MATH 744, FALL 2010
HOMEWORK ASSIGNMENT #3, PARTIAL SOLUTIONS

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Hall, Chapter 3, Problem 9. Let

$$X = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}, \quad a > 0, \quad Y = \begin{pmatrix} 0 & \pi \\ -\pi & 0 \end{pmatrix}.$$

Then

$$e^X = \begin{pmatrix} e^a & 0 \\ 0 & e^{-a} \end{pmatrix}, \quad e^Y = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

On the other hand,

$$e^X e^Y = \begin{pmatrix} -e^a & 0 \\ 0 & -e^{-a} \end{pmatrix}$$

has trace $-e^a - e^{-a} = -2\cosh a < -2$, so by problem 30 in Chapter 2, $e^X e^Y$ is not in the image of the exponential map. Thus there cannot be a $Z \in \mathfrak{sl}(2, \mathbb{R})$ with $e^X e^Y = e^Z$, and so the Campbell-Baker-Hausdorff formula can't be valid in this case.

Additional Problem. First suppose $\dim \mathfrak{g} = 3$. Since $\mathfrak{g}$ is nilpotent, its ascending central sequence terminates at $\mathfrak{g}$, and in particular, the center $\mathfrak{z}$ of $\mathfrak{g}$ is non-zero. If $\dim \mathfrak{z} = 3$, $\mathfrak{g}$ is abelian. If $\dim \mathfrak{z} = 2$, that means $\mathfrak{g}$ has a basis X, Y, Z with X and Y central. Since $[Z, Z] = 0$ and Z has 0 bracket with either X or Y, in fact Z is central also, so $\dim \mathfrak{z} = 2$ is impossible. That leaves just one more case, $\dim \mathfrak{z} = 1$. For any Lie algebra $\mathfrak{g}$, the bracket of two elements only depends on their classes mod $\mathfrak{z}$ (since $\mathfrak{z}$ has zero brackets with everything), so the bracket factors through an antisymmetric bilinear map $\mathfrak{g}/\mathfrak{z} \times \mathfrak{g}/\mathfrak{z} \to \mathfrak{g}$. The image is at most one-dimensional since if $\dot{X}$ and $\dot{Y}$ span $\mathfrak{g}/\mathfrak{z}$ and come from elements X and Y in $\mathfrak{g}$, $[X, X] = [Y, Y] = 0$ and $[Y, X] = -[X, Y]$. But if $\mathfrak{z}$ has dimension 1, that means not every element of $\mathfrak{g}$ is central, so there is at least one non-trivial bracket, say $[X, Y]$. On the other hand, the quotient Lie algebra $\mathfrak{g}/\mathfrak{z}$ is 2-dimensional. By a previous exercise, any non-abelian 2-dimensional Lie algebra has a basis T and W with $[T, W] = W$. So $\operatorname{ad} T$ has W as an eigenvector with eigenvalue 1 and so $\operatorname{ad} T$ is not nilpotent. Since $\mathfrak{g}/\mathfrak{z}$ is a quotient of a nilpotent Lie algebra, it is also nilpotent, and so $\mathfrak{g}/\mathfrak{z}$ must be abelian. That means $[\mathfrak{g}, \mathfrak{g}]$ is zero mod $\mathfrak{z}$, or $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{z}$. So if $[X, Y] \neq 0$, $Z = [X, Y]$ spans $\mathfrak{z}$ (since the latter was 1-dimensional) and $\mathfrak{g}$ is Heisenberg.

Now suppose $\dim \mathfrak{g} = 4$. Again, the center $\mathfrak{z}$ must be non-zero, and $\mathfrak{g}/\mathfrak{z}$ must be nilpotent of smaller dimension than $\mathfrak{g}$. If $\dim \mathfrak{z} = 4$, $\mathfrak{g}$ is abelian. Just as in the case of $\dim \mathfrak{g} = 3$, it is impossible for $\mathfrak{z}$ to be of codimension 1. So $\dim \mathfrak{z}$ must be either 2 or 1. In the first case, $\dim \mathfrak{z} = 2$, just as in the case of $\dim \mathfrak{g} = 3$, $\mathfrak{g}/\mathfrak{z}$ is nilpotent of dimension 2 and thus abelian, which means $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{z}$. So choose X and Y with $[X, Y] = Z$ non-zero and central. Since $\dim \mathfrak{z} = 2$, we can choose another central element W linearly independent of Z, and W has zero bracket with everything. So we see $\mathfrak{g}$ has a basis X, Y, Z, W with $[X, Y] = Z$ and no other non-trivial brackets, so $\mathfrak{g}$ is a Lie algebra direct sum of a Heisenberg algebra with a one-dimensional Lie algebra (spanned by W).

There is just one more case, $\dim \mathfrak{z} = 1$, $\mathfrak{g}/\mathfrak{z}$ is nilpotent of dimension 3. In this case, $\mathfrak{g}/\mathfrak{z}$ can't be abelian, because if it were, we'd have $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{z}$, and the Lie bracket would be a skew-symmetric bilinear map $\mathfrak{g}/\mathfrak{z} \times \mathfrak{g}/\mathfrak{z} \to \mathfrak{z}$ with one-dimensional image, which would force it to be identically zero on a one-dimensional subspace of $\mathfrak{g}/\mathfrak{z}$ by dimension counting. This would result in $\mathfrak{z}$ being 2-dimensional, a contradiction. So $\mathfrak{g}/\mathfrak{z}$

is a non-abelian nilpotent 3-dimensional Lie algebra, hence is Heisenberg. Now choose a standard basis $\dot{X}$, $\dot{Y}$, $\dot{Z}$ of the Heisenberg Lie algebra $\mathfrak{g}/\mathfrak{z}$ and pull back to basis elements X, Y, Z of $\mathfrak{g}$. We have $[X,Y] = Z$ mod $\mathfrak{z}$. Changing Z by something in the center $\mathfrak{z}$, we can always assume $[X,Y] = Z$. Since Z is not central in $\mathfrak{g}$ (only in $\mathfrak{g}/\mathfrak{z}$), it has a non-trivial bracket with either X or Y. Without loss of generality we can assume it's with X. Then $[X,Z]$ is zero mod $\mathfrak{z}$ but non-zero, hence spans the one-dimensional center $\mathfrak{z}$. So we get the bracket relations $[X,Y] = Z$, $[X,Z] = W$ with W central, as desired.