MATHEMATICS 734: ALGEBRAIC TOPOLOGY
DR. ROSENBERG
FINAL EXAMINATION
THURSDAY, MAY 16, 1991

Instructions

Show all your work in your exam booklet. The more intermediate steps you show in a problem, the greater the likelihood of your receiving partial credit if the final answer is not completely correct. Unless otherwise stated, you may appeal to any of the theorems proved in class or in the textbook. The point values of the problems are indicated. In problems with multiple parts, the parts are graded independently, so **be sure to go on to the remaining parts** even if there is one part you can't do. There are a total of 200 points on the exam.

1. (**Short answer**, 10 points each) Give brief definitions and/or statements for each of the following:
   (a) finite CW-complex;
   (b) Jordan-Brouwer Separation Theorem (for an embedding $\iota : S^{n-1} \hookrightarrow \mathbb{R}^n$);
   (c) fibration;
   (d) the Excision Theorem (for singular homology);
   (e) chain homotopy (between two chain maps $\varphi, \psi : C_. \to C'_.$).

2. (40 points) Identify S^2 with $\mathbb{C} \cup \{\infty\} = \mathbb{CP}^1$ as usual. Show that if $f(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$ is a monic polynomial with complex coefficients, then the degree n of f as a polynomial coincides with the (topological) degree of f as a map $S^2 \to S^2$. Hint: you may homotope the a_j's to 0 and thereby reduce to the case of $f(z) = z^n$.

3. (70 points, divided as indicated) **Note: In each part of this problem, you may assume the results of the previous parts** (even if you omitted one of them). Let $n \geq 2$ and let M^n be a compact connected topological n-manifold (without boundary) such that $H_j(M; \mathbb{Z}) = 0$ for $1 \leq j \leq \frac{n}{2}$.
   (a) (15 points) Show that M is orientable.
   (b) (15 points) Show using Poincaré duality that M has the same homology groups as S^n.
   (c) (20 points) Show that there is a continuous map $f : M \to S^n$ inducing an isomorphism on homology groups. (Hint: you want f to be of degree one.)
   (d) (20 points) Assuming that M has the homotopy type of a CW-complex (which can be deduced from the fact that M is a compact ANR), use the Whitehead and Hurewicz Theorems to show that if M is simply connected, the map f of part (c) is a homotopy equivalence from M to S^n. What can go wrong if M is not required to be simply connected?

4. (40 points) Show that there is no retraction from $\mathbb{CP}^3$ to $\mathbb{CP}^2$. (Hint: use either higher homotopy groups or cup-products.)