On a Law of Combination of Operators. (Second Paper.)* By J. E. CAMPBELL. Read and Received November 11th, 1897.

1. If x and y are operators which obey the ordinary laws of algebra, we know that

$$e^y e^x = e^{y+x}.$$

I propose to investigate the corresponding theorem when the operators obey the distributive and associative laws, but not the commutative.

Let y_1 denote the operator $yx - xy$,

y_2 " " $y_1 x - x y_1$,

y_r " " $y_{r-1} x - x y_{r-1}$.

Let (p, q) denote $y_p y_q - y_q y_p$,

(p, q, r) " $(p, q)\, y_r - y_r (p, q)$,

and let (p, q, r, s), &c., have similar meanings.

The theorem to be proved is this:—If $a_1, a_2, a_3, \ldots$ is the series of numerical constants† discussed in a former paper (*Proceedings*,

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* At the suggestion of the referees, and by permission of the Council, the title of this paper has been changed.

† These constants are closely associated with Bernouilli's numbers; in fact

$$a_{2n} = (-1)^{n-1} \frac{B_{2n-1}}{(2n)!}$$

(Lie, *Transformationsgruppen*, III., § 144). This may easily be verified independently thus:—

$$(e^t - 1)^{-1} = \frac{1}{t} - \frac{1}{2} + \frac{B_1}{2!} t - \frac{B_3}{4!} t^3 + \ldots .$$

Differentiate with respect to t; then

$$\frac{1}{t^2} - \frac{B_1}{2!} + \frac{3B_3}{4!} t^2 - \ldots = (e^t - 1)^{-1} + (e^t - 1)^{-2}$$

$$= \left(\frac{1}{t} - \frac{1}{2} + \frac{B_1}{2!} t - \frac{B_3}{4!} t^3 + \ldots \right) + \left(\frac{1}{t} - \frac{1}{2} + \frac{B_1}{2!} t - \frac{B_3}{4!} t^3 + \ldots \right)^2 .$$

Equating coefficients of t^{2n} on each side, we see that

$$(2n+3) \frac{B_{2n+1}}{(2n+2)!} = \left(\frac{B_1}{2!} \frac{B_{2n-1}}{(2n)!} + \frac{B_3}{4!} \frac{B_{2n-3}}{(2n-2)!} + \ldots \right),$$

from which it can easily be seen that

$$a_{2n} = (-1)^{n-1} \frac{B_{2n-1}}{(2n)!}.$$

Vol. XXVIII., p. 381), and $b_{pq} \ldots b_{pqr} \ldots b_{pqrs} \ldots$ are other series of numerical constants, derived by certain laws from the first series, then

$$e^y e^x = e^w,$$

where

$$w \equiv x+y+a_1y_1+a_2y_2+\ldots+\Sigma b_{pq}(p, q)+\Sigma b_{pqr}(p, q, r)+\ldots,$$

the summation being taken in each case for all positive integral values of $p, q, r, \ldots$ (zero included).

Having established this result, it will be shown that $x_1, x_2, \ldots x_n$ being any n operators, not commutative with respect to one another, and $\lambda_1, \lambda_2, \ldots \lambda_n$ and $\mu_1, \mu_2, \ldots \mu_n$ two sets of arbitrary constants,

$$e^{\mu_1x_1+\ldots+\mu_nx_n} \cdot e^{\lambda_1x_1+\ldots+\lambda_nx_n} \equiv e^{\Sigma\beta_px_p+\Sigma\beta_{pq}(p, q)+\Sigma\beta_{pqr}(p, q, r)+\ldots},$$

where (p, q) is now taken to mean $x_px_q-x_qx_p$,

(p, q, r) " " $(p, q)\,x_r-x_r\,(p, q)$,

with similar meanings for the other symbols, and the summation is now made by giving to $p, q, r, \ldots$ all values from 1 to n inclusive.

The β's are now, however, no longer mere numerical constants, but functions of the parameters $\lambda_1 \ldots \lambda_n$ and $\mu_1 \ldots \mu_n$.

It will be seen that the number of indices in a β function expresses its degree in λ, μ; thus β_{pqr} is a function of the third degree in $\lambda_1 \ldots \lambda_n, \mu_1 \ldots \mu_n$.

In proving these theorems it will be seen that the following more general theorem is also true :—

If $w_1 = a_1x_1+\ldots+a_nx_n+a_{pq}(p, q)+\ldots+a_{pqr}(p, q, r)+\ldots,$

$w_2 = b_1x_1+\ldots+b_nx_n+b_{pq}(p, q)+\ldots+b_{pqr}(p, q, r)+\ldots,$

then

$$e^{w_1} \cdot e^{w_2} = e^{w_3},$$

where $w_3 = c_1x_1+\ldots+c_nx_n+c_{pq}(p, q)+\ldots+c_{pqr}(p, q, r)+\ldots.$

Here the a's and the b's are any constants whatever, and the c's are other constants depending on them.

2. The identity

$$\text{(A)} \qquad x^ny = yx^n-ny_1x^{n-1}+\frac{n(n-1)}{2!}y_2x^{n-2}-\ldots,$$

obviously holds when $n=1$. Assume that it also holds for all values up to n; then

$$x^{n+1}y = xyx^n - nxy_1x^{n-1} + \frac{n(n-1)}{2!}xy_2x^{n-2} - \dots .$$

Now $\quad xy = yx - y_1, \quad xy_1 = y_1x - y_2, \quad xy_{r-1} = y_{r-1}x - y_r;$

therefore

$$\begin{aligned} x^{n+1}y = yx^{n+1} - \quad & ny_1x^n + \frac{n(n-1)}{2!}y_2x^{n-1} - \dots \\ - \quad & y_1x^n + \quad ny_2x^{n-1} - \dots \\ = yx^{n+1} - (n+1)y_1x^n & + \frac{(n+1)n}{2!}y_2x^{n-1} - \dots, \end{aligned}$$

so that the identity holds universally, since it holds when $n=1$.

