MATH 744, FALL 2010
HOMEWORK ASSIGNMENT #4, PARTIAL SOLUTIONS

JONATHAN ROSENBERG

Ch. 4, Problem #15.
(a) Checking that P_1 and P_2 are intertwining maps is straightforward:

$$P_1(g \cdot (v_1, v_2)) = P_1(g \cdot v_1, g \cdot v_2) = g \cdot v_1 = g \cdot P_1(v_2, v_2),$$

and similarly for P_2. If U is irreducible, Schur's Lemma thus implies that $P_1|_U$ is either an isomorphism or 0, and similarly for $P_2|_U$. If $P_1|_U = 0$, that means $U \subseteq V_2$, and since U and V_2 are both irreducible, they must be equal. Similarly, if $P_2|_U = 0$, then $U = V_1$. So we just have to rule out the case where $P_1|_U \neq 0$ and $P_2|_U \neq 0$. But in this case, we've seen that $P_1|_U$ and $P_2|_U$ are both isomorphisms. In other words, $U \cong V_1$ and $U \cong V_2$. Since $V_1 \not\cong V_2$, this is impossible.
(b) Suppose U is a proper, nonzero invariant subspace of $V_1 \oplus V_2$. By Exercise 13, U contains an irreducible invariant subspace. By (a), this is either V_1 or V_2. Suppose, say, that $U \supsetneq V_1$. Let $(v_1, v_2) \in U$ with $v_2 \neq 0$. (This is possible since U is strictly bigger than V_1.) Since $V_1 \subset U$, $(v_1, 0) \in U$, so $(v_1, v_2) - (v_1, 0) = (0, v_2) \in U$. But the invariant subspace generated by $(0, v_2)$ is all of V_2, so $U = V_1 \oplus V_2$, contradicting the assumption that U was proper.

Additional Problem #1. By Schur's Lemma, the commuting ring of π on V is just the scalar multiples of the identity. On the other hand, since the representation of G on W is trivial, the commuting ring of this representation is all of $\operatorname{End}_{\mathbb{C}} W$, isomorphic to $M_r(\mathbb{C})$ (the $r \times r$ matrices with entries in $\mathbb{C}$). We claim the commuting ring of the representation $\pi \otimes 1_W$ on $V \otimes W$ is just $\mathbb{C} \cdot 1_V \otimes \operatorname{End}_{\mathbb{C}} W$. One direction is trivial — it is clear that anything of the form $1_V \otimes T$ commutes with the representation. So we just need to show we've exhausted everything.

If $e_1, \cdots e_n$ is a basis for W, then a basis for $\operatorname{End} W$ is the set of rank-one operators e_{ij} (sending e_j to e_i and killing all e_k for $k \neq j$). Thus any linear operator T in $\operatorname{End}(V \otimes W)$ has a unique expansion as $T = \sum_{i,j} T_{ij} \otimes e_{ij}$, with $T_{ij} \in \operatorname{End}(V)$. Suppose T commutes with all $\pi(g) \otimes 1_W$. That means $[T, \pi(g) \otimes 1_W] = \sum_{i,j} [T_{ij}, \pi(g)] \otimes e_{ij} = 0$ for all $g \in G$. Since the e_{ij} are linearly independent, $[T_{ij}, \pi(g)] = 0$ for all i, j and $g \in G$, so each T_{ij} is a scalar multiple of the identity, and $T = \sum_{i,j} c_{ij} 1_V \otimes e_{ij}$ for some scalars c_{ij}, i.e., $T \in \mathbb{C} \cdot 1_V \otimes \operatorname{End}_{\mathbb{C}} W$.

