Math 734, Assignment #2
Relative Homology, Chain Complexes, and Exact Sequences

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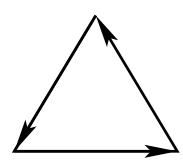
due Monday, February 11, 2008

1. (a) Show that Δ^n (the standard n-simplex) is homeomorphic to D^n (the unit disk in $\mathbb{R}^n$) and that its boundary is homeomorphic to S^{n-1}.

   (b) Show that the identity map $\Delta^n \to \Delta^n$ defines a class in $H_n(D^n, S^{n-1})$ which under the boundary map of the long exact homology sequence maps to the class in $H_{n-1}(S^{n-1})$ represented by the sum of the faces of Δ^n, with appropriate signs, like this:



2. Consider short exact sequences of $\mathbb{Z}$-modules

$$0 \to \mathbb{Z}/4 \to A \to \mathbb{Z}/4 \to 0.$$

   (a) What are the possibilities for A (up to isomorphism)?

   (b) Define an equivalence relation on such short exact sequences by saying two are equivalent if there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}/4 & \longrightarrow & A & \longrightarrow & \mathbb{Z}/4 & \longrightarrow & 0 \\ & & \| & & \downarrow \cong & & \| & & \\ 0 & \longrightarrow & \mathbb{Z}/4 & \longrightarrow & B & \longrightarrow & \mathbb{Z}/4 & \longrightarrow & 0. \end{array}$$

   Show that there are exactly four equivalence classes.

   (c) Show that if we require A to be a $\mathbb{Z}/4$-module (and not just a $\mathbb{Z}$-module), then the short exact sequence must split.

3. In this problem, let C_* be a chain complex with $C_* = 0$ for $* < 0$, and let H_* be the associated homology groups.

(a) Show that H_* itself becomes a chain complex if we define all its boundary maps to be 0, and that this chain complex has the same homology as the original complex.

(b) Show by examples (of chain complexes of $\mathbb{Z}$-modules) that there is not necessarily a chain map $C_* \to H_*$ or $H_* \to C_*$ inducing an isomorphism on homology.

(c) Show that in the case of chain complexes over a *field* (not over $\mathbb{Z}$!) that one can always construct a splitting $C_* \cong E_* \oplus H_*$, where E_* is exact (has no homology) and where the projection $C_* \to H_*$ and the inclusion $H_* \to C_*$ induce isomorphisms on homology. (You will need the fact, which you can assume, that any linearly independent set in a vector space can be extended to a basis.)