If $\quad [yx^r] \equiv yx^r + xyx^{r-1} + x^2yx^{r-2} + \dots + x^ry,$

we see that

$$\text{(B)} \quad \left[\frac{yx^r}{(r+1)!}\right] = \frac{yx^r}{1!\,r!} - \frac{y_1x^{r-1}}{2!\,(r-1)!} + \frac{y_2x^{r-2}}{3!\,(r-2)!} - \dots + \frac{(-1)^r y_r}{(r+1)!},$$

for the identity holds when $r=1$; and, assuming that it holds for all values of r up to $n-1$, then

$$\begin{aligned} \left[\frac{yx^n}{n!}\right] &= \left[\frac{yx^{(n-1)}}{n!}\right]x + \frac{x^ny}{n!} \\ &= \frac{yx^n}{1!\,(n-1)!} - \frac{y_1x^{n-1}}{2!\,(n-2)!} + \dots + (-1)^{n-1}\frac{y_{n-1}x}{n!} + \frac{x^ny}{n!}; \end{aligned}$$

but $\quad x^ny = yx^n - ny_1x^{n-1} + \frac{n(n-1)}{2!}y_2x^{n-2} - \dots,$

so that, by adding similar terms in the two series, we get

$$\left[\frac{yx^n}{n!}\right] = (n+1)\left\{\frac{yx^n}{1!\,n!} - \frac{y_1x^{n-1}}{2!\,(n-1)!} + \dots + \frac{(-1)^n y_n}{(n+1)!}\right\};$$

that is, the identity also holds when $r=n$.

3. y_1 denotes $yx - xy$, which might be written (yx), so y_2 denoting $(yx)x - x(yx)$ may be written (yxx). Similarly (yxz) is taken to denote $(yx)z - z(yx)$, and so on. From the definition of the symbol,

$$(yx) + (xy) \equiv 0,$$

by expansion we also obtain immediately

$$(xyz)+(yzx)+(zxy) \equiv 0.^*$$

The above identity may also be written

$$(yxz) = (yzx) - \{y\,(zx)\},$$

where $\{y\,(zx)\}$ means $y\,(zx)-(zx)\,y$, or

$$(yxz) = (yzx)-(yz_1),$$

z_1 denoting $zx-xz$; that is, from the ordinary identity

$$yxz = yzx - yz_1,$$

we deduce the symbolical one

$$(yxz) = (yzx)-(yz_1).$$

It follows in the same way that

$$(yxz_1) = (yz_1x)-(yz_2),$$

and
$$(yxz_{r-1}) = (yz_{r-1}x)-(yz_r),$$

where
$$z_r = z_{r-1}x - xz_{r-1}.$$

Now, by exactly the same reasoning as from the formulæ

$$xy = yx - y_1, \quad xy_1 = y_1x - y_2, \quad xy_{r-1} = y_{r-1}x - y_r,$$

we deduced that

$$yx^r + xyx^{r-1} + \ldots + x^ry \equiv \frac{(r+1)!}{1!\,r!}yx^r - \frac{(r+1)!}{2!\,(r-1)!}y_1x^{r-1} + \ldots + (-1)^r y_r;$$

so from the above symbolical formulæ we deduce that

$$\text{(B}'\text{)} \quad (yzx^{(r)}) + (yxzx^{(r-1)}) + \ldots + (yx^{(r)}z)$$

$$\equiv \frac{(r+1)!}{1!\,r!}(yzx^{(r)}) - \frac{(r+1)!}{2!\,(r-1)!}(yz_1x^{(r-1)}) + \ldots + (-1)^r(yz_r),$$

where $(yzx^{(r)})$ is a symbol used to denote $(yzx \ldots$ to r terms).

4. Now

$$\{y\,(u+v)\} = (yu)+(yv),$$

$$\{y\,(u+v)^{(2)}\} = \{y\,(u+v)\} \times (u+v) - (u+v) \times \{y\,(u+v)\}$$

$$= (yu^{(2)}) + (yuv) + (yvu) + (yv^{(2)}),$$

* This may easily be deduced as a particular case of the Jacobian identity, and to many of the theorems which follow we have corresponding ones, which are deduced from the Jacobian identity by the same method as the theorems given are deduced from the above formula.

and similarly we see that

$$\{y\,(u+v)^{(r)}\} = (yu^{(r)}) + (yu^{(r-1)}v) + (yu^{(r-2)}vu) + \ldots + (yv^{(r)}).$$

For the purpose of this paper it will be found necessary to find the coefficient of μ in $\{y\,(x+\mu z)^{(r)}\}$, where

$$z = y + a_1 y_1 + a_2 y_2 + \ldots \text{ to infinity};$$

and we see by considering the above that it is

$$(yx^{(r-1)}z) + (yx^{(r-2)}zx) + (yx^{(r-3)}zx^{(2)}) + \ldots,$$

which may also be written

$$\frac{r!}{1!\,(r-1)!}(yzx^{(r-1)}) - \frac{r!}{2!\,(r-2)!}(yz_1x^{(r-2)}) + \frac{r!}{3!\,(r-3)!}(yz_2x^{(r-3)}) - \ldots + (-1)^{r-1}(yz_{r-1}),$$

by (B′).

5. Our object being to express this result in terms of y, y_1, ... only, we must simplify such an expression as $(yz_px^{(r)})$, or $(yz_p)_r - (z_py)_r$, since

$$(yz_px^{(r)}) = (yz_p)_r - (z_py)_r,$$

where $(uv)_r$ is a symbol used to denote the result of writing uv for y in y_r.

It will first be proved that

$$\text{(C)} \quad (uv)_r = u_rv + ru_{r-1}v_1 + \frac{r\,(r-1)}{2!}u_{r-2}v_2 + \ldots + uv_r.^*$$

This theorem is analogous to Leibnitz's, and is proved by the same method, thus

$$(uv)_1 = uvx - xuv = uxv + uv_1 - xuv$$
$$= xuv + u_1v + uv_1 - xuv = u_1v + uv_1.$$

Assuming then that the theorem holds for all values of r up to n, it will be proved that it also holds for values up to $n+1$. Since

$$(uv)_n = \sum_{p=0}^{p=n} \frac{n!}{p!\,(n-p)!} u_pv_{n-p},$$

* For the theorem which bears the same relation to the Jacobian identity which this does to the identity $(xyz) + (yzx) + (zxy) \equiv 0$, see Lie, II., p. 280.

therefore

$$(uv)_{n+1} = \sum_{p=0}^{p=n} \frac{n!}{p!\,(n-p)!}\,(u_p v_{n-p})_1$$

$$= \sum_{p=0}^{p=n} \frac{n!}{p!\,(n-p)!}\,u_{p+1}v_{n-p} + \sum_{p=0}^{p=n} \frac{n!}{p!\,(n-p)!}\,u_p v_{n-p+1}$$

$$= \sum_{p=0}^{p=n+1} \frac{(n+1)}{p!\,(n+1-p)!}\,u_p v_{n+1-p};$$

that is, the theorem holds for values of r up to $n+1$, and therefore generally.