When it comes to invariant subspaces of $V \otimes W$, one has obvious invariant subspaces of the form $V \otimes U$, for U a subspace of W. To show these are all the invariant subspaces, it is easiest to think of $V \otimes W$ as $V^r = \overbrace{V \oplus \cdots \oplus V}^{r}$. Then a vector in $V \otimes W$ is just an r-tuple of vectors in V. We'll prove the result by induction on r and think of V^{r+1} as $V^r \oplus V$. Let Z be an invariant subspace of V^{r+1}. Projection p onto the final summand of V is G-equivariant, and so sends Z to an invariant subspace of V, which is either 0 or V. If it's 0, that means we can think of Z as embedded in V^r and the result follows from the inductive hypothesis. If $p(Z) = V$, we still have $Z \cap \ker p \subseteq V^r$, so $Z \cap \ker p = V \otimes U_1$ with U_1 a subspace of $\mathbb{C}^r$, by the inductive hypothesis. Now among all the invariant subspaces $Z_1 \subseteq Z$ with $p(Z_1) \neq 0$, there must be a minimal one (say by Zorn's Lemma, though if V is finite-dimensional, you don't need it). This Z_1 has to be irreducible, since if it weren't, we could contradict minimality. So $p|_{Z_1}$ is an isomorphism. Choose $v \neq 0$ in V and take its inverse image in Z_1, which must be of the form $(v_1, \cdots, v_r, v)$. The fact that this generates

Z_1, isomorphic to V, forces $v_1, \cdots, v_r$ to be multiples $\lambda_1 v, \cdots, \lambda_r v$ of v. (We'll see why in a moment, but let's assume this for now.) Thus $Z_1 = V \otimes \mathbb{C} \cdot (\lambda_1, \cdots, \lambda_r, 1)$, and

$$Z = V \otimes (U_1 \times \{0\}) + V \otimes \mathbb{C} \cdot (\lambda_1, \cdots, \lambda_r, 1) = V \otimes (U_1 \times \{0\} \oplus \mathbb{C} \cdot (\lambda_1, \cdots, \lambda_r, 1)) = V \otimes U$$

for some U.

Finally, we need to see why $v_1, \cdots, v_r$ are all multiples of v. If say v_j were not a multiple of v, then since V is irreducible, there would be a linear transformation T in the algebra generated by the action of G on V that kills v but not v_j. Then Z_1 would contain $(Tv_1, \cdots, Tv_r, Tv)$, which maps to 0 under p (since $Tv = 0$) but is nonzero since $Tv_j \neq 0$. This contradicts the assumption that $p|_{Z_1}$ is an isomorphism.

Additional Problem #2. The weights of π are ± 1, and the weights of $\pi \otimes \pi$ are obtained by adding weights of the tensor factors, so are of the form $\pm 1 \pm 1$. Thus ± 2 each occur once and 0 occurs twice (as $1-1$ and as $-1+1$). Since 2 is the highest weight, we must have a summand isomorphic to V_2 (the complexified adjoint representation). Subtracting off its weights, we still have the weight 0, so there is another summand of V_0. (Alternatively, the trivial representation occurs as a summand since π is self-contragredient, and $\pi \otimes \pi^* \cong \mathrm{Hom}_{\mathbb{C}}(\pi, \pi)$ contains a trivial summand, corresponding to the identity map $\pi \to \pi$.)

Similarly, the weights of $\pi \otimes \pi \otimes \pi$ are obtained by adding weights of the three tensor factors, so are all of the form $\pm 1 \pm 1 \pm 1$. Thus 3 and -3 each occur once, and 1 and -1 each occur three times (since we can have $1+1-1$, $1-1+1$, and $-1+1+1$, etc.). Since the weights of the irreducible representation V_n are $n, n-2, \cdots, -n$ and the weights determine the representation, the representation must be equivalent to $V_3 \oplus V_1 \oplus V_1$, which is the only representation with the correct weights. (To put it another way, since 3 is the highest weight, the representation contains a copy of V_3. Taking out the weights of V_3, what remains are two copies of the weights ± 1 of V_1.) The dimension count is right since $(3+1) + 2 \cdot (1+1) = 2^3 = 8$.