Algebraic Topology
(Mathematics 734, Prof. Rosenberg)
Solutions, Longer Problems
Mid-Term Examination
Friday, March 19, 2004

1. (30 points) **Short-Answer Problems.** Give brief definitions or statements (no proofs) for the following terms.

(a) The singular chain complex $C_\bullet(X)$ of a topological space X.

(b) The excision property (for an axiomatic homology theory $H_\bullet$).

(c) The Generalized Jordan Curve Theorem.

2. (20 points) Let M a topological n-manifold (a locally compact Hausdorff space locally homeomorphic to $\mathbb{R}^n$). Use excision to prove that if $x \in M$, $H_j(M, M \smallsetminus \{x\}) \cong \mathbb{Z}$ if $j = n$, 0 otherwise.

Solution. Since M is a topological n-manifold, there is an open neighborhood U of x with a homeomorphism $U \twoheadrightarrow \mathbb{R}^n$ sending x to 0. Let B be the complement of U. Then the closure of B is B, which is contained in $M \smallsetminus \{x\}$. Hence we may excise B to obtain

$$H_j(M, M \smallsetminus \{x\}) \cong H_j(U, U \smallsetminus \{x\}) \cong H_j(\mathbb{R}^n, \mathbb{R}^n \smallsetminus \{0\}).$$

But $\mathbb{R}^n \smallsetminus \{0\} \cong S^{n-1} \times (0, \infty)$ has a deformation retraction down to S^{n-1}, and $\mathbb{R}^n$ is contractible. So we obtain the exact sequence

$$0 = \widetilde{H}_j(\mathbb{R}^n) \to H_j(\mathbb{R}^n, \mathbb{R}^n \smallsetminus \{0\}) \to \widetilde{H}_{j-1}(\mathbb{R}^n \smallsetminus \{0\}) \to \widetilde{H}_{j-1}(\mathbb{R}^n) = 0,$$

and hence $H_j(M, M \smallsetminus \{x\}) \cong \widetilde{H}_{j-1}(S^{n-1})$, which is $\mathbb{Z}$ if $j = n$ and 0 otherwise.

3. (50 points) Let X be a CW-complex with exactly 3 cells: a 0-cell, an n-cell, and an m-cell, where $0 < n < m$.

(a) If $m \geq n+2$, show that $H_j(X)$ is $\cong \mathbb{Z}$ in dimensions 0, n, and m, and 0 otherwise.

Solution. We use the fact that the homology can be computed from the cellular chain complex, which in this case has a $\mathbb{Z}$ in dimensions 0, n, and m, and 0's elsewhere. Thus we just need to determine the cellular boundary maps. Since there are no cells in dimensions $n+1$ and $m+1$, there are no cellular boundaries in dimensions n and m. There are also no cellular boundaries in dimension 0, since either $n > 1$ and there are no cellular 1-chains, or else there is a unique cell in dimension 1 and both of its endpoints are attached to the unique 0-cell a, so that its boundary is $a - a = 0$. Furthermore, every cellular chain is a cycle, since there are no cells in dimensions $m - 1$ or -1, and there is a cell in dimension $n-1$ only if $n = 1$, in which case we've already seen the boundary map on cellular 1-chains is the 0-map. Thus the cellular homology groups are the same as the cellular chain groups, namely $\mathbb{Z}$ in dimensions m, n, and 0, and 0 in all other dimensions.

(b) If $m = n + 1$, show that there are three possibilities for the homology: either X is acyclic, or else $H_n(X) \cong H_{n+1}(X) \cong \mathbb{Z}$, or else $H_{n+1}(X) = 0$ and $H_n(X)$ is finite cyclic but non-trivial.

Solution. This case only differs in that the cellular boundary map $\mathbb{Z} = C_{n+1} \to C_n = \mathbb{Z}$ is potentially non-zero. If we identify this map with multiplication by k from $\mathbb{Z}$ to $\mathbb{Z}$, then if $k = \pm 1$, $Z_{n+1} = 0$ and $B_n = C_n = Z_n$, so that $H_{n+1} = H_n = 0$. If $k = 0$, then $H_{n+1} \cong H_n \cong \mathbb{Z}$, and if $k \notin \{-1, 0, 1\}$, then $Z_{n+1} = 0$ and B_n is a proper subgroup of C_n of finite index, so that $H_{n+1} = 0$ and $H_n \cong \mathbb{Z}/k$ is finite cyclic.

(c) Give concrete examples to show that all three cases in (b) can occur.

Solution. D^{n+1}, with viewed as obtained from attaching an $(n+1)$-cell to S^n via the identity map $S^n \to S^n$, is an acyclic example. $S^n \vee S^{n+1}$ is an example with $H_{n+1} \cong H_n \cong \mathbb{Z}$, and $\mathbb{RP}^2$ is an example of the last case (with $n = 1$).

(d) Show that every homeomorphism $f: X \to X$ has a fixed point. (You may assume X is an ENR. Show that $L_{\mathbb{Q}}(f)$ is odd and use the Lefschetz Fixed-Point Theorem.)

Solution. By (a) and (b), $H_j(X, \mathbb{Q})$ is zero for $j \neq 0, n, m$, and is $\mathbb{Q}$ for $j = 0$. Furthermore, either $H_m(X, \mathbb{Q}) \cong H_n(X, \mathbb{Q}) \cong \mathbb{Q}$ (which is the case when $H_m(X) \cong H_n(X) \cong \mathbb{Z}$), or else $H_m(X, \mathbb{Q}) \cong H_n(X, \mathbb{Q}) \cong 0$ (when $m = n + 1$ and H_n is finite). Since f must map the unique path component of X to itself, $f_* = 1$ on H_0. Furthermore, if f is a homeomorphism, then the self-map induced by f on each non-zero *integral* homology group in positive dimensions is an isomorphism, hence $f_* = \pm 1$ on any $H_j(X, \mathbb{Q})$ which is $\cong \mathbb{Q}$. Thus

$$L_{\mathbb{Q}}(f) = \pm 1 \pm 1 + 1 \text{ or } 1,$$

and in any event, $L_{\mathbb{Q}}(f)$ is odd. Since X is an ENR, the Lefschetz Fixed-Point Theorem says f has a fixed point.

(e) Deduce from (d) that if G is a group (with more than one element), then G cannot act freely on $\mathbb{CP}^2$ or on $\mathbb{RP}^2$.

Solution. Both of the spaces $\mathbb{CP}^2$ and $\mathbb{RP}^2$ have cell decompositions satisfying the hypothesis (with $m = 4$ and $n = 2$ for $\mathbb{CP}^2$, and with $m = 2$ and $n = 1$ for $\mathbb{RP}^2$). Thus by (d), any self-homeomorphism of $\mathbb{CP}^2$ or $\mathbb{RP}^2$ has a fixed point. Hence if a non-trivial group g acts on $\mathbb{CP}^2$ or $\mathbb{RP}^2$, then each $g \neq 1$ has a fixed point, which means G cannot act freely.