6. Using now this result, let us collect all the terms in the series

$$\frac{r!}{1!\,(r-1)!}\left\{(yz)_{r-1}-(zy)_{r-1}\right\} - \frac{r!}{2!\,(r-2)!}\left\{(yz_1)_{r-2}-(z_1 y)_{r-2}\right\} + \dots$$

$$\dots + (-1)^{r-1}\left\{yz_{r-1}-z_{r-1}y\right\},$$

which end in z_q. We get, by aid of (C),

$$\frac{r!\,q_{r-q-1}\,.\,z_q}{(r-q-1)!}\left\{\frac{1}{1!\,q!} - \frac{1}{2!\,(q-1)!} + \dots + \frac{(-1)^q}{(q+1)!\,0!}\right\}$$

$$= \frac{r!\,y_{r-q-1}\,z_q}{(q+1)!\,(r-q-1)!};$$

thus the terms in the series which end in z, z_1, ... or z_{r-1} are given by

$$\sum_{q=0}^{q=r-1} \frac{r!}{(q+1)!\,(r-q-1)!}\,y_{r-q-1}z_q$$

(understanding by z_0 merely z).

Collecting all the terms in the series ending with y_q, we get

$$\frac{-r!\,z_{r-q-1}y_q}{q!}\left\{\frac{1}{1!\,(r-q-1)!} - \frac{1}{2!\,(r-q-2)!} + \dots + \frac{(-1)^{r-q-1}}{(r-q)!\,0!}\right\}$$

$$= \frac{-r!\,z_{r-q-1}y_q}{q!\,(r-q)!},$$

so that the terms in the series which end in y, y_1, ... or y_{r-1} are given by

$$-\sum_{q=0}^{q=r-1} \frac{r!\,z_{r-q-1}y_q}{q!\,(r-q)!},$$

which may be written (as we see by writing $r-q'-1$ for q)

$$-\sum_{q=0}^{q=r-1} \frac{r!\,z_q y_{r-q-1}}{(q+1)!\,(r-q-1)!}.$$

Thus the total series required may be written in the simpler form

$$\sum_{q=0}^{q=r-1} \frac{r!}{(q+1)!\,(r-q-1)!}\,(y_{r-q-1}z_q).$$

7. Now $\qquad z = y + a_1 y_1 + a_2 y_2 + \ldots$ to infinity;

therefore $\qquad z_q = y_q + a_1 y_{q+1} + a_2 y_{q+2} + \ldots$ to infinity;

therefore $(y_{r-q-1}z_q) = y_{r-q-1}\,(y_q + a_1 y_{q+1} + a_2 y_{q+2} + \ldots$ to infinity)

$\qquad - (y_q + a_1 y_{q+1} + a_2 y_{q+2} + \ldots$ to infinity) y_{r-q-1}.

We thus see that, (m, n) denoting $y_m y_n - y_n y_m$, the series may be written

$$\sum_{q=0}^{q=r-1} \frac{r!}{(q+1)!\,(r-q-1)!}\,\{(r-q-1,\, q) + a_1\,(r-q-1,\, q+1)$$
$$+ a_2\,(r-q-1,\, q+2) + \ldots \text{ to infinity}\},$$

or in the simpler form

$$\sum_{q=0}^{q=r-1} \sum_{\kappa=0}^{\kappa=\infty} \frac{r!}{(q+1)!\,(r-q-1)!}\, a_\kappa\,(r-q-1,\, q+\kappa);$$

(D) or $$\sum_{m=0}^{m=r-1} \sum_{n=r-1-m}^{n=\infty} a_{m+n-r+1} \frac{r!}{m!\,(n-m)!}\,(m, n),$$

if we define a_0 as unity.

8. We have thus found the coefficient of μ in the result obtained by substituting for x, $x+\mu z$ in any term y_r. It will be necessary to find the corresponding coefficient obtained by making the same substitution in any terms such as (p, q) or (p, q, r) .... To find this consider a particular term (p, q, r, s), though the reasoning is general.

First, it is clear that the coefficient will be the sum of the coefficients obtained by substituting for each element y_r of the term the series

$$\sum_{m=0}^{m=r-1} \sum_{n=r-1-m}^{n=\infty} a_{m+n-r+1} \frac{r!}{m!\,(r-m)}\,(m, n).$$

Take a particular term of this series (m, n); then it contributes

$$\frac{r!\,a_{m+n-r+1}}{m!\,(r-m)!}\,\{p,\, q\,(m, n)\, s\},$$

where the part (m, n) is to be taken as one symbol in expanding the complex symbol $\{p,\, q\,(m, n)\, s\}$.

Now we have seen that

$$(q, m, n)-(q, n, m) \equiv \{q\,(m, n)\};$$

and it is easily seen to follow that

$$\{p, q\,(m, n)\,s\} \equiv (p, q, m, n, s)-(p, q, n, m, s),$$

so that the coefficient derived from (p, q, r, s) will consist of five-lettered symbols.

9. We now know how to obtain the coefficients derived from

$$y_1 y_2 \ldots, \quad (y_p y_q) \ldots, \quad (y_p y_q y_r) \ldots,$$

and must next obtain the coefficient derived from z, which might be called the first derived coefficient of z; from this we similarly derive a coefficient which might be called the second derived coefficient of z, and so on.

