor}_p^R(B, \varprojlim^{-q} A_i), \quad q = -1,\ 0,$$

the other has

$$E_2^{p,q} = \varprojlim^{-p} \operatorname{Tor}_q^R(B, A_i), \quad p = -1,\ 0.$$

(This theorem is due to Roos, in the paper [Roos] cited in the bibliography of Weibel's book.) **Be careful; where do you use the finite type assumption?**

4. Now specialize again to the case $R = \mathbb{Z}$. Show that for B finitely generated, you get (from the results of part (3)) a commuting diagram with exact rows and columns:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
 & & & & \left(\varprojlim A_i\right) \otimes B & & & & \\
 & & & & \downarrow & & & & \\
0 & \to & \varprojlim^1 \operatorname{Tor}_1(A_i, B) & \longrightarrow & H & \longrightarrow & \varprojlim (A_i \otimes B) & \to & 0 \\
 & & & & \downarrow & & & & \\
 & & & & \operatorname{Tor}_1\left(\varprojlim^1 A_i, B\right) & & & & \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & & ,
\end{array}
$$

where H is H_0 of the total complex.

5. Let $R = \mathbb{Z}$, let $\{A_i\}$ be as in part (1), and let $B = \mathbb{Q}$. Compute all the groups in the sequences of part (4) for this case, and show that exctness must fail. Thus the finite generation hypothesis on B is necessary. Explain how this is related to part (2).

6. Again let $\{A_i\}$ be as in part (1), this time with $B = \mathbb{Z}/p$. Make precise what you get for all the exact sequences of part (4).