# MATH 744, FALL 2010
# HOMEWORK ASSIGNMENT #6 SOLUTIONS

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(a) $\mathfrak{h}$ is a solvable ideal in $\mathfrak{g}$, so $[X, \mathfrak{h}] \subset \mathfrak{h}$, and $\mathfrak{h} + \mathbb{C}X$ contains $\mathfrak{h}$ as an ideal of codimension 1, with $(\mathfrak{h} + \mathbb{C}X)/\mathfrak{h}$ one-dimensional and abelian. Since the class of solvable Lie algebras is closed under extensions, $\mathfrak{h} + \mathbb{C}X$ is solvable. Thus by Lie's Theorem, there is a basis for $V$ (possibly depending on $X$) with respect to which $\mathfrak{h} + \mathbb{C}X$ acts by upper-triangular matrices, and $[X, \mathfrak{h}]$ acts by strictly upper-triangular matrices. Thus if $Y, Z \in \mathfrak{h}$, $Y$ is upper-triangular, $[X, Z]$ is strictly upper-triangular, and the product $Y[X, Z]$ is strictly upper-triangular, hence nilpotent, and $\mathrm{Tr}_V(Y[X, Z]) = 0$.

(b) Let $\mathfrak{h}$ be the radical of $\mathfrak{g}$ and assume the trace form $H$ is nondegenerate. By (a), $H(Y, [X, Z]) = 0$ for $Y, Z \in \mathfrak{h}$, $X \in \mathfrak{g}$. Thus $H([Y, Z], X) = H(Y, [Z, X]) = 0$, and $[\mathfrak{h}, \mathfrak{h}]$ is orthogonal to everything in $\mathfrak{g}$, which by nondegeneracy means that $[\mathfrak{h}, \mathfrak{h}] = 0$. In other words, $\mathfrak{h}$ is abelian.

(c) Let $\mathfrak{h}$ be the radical of $\mathfrak{g}$. By (b), $\mathfrak{h}$ is abelian, and by (a), $[\mathfrak{g}, \mathfrak{h}] \subseteq \mathfrak{h} \cap \mathfrak{h}^{\perp}$, where the orthogonal complement is taken with respect to $H$. Let $\mathfrak{s} = \mathfrak{h}^{\perp}$. By the associativity property $H([Y, Z], X) = H(Y, [Z, X])$ with $Y \in \mathfrak{s}$, $Z \in \mathfrak{g}$, and $X \in \mathfrak{h}$, $\mathfrak{s}$ is an ideal. Let $\mathfrak{r}$ be the radical of $\mathfrak{s}$. This is a characteristic ideal in the ideal $\mathfrak{s}$, so it is a solvable ideal of $\mathfrak{g}$. Since any solvable ideal lies in the radical, $\mathfrak{r} \subseteq \mathfrak{h}$, but also $\mathfrak{r} \subseteq \mathfrak{s} = \mathfrak{h}^{\perp}$, so $\mathfrak{r} \subseteq \mathfrak{h} \cap \mathfrak{h}^{\perp}$. Thus assuming that $\mathfrak{h} \cap \mathfrak{h}^{\perp} = 0$, this will imply both that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{s}$ and that $\mathfrak{s}$ is semisimple. So $\mathfrak{h}$ is central and $\mathfrak{g}$ is reductive.

(d) If $\mathfrak{g} = \mathfrak{gl}(V) = \mathrm{End}(V)$, then $\mathfrak{g}$ is reductive but not semisimple, since it has a one-dimensional center (the multiples of the identity matrix). The form $H$ is non-degenerate on $\mathfrak{g}$ since if $e_{ij}$ is the matrix with a 1 in the $(i, j)$-entry and 0's everywhere else, then

$$H(e_{ij}, e_{kl}) = \mathrm{Tr}(e_{ij} e_{kl}) = \mathrm{Tr}(\delta_{jk} e_{il}) = \delta_{jk} \delta_{il}.$$

Thus $H$ is nondegenerate, with $\{e_{ji}\}$ the basis dual to $\{e_{ij}\}$. Note that $H$ is also nondegenerate on the center, since $H(1, 1) = \dim V$.

(e) If $\mathfrak{g}$ is semisimple, it is a direct sum of simple ideals $\mathfrak{s}_i$. For each $i$, $\mathfrak{s}_i \cap \mathfrak{s}_i^{\perp}$ is an ideal in $\mathfrak{s}_i$, hence since $\mathfrak{s}_i$ has no nontrivial ideals, must be either 0 (meaning $H$ is nondegenerate on $\mathfrak{s}_i$) or all of $\mathfrak{s}_i$. But if there is some $\mathfrak{s}_i$ for which $\mathfrak{s}_i \cap \mathfrak{s}_i^{\perp} = \mathfrak{s}_i$, then $\mathrm{Tr}_V(XY) = 0$ for all $X, Y \in \mathfrak{s}_i$, and by Cartan's Criterion, $\mathfrak{s}_i$ is solvable, a contradiction. Hence $H$ is nondegenerate on each $\mathfrak{s}_i$, and so is nondegenerate on all of $\mathfrak{g}$.