10. $$z = y + a_1 y_1 + a_2 y_2 + \ldots \text{ to infinity},$$

so that the coefficient derived from it will be

$$\Sigma\, b_{pq}\,(p, q),$$

where p and q may have any positive or zero integral values, and b_{pq} denotes

$$\sum_{r=p+1}^{r=p+q+1} a_r a_{p+q+1-r} \frac{r!}{p!\,(r-p)!}.$$

11. What will now be the coefficient which is similarly derived from

$$\Sigma\, b_{pq}\,(p, q)\,?$$

From what we have seen it will be of the form

$$\Sigma\, b_{pqr}\,(p, q, r),$$

where the summation is taken for all positive and zero integral values of p, q, r. Similarly the coefficient derived from this will be of the form

$$\Sigma\, b_{pqrs}\,(p, q, r, s),$$

where the summation is taken for all positive and zero integral values of p, q, r, s; and so generally. We have, however, to show how to calculate the numerical constants $b_{pqr}, b_{pqrs}, \ldots$.

12. We saw that the term (p, q, r, s, t) was made up by contributions from the terms (x, r, s, t), (p, x, s, t), (p, q, x, t), and (p, q, r, x).

Take (x, r, s, t): the term (p, q, r, s, t) was derived from this by

$$a_{p+q-x+1}\frac{x!}{p!\,(x-p)!}(p, q)$$

replacing x in $b_{xrst}(x, r, s, t)$, and by

$$a_{q+p-x+1}\frac{x!}{q!\,(x-q)!}(q, p)$$

replacing x in $b_{xrst}(x, r, s, t)$.

The first of these, then, contributes to the coefficient b_{pqrst} the series

$$\sum_{x=p+1}^{x=p+q+1} a_{p+q-x+1}\frac{x!}{p!\,(x-p)!}\,b_{xrst};$$

and the second contributes

$$-\sum_{x=q+1}^{x=p+q+1} a_{p+q-x+1}\frac{x!}{q!\,(x-q)!}\,b_{xrst};$$

or, if we take $x^{(p)}$ as usual to denote $x(x-1)\ldots(x-p+1)$, and remember that $x^{(p)}$ is zero if $x<p$, the two together contribute to b_{pqrst} the series

$$\sum_{x=0}^{x=p+q+1}\left(\frac{x^{(p)}}{p!}-\frac{x^{(q)}}{q!}\right)a_{p+q+1-x}b_{xrst}+a_{p+1}b_{qrst}-a_{q+1}b_{prst}.$$

Similarly, by considering the series which the other terms contribute, we see that

$$\begin{aligned}b_{pqrst} = {} & \sum_{x=0}^{x=p+q+1}\left(\frac{x^{(p)}}{p!}-\frac{x^{(q)}}{q!}\right)a_{p+q+1-x}b_{xrst}+\sum_{x=0}^{x=q+r+1}\left(\frac{x^{(q)}}{q!}-\frac{x^{(r)}}{r!}\right)a_{q+r+1-x}b_{pxst}\\ & +\sum_{x=0}^{x=r+s+1}\left(\frac{x^{(r)}}{r!}-\frac{x^{(s)}}{s!}\right)a_{r+s+1-x}b_{pqxt}+\sum_{x=0}^{x=s+t+1}\left(\frac{x^{(s)}}{s!}-\frac{x^{(t)}}{t!}\right)a_{s+t+1-x}b_{pqrx}\\ & +a_{p+1}b_{qrst}-a_{t+1}b_{pqrs}.\end{aligned}$$

The coefficient of any symbol of $n+1$ letters is deduced by a similar rule from the coefficients of the symbols of n letters, obtained by crossing out two successive letters in the given symbol of $(n+1)$ letters.

13. We now proceed to obtain the theorem $e^y e^x = e^w$. In the paper previously referred to (*Proceedings*, Vol. XXVIII., p. 381) it was proved that, if μ is a constant, $(1+\mu y)\,e^x$ is equal to $e^{x+\mu z}$ + terms involving powers of μ of the second and higher orders.

We might express this in the form of the equation

$$(1+\mu y)\, e^{x} = e^{x+\mu z}+\mu^2 R,$$

where R is some operator formed by combinations of x and y.

Let the result of substituting $x+\mu z$ for x in z, when z is expressed explicitly in terms of x and y, be called P_1; and let P_2 be obtained from P_1 by expressing P_1 explicitly in terms of x and y, and substituting $x+\mu z$ for x in it, and let P_3 be similarly obtained from P_2, P_4 from P_3, and so on.

Defining x_1 as $x+\mu z$, x_2 as $x_1+\mu P_1$, x_n as $x_{n-1}+\mu P_{n-1}$, we see that the result of substituting $x+\mu z$ for x in x_1 (when explicitly expressed in terms of x and y) is x_2; and generally that the result of substituting $x+\mu z$ for x in x_{n-1} (when explicitly expressed in terms of x and y) is x_n. We have therefore

$$\begin{aligned}
(1+\mu y)\, e^{x} &= e^{x_1}+\mu^2 R,\\
(1+\mu y)\, e_{2}^{x_1} &= e^{x}+\mu^2 R_1,\\
\ldots\quad \ldots &\quad \ldots \quad \ldots\\
(1+\mu y)\, e^{x_{n-1}} &= e^{x_n}+\mu^2 R_{n-1},
\end{aligned}$$

where R_1, R_2, ... R_{n-1} mean the result of substituting for x, x_1, x_2, ... x_{n-1} respectively in R.

Multiplying the first of the above equations by $(1+\mu y)^{n-1}$, the second by $(1+\mu y)^{n-2}$, and so on, and adding, we obtain

$$(1+\mu y)^n e^{x} = e^{x_n}+\mu^2 \left\{(1+\mu y)^{n-1} R+(1+\mu y)^{n+2} R_1+\ldots+R_{n-1}\right\}.$$

Let us now take $\mu n = 1$, and let n increase indefinitely, then, as in ordinary algebra,

$$(1+\mu y)^n = e^{y}.$$

The subject on which the operators x, y are to act will be supposed such that R operating on it gives a finite result, and the like will be supposed to hold for each of the operators R_1, R_2, ... It will follow that the effects of $(1+\mu y)^{n-1} R$, $(1+\mu y)^{n-2} R_1$, ... will also be severally finite, and if A is the greatest effect any one of them can have, then the greatest effect of

$$\mu^2 \left\{(1+\mu y)^{n-1} R+(1+\mu y)^{n-2} R_1+\ldots+R_{n-1}\right\}$$

will not exceed $\dfrac{1}{n^2}\, nA$, that is, their effect will ultimately (as n increases) be zero.

