MATH 744, FALL 2010
HOMEWORK ASSIGNMENT #5, PARTIAL SOLUTIONS

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Ch. 5, Problem #2.

Suppose $v_1 \cdots, v_n$ is a basis for V consisting of weight vectors with weights λ_j. Let $v_1^*, \cdots, v_n^*$ be the dual basis for the dual space V^*. By definition, $\pi^*(X) = -\pi(X)^t$ acting on the dual space. So if $X \in \mathfrak{h}$, we have

$$\begin{aligned}\langle v_j, \pi^*(X)v_k^*\rangle &= -\langle v_j, \pi^t(X)v_k^*\rangle \\ &= -\langle \pi(X)v_j, v_k^*\rangle = -\lambda_j(X)\langle v_j, v_k^*\rangle = -\lambda_j(X)\delta_{jk} = -\lambda_k(X)\delta_{jk}.\end{aligned}$$

By the definition of the dual basis, that means $\pi^*(X)v_k^* = -\lambda_k(X)v_k^*$, so v_k^* is a weight vector with weight $-\lambda_k$.

Ch. 5, Problem #6.

The more interesting part is the decomposition of $V \otimes V$, where we take two copies of the standard representation $V = \mathbb{C}^3$ with highest weight $(1, 0)$. The general fact is that for any group G and any representation V, $V \otimes V$ always decomposes canonically into a direct sum of the symmetric tensors S^2V and the antisymmetric tensors $\bigwedge^2 V$. Those pieces in general might still not be irreducible, but in this case they are. Since V has dimension 3, S^2V has the same dimension as the space of homogeneous quadratic polynomials in 3 variables, or 6, and $\bigwedge^2 V$ has dimension $\binom{3}{2} = 3$. In fact, $\bigwedge^2 V$ is canonically isomorphic to the dual space V^*, since the wedge product gives a dual pairing between V and $\bigwedge^2 V$ with values in $\bigwedge^3 V \cong \mathbb{C}$. So one component of $V \otimes V$ is the 3-dimensional irreducible representation V^*, with highest weight $(0, 1)$. Another component is S^2V, which has as highest weight $(2, 0)$, corresponding to the (symmetric) tensor product $v \otimes v$, v a highest weight vector in V. Recall that the weights of V are $(1, 0)$, $(-1, 1)$, and $(0, -1)$, while the weights of V^* are $(0, 1)$, $(1, -1)$, and $(-1, 0)$. The weights of $V \otimes V$ are all possible sums of two weights of V, so these are $(2, 0)$, $(-2, 2)$, $(0, -2)$, each with multiplicity 1, and $(1, 0) + (-1, 1) = (0, 1)$, $(1, 0) + (0, -1) = (1, -1)$, $(-1, 1) + (0, -1) = (-1, 0)$, each with multiplicity 2 (since for each pair of distinct weight vectors, we can take the tensor product in either order). After pulling out the weights $(0, 1)$, $(1, -1)$, and $(-1, 0)$ of V^*, we have six weights, each with multiplicity 1, and these are all weights of the irreducible representation with highest weight $(2, 0)$. So $V \otimes V \cong V_{(2,0)} \oplus V_{(0,1)}$, with one irreducible summand of dimension 6 and one of dimension 3.

Ch. 5, Problem #12.

(a) We need to show that there is no $g \in G$ with $\mathrm{Ad}(g)(X) = -X$ for all $X \in \mathfrak{h}$. One quick method is to use the fact that the adjoint action is conjugation of matrices, i.e., $\mathrm{Ad}(g)(X) = gXg^{-1}$. So if X is invertible, which we can arrange (H_1 and H_2 are not invertible, but $2H_1 + H_2$ is invertible, for instance), then we get $\det \mathrm{Ad}(g)(X) = \det(gXg^{-1}) = \det X = \det(-X) = -\det X$, which is a contradiction since $\det X \neq 0$. This same argument shows -1 does not lie in the Weyl group of $SU(n)$ for any odd n.

(b) The irreducible representation V with highest weight (m, n), m, $n \geq 0$, turns out to have weights invariant under multiplication by -1 if and only if $m = n$. We can see this as follows. First suppose the highest weight is of the form (m, m), $m \geq 0$. Since the highest weight is invariant under interchange of the two coordinates, so is the set of all the weights. So if (n_1, n_2) is a weight of V, so is $\lambda = (n_2, n_1)$. But the weights are also invariant under the Weyl group, which is generated by the permutations $s_1 = (1, 2)$ and $s_2 = (2, 3)$. Note that $s_1(H_1) = -H_1$ and $s_1(H_2) = \mathrm{diag}(1, 0, -1) = H_1 + H_2$. Similarly $s_2(H_2) = -H_2$ and $s_2(H_1) = H_1 + H_2$. The action of $w = s_1s_2s_1$ on λ is as follows:

$$\begin{aligned}
w \cdot \lambda(H_1) &= \lambda(w^{-1} \cdot H_1) \\
&= \lambda(s_1s_2s_1 \cdot H_1) = -\lambda(s_1s_2 \cdot H_1) \\
&= -\lambda(s_1 \cdot (H_1 + H_2)) = -\lambda(-H_1 + (H_1 + H_2)) \\
&= -\lambda(H_2) = -n_1. \\
w \cdot \lambda(H_2) &= \lambda(w^{-1} \cdot H_2) \\
&= \lambda(s_1s_2s_1 \cdot H_2) = \lambda(s_1s_2 \cdot (H_1 + H_2)) \\
&= \lambda(s_1 \cdot ((H_1 + H_2) - H_2)) = \lambda(s_1 \cdot H_1) \\
&= -\lambda(H_1) = -n_2.
\end{aligned}$$

Thus w sends λ to $-(n_1, n_2)$, and the weights of V are invariant under multiplication by -1.

In the other direction, suppose the weights are invariant under -1. Then if (n_1, n_2) is a weight, $(-n_1, -n_2)$ is also a weight. As we just saw, this is conjugate under $w \in W$ to (n_2, n_1). So (n_2, n_1) is also a weight. In other words, the weights are invariant under interchange of the two coordinates. Now if (m, n) is the highest weight of an irreducible representation, then $w \cdot (m, n) = (-n, -m)$ is the lowest weight, and $(n, m) = -w \cdot (m, n)$ is also a highest weight. Since the highest weight is unique, it has to be of the form (m, m).