We have thus obtained the equation

$$e^y e^x = e^{x_\infty},$$

where x_∞ denotes the limit to which x_n tends as n increases indefinitely.

14. Now $$x_n = x+\mu z+\mu\,(P_1+P_2+\ldots+P_{n-1}),$$

so that to obtain the form of x_n as n approaches infinity it will be necessary to evaluate $P_1, P_2, \ldots P_{n-1}$.

P_1 being the result of substituting $x+\mu z$ for x in z, we have

$$P_1 = z+\mu z_1+\mu^2 z_2+\ldots+\mu^r z_r+\ldots \text{ to infinity.*}$$

So that we obtain

$$\begin{aligned} P_2 = P_1 &+\mu z_1+\mu^2\ z_{11}+\mu^3\ z_{12}+\ldots \\ &+\mu^2 z_2+\mu^3\ z_{21}+\mu^4\ z_{22}+\ldots \\ &+\ldots \quad \ldots \quad \ldots \quad \ldots \\ &+\mu^r z_r+\mu^{r+1} z_{r1}+\mu^{r+2} z_{r2}+\ldots \\ &+\ldots \quad \ldots \quad \ldots \quad \ldots .\dagger \end{aligned}$$

We notice that in P_2 the coefficient of μ^r is

$$2z_r+z_{1,r-1}+z_{2,r-2}+\ldots+z_{r-1,1},$$

that is, the z's which have the single suffix are counted twice, those which have the double suffix are counted once.

Expressing P_3 similarly, we notice that the coefficient of μ^r is

$$3z_r+3\,(z_{1,r-1}+z_{2,r-}\ +\ldots+z_{r-1,1})+\Sigma z_{p_1\,p_2\,p_3},$$

where the summation is taken for all positive integral values (excluding zeros) of p_1, p_2, p_3 which make

$$p_1+p_2+p_3 = r.$$

Let $\Sigma z_{p_1 p_2 \ldots p_\lambda}$, where the summation is taken for all positive integral values of $p_1, p_2, \ldots p_\lambda$ which make

$$p_1+p_2+\ldots+p_\lambda = r,$$

be written $$\overset{\lambda . r}{C}.$$

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* The notation z_r is here employed for temporary convenience to denote the coefficient of μ^r, and has no connexion with the notation z_r meaning $z_{r-1}x-xz_{r-1}$.

† Here z_{pq} is taken to denote the coefficient μ^q when $x+\mu z$ is substituted for x in z_p (explicitly expressed in terms of x and y), and the result expanded in powers of μ.

Then it is suggested by the above two particular cases that the coefficient of μ^r in P_n is

$$nz_r + \frac{n(n-1)}{2!}\overset{2.r}{C} + \dots \frac{n!}{\lambda!\,(n-\lambda)!}\overset{\lambda.r}{C} + \dots + \overset{n.r}{C}.$$

The summation is for n terms, but if $r<n$ it need not be taken to more than r terms, for from the definition

$$\overset{r+\kappa.r}{C} \equiv 0.$$

Assuming that this law holds for P_n (we have seen its truth when $n=1$, 2, or 3), we shall prove that it holds also for P_{n+1}.

15. We can write P_n in the following form :—

$$P_n = z + \mu n z_1 + \mu^2 \left\{ nz_2 + \frac{n(n-1)}{2!} z_{11} \right\}$$
$$+ \mu^3 \left\{ nz_3 + \frac{n(n-1)}{2!}(z_{12}+z_{21}) + \frac{n(n-1)(n-2)}{3!} z_{111} \right\}$$
$$+ \dots .$$

Consider now how the separate terms of this series contribute to the generation of the coefficient of μ^r in P_{n+1}. A term such as $\mu^\kappa z_{p_1p_2\dots p_m}$ becomes, in generating P_{n+1},

$$\mu^\kappa z_{p_1p_2\dots p_m} + \mu^{\kappa+1} z_{p_1p_2\dots p_m 1} + \mu^{\kappa+2} z_{p_1p_2\dots p_m 2} + \dots \mu^{\kappa+s} z_{p_1p_2\dots p_m s} + \dots .$$

If, then, we use the symbol $\underset{s}{\overset{\lambda.m}{C}}$ to denote the result of changing every term such as $z_{p_1p_2\dots p_m}$ in $\overset{\lambda.m}{C}$ into $z_{p_1p_2\dots p_m s}$, we see that to the coefficient of μ^r in P_{n+1},

z	has contributed	z_r,
$\mu n z_1$	" "	$n \underset{r-1}{\overset{1.1}{C}}$,
$\mu^2 \left\{ nz_2 + \frac{n(n-1)}{2!} z_{11} \right\}$	" "	$n \underset{r-2}{\overset{1.2}{C}} + \frac{n(n-1)}{2!} \underset{r-2}{\overset{2.2}{C}}$,
the κ^{th} term	" "	$n \underset{r-\kappa+1}{\overset{1.\kappa-1}{C}} + \frac{n(n-1)}{2!} \underset{r-\kappa+1}{\overset{2.\kappa-1}{C}} + \dots$,

the summation going till either the numerical coefficient or $\underset{r-\kappa+1}{\overset{\lambda.\kappa-1}{C}}$ vanishes; that is, either for n terms or $\kappa-1$ terms, whichever may be the lesser number.

The limiting value of κ will be $r+1$, if we understand by $\overset{p.q}{\underset{0}{C}}$ merely $\overset{p.q}{C}$.

Hence the coefficient of μ^r in P_{n+1} is

$$z_r+n\left(\overset{1.1}{\underset{r-1}{C}}+\overset{1.2}{\underset{r-2}{C}}+\ldots+\overset{1.r-1}{\underset{1}{C}}\right)+\frac{n(n-1)}{2!}\left(\overset{2.2}{\underset{r-2}{C}}+\overset{2.3}{\underset{r-3}{C}}+\ldots+\overset{2.r-1}{\underset{1}{C}}\right)+\ldots$$

$$\ldots+\frac{n!}{\lambda!\,(n-\lambda)}\left(\overset{\lambda.\lambda}{\underset{r-\lambda}{C}}+\overset{\lambda.\lambda+1}{\underset{r-\lambda-1}{C}}+\ldots+\overset{\lambda.r-1}{\underset{1}{C}}\right)+\ldots$$

(the limit of λ being n or $r-1$, whichever may be the lower integer),

$$+n\overset{1.r}{\underset{0}{C}}+\frac{n(n-1)}{2!}\overset{2.r}{\underset{0}{C}}+\ldots+\frac{n!}{\lambda!\,(n-\lambda)}\overset{\lambda.r}{\underset{0}{C}}+\ldots,$$

(the limit of λ being n or r, whichever may be the lower integer).

Now it is at once seen from the definition of the symbol $\overset{\lambda.r}{C}$ that

$$\overset{\lambda.r}{C}=\overset{\lambda-1.r-1}{\underset{1}{C}}+\overset{\lambda-1.r-2}{\underset{2}{C}}+\ldots+\overset{\lambda-1.\lambda-1}{\underset{r-\lambda+1}{C}};$$

therefore

$$\overset{2.r}{C}=\overset{1.1}{\underset{r-1}{C}}+\overset{1.2}{\underset{r-2}{C}}+\ldots+\overset{1.r-1}{\underset{1}{C}},$$

$$\overset{3.r}{C}=\overset{2.2}{\underset{r-2}{C}}+\overset{2.3}{\underset{r-3}{C}}+\ldots+\overset{2.r-1}{\underset{1}{C}},$$

$$\ldots\qquad\ldots\qquad\ldots\qquad\ldots$$

$$\overset{\lambda+1.r}{C}=\overset{\lambda.\lambda}{\underset{r-\lambda}{C}}+\overset{\lambda.\lambda+1}{\underset{r-\lambda+1}{C}}+\ldots+\overset{\lambda.r-1}{\underset{1}{C}};$$

the coefficient of μ^r in P_{n+1} is therefore

$$z_r+n\overset{2.r}{C}+\frac{n(n-1)}{2!}\overset{3.r}{C}+\ldots+\frac{n!}{(\lambda-1)!\,(n+1-\lambda)}\overset{\lambda.r}{C}+\ldots$$

(the limit of λ being the smaller of the two, $n+1$ and r)

$$+n\overset{.r}{C}+\frac{n(n-1)}{2!}\overset{2.r}{C}+\frac{n(n-1)(n-2)}{3!}\overset{3.r}{C}+\ldots\frac{n!}{\lambda!\,(n-\lambda)}\overset{\lambda.r}{C}+\ldots$$

(the limit of λ being the smaller of the two, n and r)

$$=(n+1)\,z_r+\frac{(n+1)\,n}{2!}\overset{2.r}{C}+\ldots+\frac{(n+1)!}{\lambda!\,(n+1-\lambda)!}\overset{\lambda.r}{C}+\ldots$$

(the limit of λ being the smaller of the two, $n+1$ and r);

that is, the law holds also for P_{n+1}, and therefore universally.

16. We have thus found the form of P_n. Our object is, however, to find the form of x_n; this may easily be deduced. First, let us find the coefficient of μ^r $(r>0)$ in $P_1+P_2+\ldots+P_{n-1}$; this is made up of contributions

from P_1, z_r,

" P_2, $2z_r+\overset{2.r}{C}$,

" P_3, $3z_r+3\overset{2.r}{C}+\overset{3.r}{C}$,

" P_{n-1}, $(n-1)\,z_r+\dfrac{(n-1)(n-2)}{2!}\overset{2.r}{C}+\ldots+\dfrac{(n-1)!}{\lambda!\,(n-1-\lambda)}\overset{\lambda.r}{C}+\ldots$

(the limit of λ being the smaller of the two, $n-1$ and r).

Adding, we see that the coefficient of μ^r is

$$\frac{n\,(n-1)}{2!}z_r+\frac{n\,(n-1)(n-2)}{3!}\overset{2.r}{C}+\ldots+\frac{n!}{(\lambda+1)!\,(n-\lambda-1)!}\overset{\lambda.r}{C}+\ldots$$

(the limit of λ being the smaller of the two, $n-1$ and r).

Call this series D_{r+1}.

Now the term independent of μ in $P_1+P_2+\ldots+P_{n-1}$ is obviously $(n-1)\,z$; therefore

$$\begin{aligned} x_n &= x+\mu z+\mu\,(P_1+P_2+\ldots+P_{n-1}) \\ &= x+n\mu z+\mu^2 D_2+\mu^3 D_3+\ldots \text{ to infinity.} \end{aligned}$$

17. What, then, does this approach when we take $\mu n=1$, and let n increase indefinitely?

The only coefficient in the series $\mu^r D_r$ which does not vanish when $\mu=\dfrac{1}{n}$ and n is indefinitely increased, is the coefficient of $\overset{r-1.r-1}{C}$; for

$$\frac{n!\left(\dfrac{1}{n}\right)^r}{(\lambda+1)!\,(n-\lambda-1)!}=\frac{n\,(n-1)\ldots(n-\lambda)}{n^r\,(\lambda+1)!},$$

and $$r-1\geqq\lambda;$$

therefore the expression vanishes when n increases indefinitely, unless

$$r=\lambda+1,$$

in which case $\mu^r D_r$ becomes $$\frac{\overset{r-1.r-1}{C}}{r!}.$$

The limiting value of x_n, as n approaches infinity, is therefore

$$x+z+\frac{\overset{1.1}{C}}{2!}+\frac{\overset{2.2}{C}}{3!}+\frac{\overset{3.3}{C}}{4!}+\dots \text{ to infinity.}$$

But the only term in $\overset{1.1}{C}$ is z_1, the only term in $\overset{2.2}{C}$ is z_{11}, the only term in $\overset{r.r}{C}$ is $z_{1,1,\dots}$ to r suffixes.

Now z_1 denotes the coefficient of μ in the result of substituting $x+\mu z$ for x in z, and we have seen that it is equal to $\Sigma b_{pq}(p, q)$.

z_{11}, then, denoting the coefficient of μ when for x we write $x+\mu z$ in $\Sigma b_{pq}(p, q)$, has been shown to be $\Sigma b_{pqr}(p, q, r)$, and so on; so that we see finally that

$$x_\infty = x+y+a_1y_1+a_2y_2+\dots+\frac{1}{2!}\Sigma b_{bq}(p, q)+\frac{1}{3!}\Sigma b_{pqr}(p, q, r)+\dots,$$

and $$e^ye^x = e^{x_\infty}.$$

It will be noticed that the terms in b_{pq} are of the second degree in the set $a_1a_2a_3\dots$, and of weight $p+q+1$ in the suffixes of that set; and we easily see the more general fact that the weight of every term is constant; *e.g.*, b_{pqrst} has weight

$$p+q+r+s+t+4;$$

thus the functions are isobaric in the set $a_1, a_2, a_3, \dots$.

Also we see that the number of letters in a symbol $(pqrst)$ expresses the degree of the symbol in y, and $p+q+r+s+t$ expresses the degree in x.

18. We have seen how to express e^ye^x in the form e^w, and have obtained an expression for w, but for development of the result it will be found useful to change the form in which w is expressed.

y_p means the same as $(yx^{(p)})$; it will be advantageous to express (p, q) also in the form of a sum of terms $(yx^{(m)}\dots yx^{(n)})$, and similarly to express $(p, q, r)\dots$.

We can easily see that

$$-(y_py_q) = (y_{p+1}y_{q-1})-(y_py_{q-1}x),$$

and, by applying the same transformation a second time, we see that

$$(y_py_q) = (y_{p+2}y_{q-2})-2(y_{p+1}y_{q-2}x)+(y_py_{q-2}xx).$$

Assume, then,

$$(-1)^r (y_p y_q)$$

$$= (y_{p+r} y_{q-r}) - r\,(y_{p+r-1}\, y_{q-2} x) + \frac{r\,(r-1)}{2!}\,(y_{p+r-2} y_{q-r} x^{(2)}) - \dots .$$

We shall prove that the expression on the right is unaltered in value when for r we write $r+1$; that is, if the identity holds for the value r, it also holds for the value $r+1$, and, holding for $r=1$, it will then be seen to hold generally.

Since $\qquad -(y_p y_q) = (y_{p+1} y_{q-1}) - (y_p y_{q-1} x),$

we see that $\quad -(y_p y_q x^{(r)}) = (y_{p+1} y_{q-1} x^{(r)}) - (y_p y_{q-1} x^{(r+1)})$;

therefore $\quad -(y_{p+r} y_{q-r}) = (y_{p+r+1} y_{q-r-1}) - (y_{p+r} y_{q-r-1} x)$

$$+r\,(y_{p+r-1} y_{q-r} x) = -r\,(y_{p+r} y_{q-r-1} x) + r\,(y_{p+r-1} y_{q-r-1} x^{(2)})$$

$$-\frac{r\,(r-1)}{2!}\,(y_{p+r-2} y_{q-r} x^{(2)}) = \frac{r\,(r-1)}{2!}\,(y_{p+r-1} y_{q-r-1} x^{(2)})$$

$$-\frac{r\,(r-1)}{2!}\,(y_{p+r-2} y_{q-r-1} x^{(3)})$$

$$+ \quad \dots \quad \dots \quad \dots \quad = \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots$$

Adding, we see that the sum of the members on the left, which we know, is

$$(-1)^{r+1} (y_p y_q)$$

$$= (y_{p+r+1} y_{q-r-1}) - (r+1)(y_{p+r} y_{q-r-1} x) + \frac{(r+1)\,r}{2!}\,(y_{p+r-1} y_{q-r-1} x^{(2)}) - \dots,$$

and this proves the theorem required.

19. We can therefore write $(-1)^q (y_p y_q)$ in the form

$$(y_{p+q} y) - q\,(y_{p+q-1} y x) + \frac{q\,(q-1)}{2!}\,(y_{p+q-2} y x^{(2)}) - \dots$$

$$= (y x^{(p+q)} y) - q\,(y x^{(p+q-1)} y x) + \frac{q\,(q-1)}{2!}\,(y x^{(p+q-2)} y x^{(2)}) - \dots,$$

the general term being

$$(-1)^s \frac{q!}{s!\,(q-s)!}\,(y x^{(p+q-s)} y x^{(s)}),$$

where s is any integer $\leqq q$.

By taking the separate terms of this series we see in the same way that the general term in $(y_p y_q y_r)$ is

$$\frac{(-1)^{s+t+q+r}\, q!\; r!}{s!\; t!\; (q-s)!\; (r-t)!}\,(y x^{(p+q-s)} y x^{(r+s-t)}\, y x^{(t)}),$$

and the summation is to be taken for all integral values

$$s \leqq q \quad \text{and} \quad t \leqq r.$$

Any other term in w may be similarly expressed.

If, now, instead of y we write x_1, and instead of x write x_2, and if we denote such a term as $(x_1 x_2^{(p)} x_1^{(q)} x_2^{(r)})$ by $(12^{(p)} 1^{(q)} 2^{(r)})$, we see that

$$e^{x_1} e^{x_2} = e^{w},$$

where $\quad w = x_1 + x_2 + \beta_{12}(12) + \beta_{122}(122) + \beta_{121}(121) + \ldots,$

where the β's are a new set of numerical constants.

20. How are these constants to be determined? By expressing the old symbols $(y_p y_q y_r)$ in terms of the new; *e.g.*, because

$$\Sigma b_{pqr}(y_p y_q y_r) = \Sigma (-1)^{s+t+q+r} b_{pqr} \frac{q!\, r!\, (12^{(p+q-s)} 12^{(r+s-t)} 12^{(t)})}{s!\, t!\, (q-s)!\, (r-t)!},$$

writing $\quad p+q-s = \lambda, \quad r+s-t = \mu, \quad t = \nu,$

we see that

$$\beta_{12^{(\lambda)}12^{(\mu)}12^{(\nu)}} = \Sigma (-1)^{\mu+q} \frac{b_{pqr}(\lambda+\mu+\nu-q)!\, r!}{(\lambda-p)!\, (r-\nu)!\, (\mu+\nu-r)!\, \nu!},$$

where the summation on the right is to be taken for all possible (positive integral or zero) values of p, q, and r which make no factorial $(\lambda-p)!$, &c., negative.

21. Let $x_1, x_2, \ldots x_n$ be n operators which obey the distributive and associative laws, but not the commutative.

$(1, 2)$ will be used to denote $x_1 x_2 - x_2 x_1$,

$(1, 2, 3)$ " " $(1, 2)\, x_3 - x_3 (1, 2)$,

and so generally. We thus get a series of symbols; X_1 will be used to denote any symbol of the first order, *i.e.*, $x_1, x_2, \ldots$ or x_n, X_2 any symbol of the form (p, q), and so on.

Suppose that from these symbols $X_1, X_2, \ldots$ we were to try and form others, such as

$$(X_p X_q) = X_p X_q - X_q X_p;$$

it will be proved that we get no new symbols, but merely a linear combination of symbols of the form X_{p+q}, so that the symbols themselves may be said to form a linear group.

If one of the suffixes p or q in $(X_p X_q)$ is unity, this is evident, for $X_p x_r - x_r X_p$ is by definition a symbol of form X_{p+1}.

Suppose, now, that we have proved that, if one of the suffixes p or q does not exceed r, $(X_p X_q)$ can be expressed as a linear combination of symbols of the form X_{p+q}; then it will be shown that the same result holds if one of the suffixes does not exceed $r+1$; and the theorem, having been proved when $r = 1$, will then hold generally.

By definition $\quad X_{r+1} \equiv X_r x_s - x_s X_r$;

therefore

$$\begin{aligned} X_n X_{r+1} - X_{r+1} X_n &\equiv X_n (X_r x_s - x_s X_r) - (X_r x_s - x_s X_r) X_n \\ &\equiv (X_n X_r - X_r X_n) x_s - x_s (X_n X_r - X_r X_n) \\ &\quad + X_r (X_n x_s - x_s X_n) - (X_n x_s - x_s X_n) X_r. \end{aligned}$$

Now, by hypothesis, $X_n X_r - X_r X_n$ is expressible in terms of symbols of the form X_{n+r}; therefore

$$(X_n X_r - X_r X_n) x_s - x_s (X_n X_r - X_r X_n)$$

is expressible as a linear combination of symbols of the form X_{n+r+1}.

Again, $\quad X_r (X_n x_s - x_s X_n) - (X_n x_s - x_s X_n) X_r$

is of the form $X_r X_{n+1} - X_{n+1} X_r$; and therefore (since one suffix does not exceed r) in terms of symbols of the form X_{n+r+1}; that is, $(X_n X_{r+1})$ is expressible as a linear combination of symbols of the form X_{n+r+1}.

22. There are n independent symbols of the first order; there are n^2 symbols of the second order, but only $\frac{n(n-1)}{2}$ of these will be independent, owing to the identities

$$(p, q) + (q, p) \equiv 0, \quad (p, p) \equiv 0.$$

The n^3 symbols of the third order will be still further reduced by reason of the additional identity

$$(p, q, r) + (q, r, p) + (r, p, q) \equiv 0;$$

and there are doubtless additional identities for symbols of the fourth and higher orders. Without entering on the question of the number of independent symbols of any given order, we shall suppose that a system of independent symbols has been obtained.

23. If, now,
$$x_1' = a_{11}x_1 + a_{21}x_2 + \dots + a_{n1}x_n,$$
$$\vdots$$
$$x_n' = a_{1n}x_1 + a_{2n}x_2 + \dots + a_{nn}x_n,$$

where the set of letters $a_{p,q}$ denote any arbitrary constants, we can express the symbols $X_1'X_2' \dots$, which correspond to $X_1 X_2 \dots$, as linear combinations of the latter, the coefficients in the identities being functions of the set of constants a_{pq}.

In particular, if
$$x_1' = \lambda_1 x_1 + \dots + \lambda_n x_n,$$
$$x_2' = \mu_1 x_1 + \dots + \mu_n x_n,$$

we can express any such symbol as $(1, 2, 2)'$ in terms of the complete set of independent symbols of the third order, and the parameters

$$\lambda_1 \dots \lambda_n, \quad \mu_1 \dots \mu_n;$$

but we have proved that $\quad e^{x_1'} e^{x_2'} = e^w$,

where $\quad w = x_1' + x_2' + \beta_{12}(12)' + \beta_{122}(122)' + \beta_{121}(121)'$,

and the β's are numerical constants whose values we know how to determine; therefore

$$e^{\lambda_1 x_1 + \dots + \lambda_n x_n} e^{\mu_1 x_1 + \dots + \mu_n x_n} = e^{\Sigma a_p x_p + \Sigma a_{pq}(p, q) + \Sigma a_{pqr}(p, q, r) + \dots},$$

where the a's are now functions of the parameters $\lambda_1 \dots \lambda_n$, $\mu_1 \dots \mu_n$, and known numerical constants; thus $a_{p,q,r}$ is the numerical constant $\beta_{122} \times$ coefficient of the term (p, q, r) which occurs in expressing $(122)'$ in terms of symbols of the third order derived from $x_1 \dots x_n + \beta_{121} \times$ coefficient similarly obtained from $(121)' + \dots$.

$a_{p,q,r}$ is clearly of the third degree in $\lambda_1 \dots \lambda_n$, $\mu_1 \dots \mu_n$, and so generally.

24. The more general theorem which has been enunciated is seen to be implicitly proved in what has gone before; for we have proved that
$$e^{w_1} e^{w_2} = e^{w_3},$$

where
$$w_3 = w_1 + w_2 + a_1(w_1 w_2) + a_2(w_1 w_2 w_2) + \dots$$
$$+ \tfrac{1}{2}\Sigma b_{pq}\left\{(w_1 w_2^{(p)})(w_1 w_2^{(q)})\right\} + \dots,$$

and we see that every term on the right is of the form

$$c_1 x_1 + \dots + c_n x_n + c_{pq}(p, q) + \dots + c_{pqr}(p, q, r) + \dots.$$